Fluid Kinematics and Constitutive Laws

In using the differential balance equations derived previously, it will be insightful to better understand the basic types of motion that a fluid element may experience. This is the area of "fluid kinematics." In particular, we will be interested in relating the *deformation* of a fluid element to the *stresses* that act on it. By definition, a fluid cannot sustain a shear stress no matter how small. The following discussion will describe the relationship between the magnitude of a shear stress and the accompanying fluid deformation for so-called "Newtonian fluids." We will also briefly consider some models of Non-Newtonian fluids. Also, Fourier's Law, which describes the transport of internal energy by heat conduction, will be introduced.

Fluid Kinematics.
The possible motions of a small element of fluid are illustrated in Figure 1. The element contains a given set of fluid particles, with no transfer of matter in or out. The motions are translation, rotation, shear strain, and dilatation. Since the translation and rotation are both solid body type motions, in which no part of the element moves relative to any other part, these types of motion do not, by themselves, give rise to "viscous" stresses. A viscous stress is one that is generated due to relative motion between different parts of a fluid. Pressure is not a viscous stress, since pressure does not originate from relative motion between different parts of a fluid. On the other hand, it is clear from Figure 1 that shear strain and dilatation involve motions in which parts of the fluid element move relative to one another, resulting in a *deformation* of the fluid element. Therefore, these latter types of motion are expected to be responsible for the generation of viscous stress in a flowing fluid.

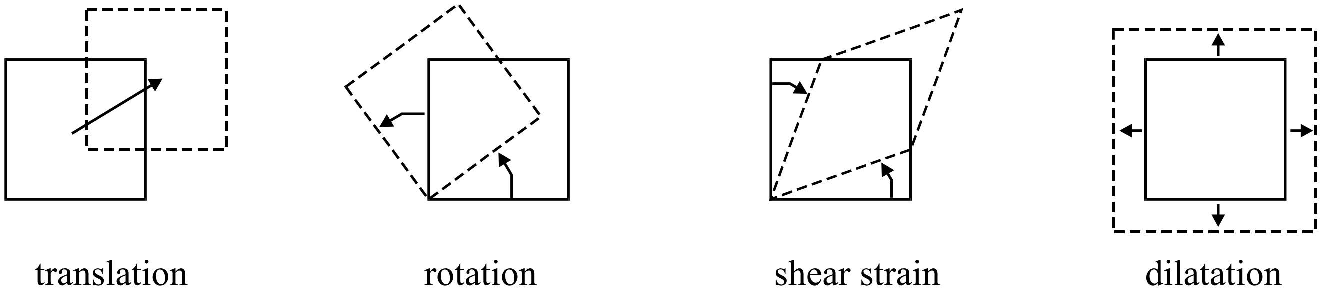

Figure 1.

By considering a differential fluid element, expressions will be derived for the rates of rotation, shear strain, and dilatation in terms of the flow velocity field **v**. The expression for the rate of translation is trivial, since the rate of translation is simply equal to **v**.

Consider a fluid element that is initially in the state illustrated on the left hand side of Figure 2. A short time dt later, the bottom left corner of the fluid element has been *translated* along the x_1 axis by a distance that is equal to v_1dt and along the x_2 axis by v_2dt. Furthermore, the center of the fluid element has *rotated* from an initial angle θ_i with respect to the x_1 axis to a final angle $\theta_f = \theta_i + d\theta$. Also, the left and bottom sides of the fluid element have been *shear strained* from being initially at 90^o with respect to one another to a final angle of 90^o - dβ - dα, where the angles dα and dβ are indicated in Figure 2. To avoid crowding the figure even more, *dilatation* of the fluid element is not illustrated in Figure 2. Later, we will look at dilatation separately.

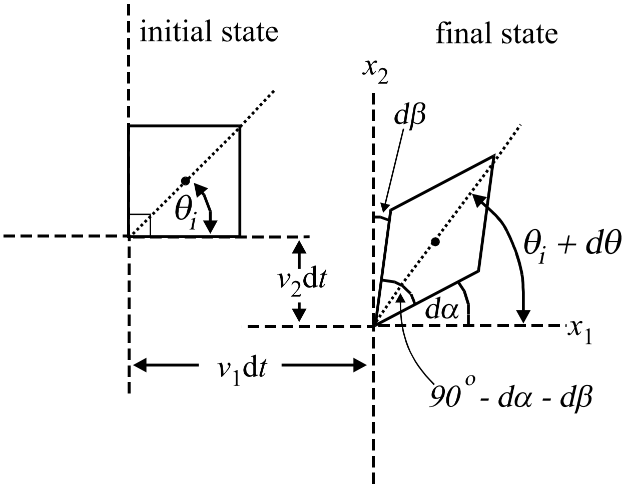

Figure 2.

It will be helpful to express the angles dα and dβ in terms of velocity gradients. If the length ds of the arc spanned by the angle dα (Figure 3) was known, then dα could be calculated using dα = ds/dl_1, where dl_1 is the length of the bottom side of the fluid element. Since the angle dα is infinitesimal, the length of the arc ds can be approximated by the vertical distance between the points labeled P1 and P2 in Figure 3. Now, this distance equals the *difference* in vertical displacement of the left and right ends of the fluid element's bottom side as the element passes from the initial to the final state. The bottom left corner of the fluid element moved *vertically* through a distance equal to v_2 dt (Figure 3), since v_2 is the velocity of the bottom left corner in the x_2 direction. The bottom *right* corner of the element moved vertically through a distance equal to ($v_2 + \frac{\partial v_2}{\partial x_1}$ dl_1) dt since, for the bottom right corner, the velocity in the x_2 direction is $v_2 + \frac{\partial v_2}{\partial x_1}$ dl_1. The length of the arc ds is, therefore

$$\mathrm{d}s = (v_2 + \frac{\partial v_2}{\partial x_1}\mathrm{d}l_1)\,\mathrm{d}t - v_2\,\mathrm{d}t$$

$$\mathrm{d}s = \frac{\partial v_2}{\partial x_1}\mathrm{d}l_1\,\mathrm{d}t \qquad (1)$$

Now it is straightforward to calculate dα,

$$\mathrm{d}\alpha = \mathrm{d}s\,/\,\mathrm{d}l_1 = \frac{\partial v_2}{\partial x_1}\,\mathrm{d}t \qquad (2)$$

Through similar arguments, one could show that

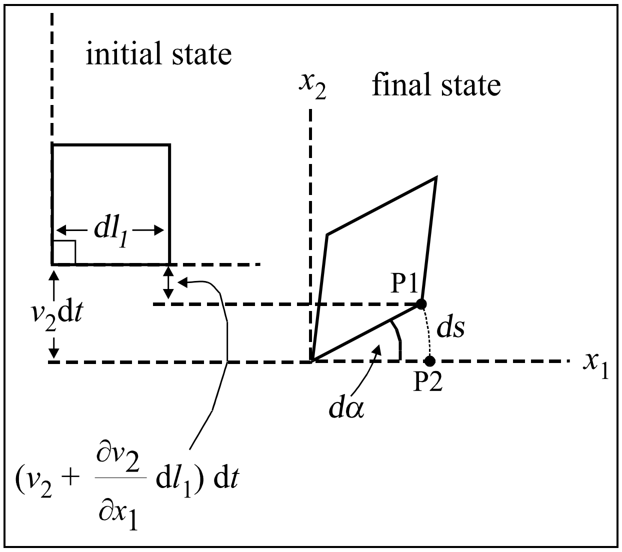

Figure 3.

$$d\beta = \frac{\partial v_1}{\partial x_2}\, dt \qquad (3)$$

Equations (2) and (3) will be useful in deriving mathematical expressions for the rate of rotation and the rate of shear strain of the fluid element.

1). Rate of Rotation.

The rate of rotation equals the angular velocity of the center of the fluid element. The angular velocity is given by the rate at which the angle θ (see Figure 2) changes with time. In other words, we need an expression for $d\theta/dt = (\theta_f - \theta_i)/dt$. In the initial state depicted in Figure 2, $\theta_i = 45^o$. In the final state depicted in Figure 2, $\theta_f = d\alpha + (1/2)(90^o - d\alpha - d\beta) = 45^o + (1/2)(d\alpha - d\beta)$. Therefore,

$$d\theta/dt = (\theta_f - \theta_i)/dt = \{45^o + (1/2)(d\alpha - d\beta) - 45^o)\}/dt$$

$$d\theta/dt = (1/2)(d\alpha - d\beta)/dt \qquad (4)$$

Substituting in equations (2) and (3) for $d\alpha$ and $d\beta$,

$$d\theta/dt = (1/2)(\ \frac{\partial v_2}{\partial x_1}\, dt - \frac{\partial v_1}{\partial x_2}\, dt\ ) / dt$$

$$d\theta/dt = (1/2)(\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}) \qquad (5)$$

Equation (5) gives the rate of rotation of the fluid element about the x_3 axis, which points up out of the plane of Figure 2. Usually, angular velocity is denoted by the symbol $\mathbf{\Omega}$. Equation (5) then equals the Ω_3 component of the angular velocity. By identical arguments, we could derive expressions for Ω_1 and Ω_2 corresponding to rotation of the fluid element about the x_1 and x_2 axes. The final result for the angular velocity would be

$$\mathbf{\Omega} = \Omega_1\, \boldsymbol{\delta_1} + \Omega_2\, \boldsymbol{\delta_2} + \Omega_3\, \boldsymbol{\delta_3} \qquad (6)$$

or

$$\mathbf{\Omega} = (1/2)(\ \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3})\, \boldsymbol{\delta_1} + (1/2)(\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1})\, \boldsymbol{\delta_2} + (1/2)(\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2})\, \boldsymbol{\delta_3} \qquad (7)$$

Comparing equation (7) with the curl of the velocity,

$$\nabla \times \mathbf{v} = (\ \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3})\, \boldsymbol{\delta_1} + (\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1})\, \boldsymbol{\delta_2} + (\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2})\, \boldsymbol{\delta_3} \qquad (8)$$

it is apparent that the angular velocity is just one half the curl, i.e.

$$\mathbf{\Omega} = (1/2)\,(\nabla \times \mathbf{v}) \tag{9}$$

The curl of the velocity, $\nabla \times \mathbf{v}$, is often referred to as the "vorticity" (symbol $\boldsymbol{\omega}$), so that $\mathbf{\Omega} = (1/2)\,\boldsymbol{\omega}$. The above derivation explains why flows for which $\nabla \times \mathbf{v} = 0$ are referred to as *irrotational*. The reason is that if we were to look at a point inside an irrotational flow, we would find that an infinitesimal fluid element at that point has zero angular velocity (i.e. is not rotating). Some examples of irrotational flows are uniform flow and "potential vortex" flow. Some of these flows will be encountered later in the course. **Rotation does not bring about a relative motion between different parts of the fluid and therefore does not contribute to the generation of viscous stress.** Although equation (7) was written in Cartesian coordinates, the tensor expression in equation (9) can be transcribed into another coordinate system of choice by substituting the proper expressions for ∇ and $\mathbf{v}$.

2). Rate of Shear Strain. "Rate of strain" refers to the rate at which parts of a fluid are being separated. *Shear* strain rate is defined as the rate at which two opposite sides of a fluid element are separated due to shearing motion, *divided by the perpendicular distance between them*. Figure 4 illustrates the concept of shear strain for a cubic fluid element immersed in the sketched velocity field. In the time dt, the bottom edge of the element is translated to the right by a distance v_1dt, while the top edge is translated to the right by a distance $(v_1 + \frac{\partial v_1}{\partial x_2}\mathrm{d}l_2)\,\mathrm{d}t$. Note that $(v_1 + \frac{\partial v_1}{\partial x_2}\mathrm{d}l_2)$ is the speed of the fluid in the x_1 direction at the top of the fluid element, and dl_2 is the fluid element's length in the x_2 direction. Because the top side of the fluid element moves faster to the right than the bottom side, a shearing motion is present. As time progresses, the fluid element becomes more distorted. The rate at which shearing motion separates the top and bottom sides is

$$[(v_1 + \frac{\partial v_1}{\partial x_2}\mathrm{d}l_2)\,\mathrm{d}t - v_1\mathrm{d}t]\,/\,\mathrm{d}t = \frac{\partial v_1}{\partial x_2}\mathrm{d}l_2$$

The term in the [ ] brackets is the shear separation, measured along the x_1 direction, that occurs between the top and bottom sides in the interval dt (i.e. this term represents how much farther the top side traveled to the right than the bottom side during the time dt). The division by time dt converts the above expression to a *rate* of shear separation. The perpendicular distance between the top and bottom sides of the fluid element is dl_2; therefore, the *rate of shear strain*, written γ_{12}, is

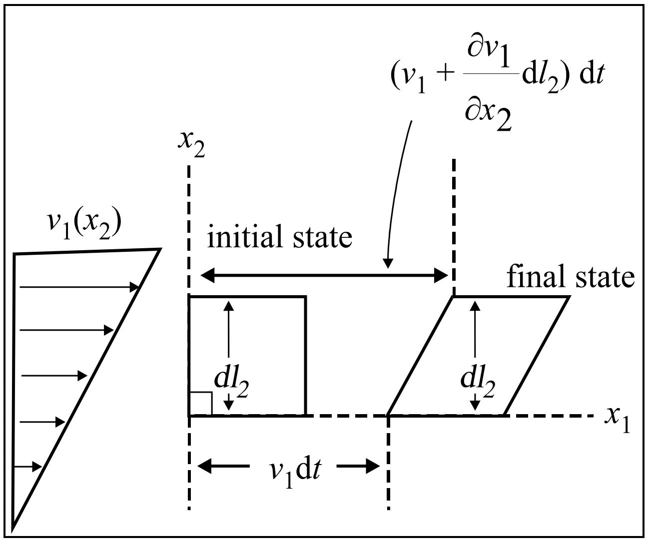

Figure 4.

$$\gamma_{12} = \frac{\partial v_1}{\partial x_2}\mathrm{d}l_2\,/\,\mathrm{d}l_2 = \frac{\partial v_1}{\partial x_2} \tag{10}$$

The subscripts "12" indicate that the rate of shear strain refers to shear occurring in the x_1-x_2 plane. Comparing equation (10) with equation (3) for the angle dβ,

$$\gamma_{12} = \mathrm{d}\beta / \mathrm{d}t \tag{11}$$

In figure 4, the fluid element is sheared in the x_1 direction. In general, a fluid element may be sheared along both the x_1 and x_2 directions as illustrated in Figure 5. Then the rate of shear strain equals the sum of $\mathrm{d}\alpha/\mathrm{d}t$ and $\mathrm{d}\beta/\mathrm{d}t$

$$\gamma_{12} = \mathrm{d}\beta / \mathrm{d}t + \mathrm{d}\alpha / \mathrm{d}t \tag{12}$$

Substituting in equations (2) and (3),

$$\gamma_{12} = \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) \tag{13}$$

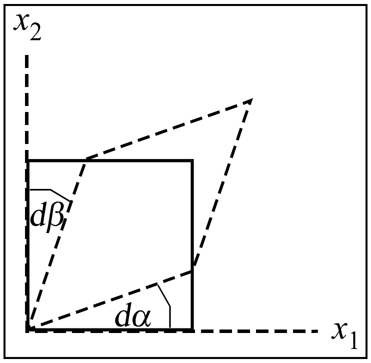

Fig. 5.

Equation (13) is the final expression for the rate of shear strain in the x_1-x_2 plane. By inspection, $\gamma_{12} = \gamma_{21}$. Similar expressions could be derived for the rates of shear strain in the x_2-x_3 and x_1-x_3 planes. The results can be summarized in the expression

$$\gamma_{ij} = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{14}$$

Equation (14) is expressed in Cartesian coordinates, but could be transcribed into other coordinate systems by appropriate change of coordinate variables. We note that **since a shear strain rate brings about a separation of different parts of the fluid, so that parts of the fluid are in relative motion with respect to one another, a viscous stress will be generated.** The following section on Newtonian fluids discusses how the rates of shear strain relate to the stress tensor.

3). Rate of Dilatation.

The rate of dilatation can be thought of as the rate of expansion, *per unit volume*, that a fluid element is experiencing. An expansion imposes strain on a fluid element, since parts of the fluid are being separated. **Since parts of the fluid are being separated they are in relative motion to one another, and dilatation therefore contributes to the viscous stress in a flowing fluid.** Consider a small fluid element with a volume equal to $l_1 l_2 l_3$, where l_i is the length of the fluid element in the x_i direction. To emphasize that the fluid element is very (i.e. infinitesimally) small, one could write $\mathrm{d}l_i$ instead of l_i. However, the notation of the following derivation would then be more confusing. Therefore, for simplicity, l_i will be used to denote the length of the fluid element along the x_i axis.

The rate of dilatation ϕ is defined as

$$\phi = \frac{1}{l_1 l_2 l_3} \frac{\mathrm{d}}{\mathrm{d}t}(l_1 l_2 l_3) \tag{15}$$

Since $l_1 l_2 l_3$ is the volume V_E of the element, $\phi = (1/V_E)\,\mathrm{d}V_E/\mathrm{d}t$. According to the product rule of differentiation,

$$\phi = \frac{1}{l_1 l_2 l_3}\left(l_2 l_3 \frac{dl_1}{dt} + l_1 l_3 \frac{dl_2}{dt} + l_1 l_2 \frac{dl_3}{dt}\right) = \frac{1}{l_i}\frac{dl_i}{dt} \qquad (16)$$

Now, the change dl_i that occurs over a time dt is given by $dl_i = (l_{if} - l_{ii})$, where l_{ii} is the initial value of l_i (at time t) and l_{if} is the final value of l_i (at time $t + dt$). Let's consider the bottom side of the fluid element in Figure 6. Over the time dt, the left end of the bottom side moves to the right by a distance of $v_1 dt$. The right end of the bottom side moves to the right by a distance of $(v_1 + \frac{\partial v_1}{\partial x_1} l_1)\, dt$, where $v_1 + \frac{\partial v_1}{\partial x_1} l_1$ is the speed of the right end in the x_1 direction. The change dl_1 in the length of the bottom side of the fluid element, as the fluid element passes from the initial to the final state, equals

$$dl_1 = (v_1 + \frac{\partial v_1}{\partial x_1} l_1)\, dt - v_1 dt = \frac{\partial v_1}{\partial x_1} l_1\, dt \qquad (17)$$

Thus

$$\frac{1}{l_1}\frac{dl_1}{dt} = \frac{\partial v_1}{\partial x_1} \qquad (18)$$

Similar arguments could be used to show that

$$\frac{1}{l_2}\frac{dl_2}{dt} = \frac{\partial v_2}{\partial x_2} \qquad \frac{1}{l_3}\frac{dl_3}{dt} = \frac{\partial v_3}{\partial x_3}$$

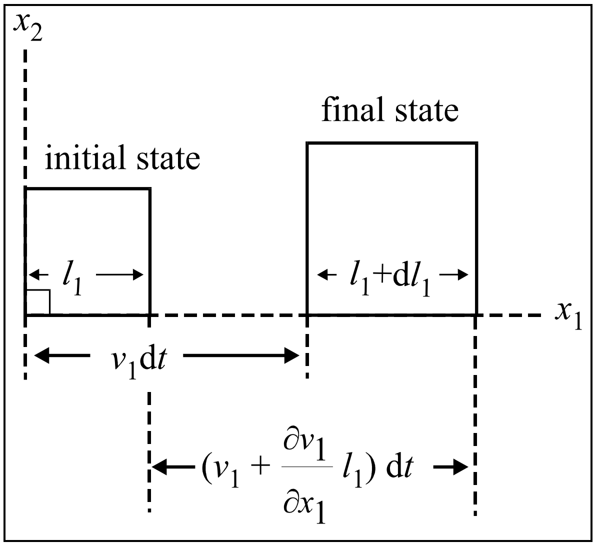

Figure 6

Inserting these expressions into equation (16) yields the rate of dilatation

$$\phi = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

But the above is just the definition of the divergence of velocity, thus

$$\phi = \nabla \bullet \mathbf{v} \qquad (19)$$

Equation (19) attaches a physical meaning to the divergence of velocity; specifically, $\nabla \bullet \mathbf{v}$ is seen to equal the rate at which a fluid element expands on a per unit volume basis. A positive value of $\nabla \bullet \mathbf{v}$ corresponds to expansion, while a negative one indicates contraction. Since the amount of matter inside the fluid element is fixed, if $\nabla \bullet \mathbf{v}$ is positive the density of the fluid element must decrease (since its volume increases). If a flow is incompressible, so that the density of the fluid cannot change, $\nabla \bullet \mathbf{v} = 0$ and no dilatation occurs.

The above introduction to the kinematics of fluid flow will be helpful for the next section, in which the relationship between types of deformation (in particular, shear strain and dilatation) and viscous stress will be examined in detail.

Newtonian Fluids

For many fluids, it is an experimental fact that:

i). viscous stresses are linearly related to the rates of strain arising from shear and dilatation. The rate of strain due to shear is given in equation (14); that due to dilatation is given in equation (19). Note: a function f is *linear* in a variable b if the expression $f(b_1) + f(b_2) = f(b_1 + b_2)$ holds. An example of a linear function is $f(x) = cx$, where c is a constant.

ii). viscous stresses vanish when all rates of strain vanish

iii). the fluid structure is isotropic, so that the state of stress does not depend on how the fluid body is oriented with respect to the direction of deformation.

Fluids that obey these criteria are generally referred to as "Newtonian." An additional criterion is that the stress tensor must be symmetric, so that $\sigma_{ij} = \sigma_{ji}$, as was argued in the previous handout. It can be proven, though we will omit the proof, that for Newtonian fluids a general relationship between the stresses σ_{ij} and the rates of strain must have the form

$$\sigma_{ij} = \lambda\,(\nabla \bullet \mathbf{v})\delta_{ij} + c\delta_{ij} + 2\mu\,(e_{ij}) \qquad (20)$$

In equation (20), λ, μ and c are three scalar parameters that are independent of rates of strain but may depend on temperature, pressure, fluid density etc. μ is called the shear viscosity, or more often, simply "viscosity". Note that the stress depends on rates of dilatation and shear, as anticipated from our discussion of fluid kinematics. λ is called the "second coefficient of viscosity". It will be shown that c can be identified with pressure. δ_{ij} is the Kronecker delta. In Cartesian notation, the "rate of strain tensor" e_{ij} is defined as

$$e_{ij} = (1/2)\,\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (21)$$

By comparing equation (21) with equation (14), we see that if i ≠ j then e_{ij} equals one half the rate of shear strain γ_{ij}. When i = j, then $e_{ij} = e_{ii} = \partial v_i/\partial x_i$, which is the ith term of the divergence of the velocity $\mathbf{v}$ and can be thought of as the rate of "dilatational" strain in the ith direction. Expressions such as equation (20) that relate strain rates to stress based on the behavior of a material are usually referred to as "**constitutive relations**" or as "**rheological equations of state**." Equation (20) is the constitutive relation for a Newtonian fluid.

To consider equation (20) in more detail, we start by specializing to the case when the fluid is in static equilibrium. Then all rates of deformation ($\nabla \bullet \mathbf{v}$ and e_{ij}) vanish, and

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = c \qquad \text{(normal stresses)} \tag{22}$$

and

$$\sigma_{12} = \sigma_{13} = \sigma_{21} = \sigma_{23} = \sigma_{31} = \sigma_{32} = 0 \qquad \text{(shear stresses)} \tag{23}$$

The fact that shear stresses in a fluid in static equilibrium are zero is already familiar from the earlier discussion of hydrostatics, and is a consequence of the fact that a fluid in static equilibrium is not being strained either by shear or by dilatational deformations. Furthermore, in static equilibrium, the normal stress on a surface is equal to the "thermodynamic" pressure - p_{th}. The minus sign indicates that the stress is compressive on the surface. The thermodynamic pressure is that which is found from an equation of state (e.g. for an ideal gas, $p_{th} = nRT/V$, with n the number of moles, R the gas constant, T the absolute temperature, and V the volume occupied by the gas). These considerations allow c in equation (22) to be identified as the negative of the thermodynamic pressure,

$$c = - p_{th} \tag{24}$$

In general, a fluid will be in motion and undergo deformation. Since the parameter c does not depend on such deformations (i.e. rates of strain), it retains its equality to the thermodynamic pressure. For a Newtonian fluid that is undergoing deformation, equation (20) states that the normal stresses become

$$\sigma_{11} = \lambda\,(\nabla \bullet \mathbf{v}) - p_{th} + 2\mu\,(\partial v_1/\partial x_1) \tag{25a}$$
$$\sigma_{22} = \lambda\,(\nabla \bullet \mathbf{v}) - p_{th} + 2\mu\,(\partial v_2/\partial x_2) \qquad \text{(normal stresses)} \tag{25b}$$
$$\sigma_{33} = \lambda\,(\nabla \bullet \mathbf{v}) - p_{th} + 2\mu\,(\partial v_3/\partial x_3) \tag{25c}$$

Therefore, when a Newtonian fluid is being deformed (strained), the normal stresses are a sum of three contributions: the thermodynamic pressure, and two terms due to "dilatational" rates of strain. The first of the dilatational terms is proportional to the second coefficient of viscosity λ, and the other is proportional to the viscosity μ. In fluid mechanics, a common convention is to define a so-called "**mechanical pressure**" p as the negative average of the normal stresses,

$$\begin{aligned} p &= -1/3\ (\sigma_{11} + \sigma_{22} + \sigma_{33}) \qquad (26)\\ &= -1/3\,[3\,\lambda\,(\nabla \bullet \mathbf{v}) - 3p_{th} + 2\mu\,(\partial v_1/\partial x_1 + \partial v_2/\partial x_2 + \partial v_3/\partial x_3)] \\ &= -1/3\,[3\,\lambda\,(\nabla \bullet \mathbf{v}) - 3p_{th} + 2\mu\,(\nabla \bullet \mathbf{v})] \end{aligned}$$

$$p = p_{th} - (\lambda + 2/3\ \mu)\,(\nabla \bullet \mathbf{v}) \tag{27}$$

The mechanical pressure p is what is customarily referred to as "pressure" in fluid mechanics equations and calculations. It lumps together (due to convention) the "true" pressure plus a contribution that arises from dilatational rates of strain, thus representing the average normal stress. From equation (27), for a fluid that is expanding the mechanical pressure will be less than the thermodynamic pressure. For nearly incompressible (i.e. liquid) flows, $\nabla \bullet \mathbf{v}$ is essentially zero and the distinction between p and p_{th} becomes insignificant. However, the distinction may be important when the flow is compressible and very pronounced dilatation of the fluid is occurring as, for example, in a shockwave or an explosion. For an ideal monatomic gas, it can also be shown that ($\lambda + 2/3\ \mu$) is equal to zero. Using p as defined by equation (27), the normal stresses (equations (25)) for a Newtonian fluid may be rewritten as

$$\sigma_{11} = -p - (2/3)\ \mu\,(\nabla \bullet \mathbf{v}) + 2\mu\,(\partial v_1/\partial x_1) \qquad (28a)$$
$$\sigma_{22} = -p - (2/3)\ \mu\,(\nabla \bullet \mathbf{v}) + 2\mu\,(\partial v_2/\partial x_2) \qquad \text{(normal stresses)} \qquad (28b)$$
$$\sigma_{33} = -p - (2/3)\ \mu\,(\nabla \bullet \mathbf{v}) + 2\mu\,(\partial v_3/\partial x_3) \qquad (28c)$$

According to equations (28), the normal stresses in a Newtonian fluid can be viewed as arising from a mechanical pressure contribution as well as viscous stresses due to dilatational rates of strain.

The shear stresses σ_{ij}, $i \neq j$, for a Newtonian flow are (see equation (20))

$$\sigma_{12} = \sigma_{21} = \mu\,(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}) \qquad (29a)$$
$$\sigma_{23} = \sigma_{32} = \mu\,(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}) \qquad \text{(shear stresses)} \qquad (29b)$$
$$\sigma_{31} = \sigma_{13} = \mu\,(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}) \qquad (29c)$$

By comparing to equation (14), we see that the shear stresses solely arise from a viscous contribution due to shear rates of strain. The proportionality parameter between the shear rate of strain and the resultant shear stress is the viscosity μ. In summary, equation (20) can be restated as

$$\sigma_{ij} = -p\delta_{ij} - (2/3)\ \mu\,(\nabla \bullet \mathbf{v})\ \delta_{ij} + 2\mu\,(e_{ij}) \qquad (30)$$

Note that viscous contributions, both to normal and shear stresses, vanish if the rates of strain (both shear and dilatational) vanish. Furthermore, the relationship between the viscous contributions to stress and the rates of strain is linear (i.e. σ_{ij} depends linearly on the rates of deformation $\nabla \bullet \mathbf{v}$ and e_{ij}). Finally, no specific orientation of the fluid body with respect to the direction in which it is deformed has been implied (i.e. the fluid behaves isotropically). As stated at the outset of this section, these qualities are the hallmarks of Newtonian fluids. Below we consider some simple cases of Non-Newtonian behavior.

Equation (30) states how the stress tensor $\boldsymbol{\sigma}$, existing at a point inside a flowing Newtonian fluid, is related to the rates of strain at that point. Furthermore, we also know how to express the rates of strain (shear and dilatational) in terms of the derivatives of the velocity $\mathbf{v}$. In a previous handout, differential balances for momentum and energy were written in terms of the stress tensor $\boldsymbol{\sigma}$. During those derivations, no particular constitutive expression for the stress tensor was assumed and the resultant expressions were valid for any type of fluid (not necessarily a Newtonian fluid). In the next handout, equation (30) for the Newtonian stress tensor will be substituted into the differential balance equations. Once that is accomplished, the differential balance equations can be applied to address problems involving the flow of Newtonian fluids.

An Introduction to Non-Newtonian Fluids

Non-Newtonian fluids are common. Toothpaste, cream cheese, slurries, and polymer solutions are just a few examples. In what follows, we will assume that they are incompressible, with normal stresses characterized by a mechanical pressure p, but different relationships between the shear stress σ_{ij} and the velocity gradients depending on the type of fluid. There are several classifications of Non-Newtonian fluids. For **time-independent fluids**, the shear stress depends on the rate of shear strain in a nonlinear (i.e. non-Newtonian) fashion but does not depend on the history of the deformation applied to the fluid. The fluid has no "memory." For **time-dependent fluids**, the history of deformation applied to the fluid affects the relationship between the shear stress and the deformation. For example, it is possible that by applying a stress to the fluid, thus causing it to flow, a break up or rearrangement of its molecular level structure takes place. This structural rearrangement, in turn, affects the relationship between the applied stress and the resultant deformation of the fluid. The extent of rearrangement of the structure of the fluid depends on how long and how forcefully the stress was applied; in other words, on the history of the flow. It is in this sense that a fluid can be said to possess "memory." While time-dependent fluids are a very interesting topic, they are also sufficiently specialized to place them outside of our scope. However, in order to provide a glimpse into non-Newtonian fluids, we will consider two of the best known examples: **Bingham plastics fluids** and **power-law fluids**. Both are examples of time-independent, non-Newtonian fluids.

Bingham fluids. Bingham plastics are distinguished by the existence of a **yield stress** σ_0. In order to cause a Bingham fluid to flow, the applied shear stress must exceed the yield stress. Once the yield stress is exceeded and the fluid begins to flow, the total shear stress increases following a linear dependence on the rate of shear strain, similar to a Newtonian fluid. In equation form,

$$\sigma_{ij} = \sigma_0 + \mu_P \, dv_i/dx_j \tag{31}$$

where μ_P is the plastic viscosity. Note that, for a Bingham plastic flow, if there is any shear deformation taking place the shear stress cannot be lower than the yield stress. Figure 7 depicts the σ_{ij} relationship to the shear strain rate dv_i/dx_j graphically. Examples of materials that can exhibit Bingham type behavior are plastics, paint emulsions, and clay suspensions.

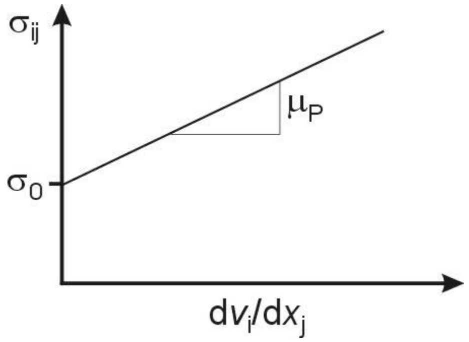

Figure 7.

Power law fluids. For a power law fluid, the shear stress is related to the rate of shear strain by equation (32)

$$\sigma_{ij} = k\,(dv_i/dx_j)^n = k\,(dv_i/dx_j)^{(n-1)}\,(dv_i/dx_j) = \mu_{APP}\,(dv_i/dx_j) \tag{32}$$

where k and n are constants characteristic of a particular fluid. A power law fluid reduces to a Newtonian fluid for $n = 1$, in which case k is the viscosity μ. The combination $k\,(dv_i/dx_j)^{(n-1)}$ is referred to at times as the apparent viscosity μ_{APP}.

For $n < 1$, the σ_{ij} relationship to dv_i/dx_j appears as in Fig. 8. Note that, at zero shear rate, the slope and the apparent viscosity approach infinity, indicating that the power law model has some unphysical characteristics. Nevertheless, this model serves as a good approximation for a number of fluids. For $n < 1$, the slope of σ_{ij} versus dv_i/dx_j decreases as dv_i/dx_j increases (Fig. 8). Fluids or flows exhibiting this type of qualitative behavior, namely decreased rate of rise of shear stress at higher shear strain rates, are termed **pseudoplastic**.

Fig. 8 also plots the qualitative dependence of σ_{ij} on dv_i/dx_j when $n > 1$. When $n > 1$, the rise of shear stress with dv_i/dx_j becomes more pronounced at higher values of dv_i/dx_j. The initial slope of the stress versus rate-of-strain curve starts at zero, but it becomes progressively higher at higher flow rates as the apparent viscosity increases. These types of flows are termed **dilatant**.

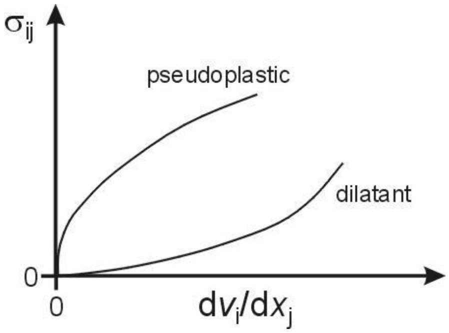

Figure 8

Fourier's Law of Heat Conduction

What is heat? Simply put, heat is energy stored in the microscopic state of agitation of matter. Matter can store energy in different ways. For example, as you may recall, electrons can possess different amounts of energy within an atom. In a "hot" atom, electrons may occupy higher energy levels than in a cold one. Similarly, nuclei, and indeed the entire atom and molecules made of atoms, have additional types of energy. For instance, molecules can translate (i.e. move about), as well as rotate, and their bonds can vibrate more or less forcefully. Such motions have kinetic (e.g. due to motion or rotation of a molecule) and potential energy (e.g. due to stretching or deformation of a chemical bond) associated with them. Hotter molecules are more agitated in their microscopic motions, thereby possessing more energy than colder molecules. Heat capacities, which you may recall from thermodynamics, measure the extent to which a material can make "use" of such higher energy states available to it to store heat energy. Why, and how, can heat energy move about, giving rise to heat transfer?

For single component systems, to which we specialize for now, there are three recognized processes: **convection**, **radiation**, and **conduction**. Convection refers to transfer by virtue of mass flow. When some fluid flows (i.e. is convected) into or out of a control volume, it brings with it its energy and thus energy transfer "by convection" occurs. If we have a cup that defines a control volume, and we pour some liquid into this control volume, energy transfer by convection has occurred. Second, heat can be transferred by "**radiation**." Radiative heat transfer is *transfer at a distance* that is mediated by electromagnetic radiation. For instance, the Sun sends out electromagnetic radiation across the vacuum of space to earth. When the radiation encounters matter, it may be absorbed by a molecule or atom of the matter, thereby exciting molecular or atomic motions that correspond to an increase in the matter's internal energy. We would say that the internal energy of the system has increased because radiative heat flowed into the system via electromagnetic radiation. We consider radiative transfer in more detail below. Third, heat may be transferred by "**conduction**." Heat conduction is independent of any velocity field **v** associated with bulk motion, or of the presence of electromagnetic radiation. One way in which heat conduction occurs is by molecules directly interacting (colliding) with one another. Due to these interactions, a transfer of thermal energy from the hotter side (more agitated molecules) to the colder side occurs. Heat conduction may also occur by exchange of molecules between hot and cold regions, such that the exchange is not associated with a *net* convection of matter. An example of heat transfer by conduction is the gradual warming of the free end of a spoon that is immersed into hot water at its other end.

According to the empirical **Fourier's Law** of heat conduction,

$$\mathbf{q_c} = -\kappa \nabla T \tag{33}$$

where $\mathbf{q_c}$ is the conductive heat flux vector (energy/(area time); e.g. Joules/(m^2 sec)), κ is the "thermal conductivity" which is a material property (in SI units of Joules/(sec m ^{o}K)), and ∇T is the gradient of the temperature field (in SI units of ^{o}K/m). From equation (33), it is evident that heat flows down (hence the minus sign) a temperature gradient. In other words, heat flows from hotter to colder regions. A good heat conductor, such as many metals, possesses a high value of κ (iron at 300 ^{o}K: $\kappa = 80$ W/(m ^{o}K)), while a poor heat conductor such as wood exhibits a low κ (for wood, $\kappa \approx 0.2$ W/(m ^{o}K)). The conductive flux of heat in a poor conductor is slower and the magnitude of the vector $\mathbf{q_c}$ will be less. In the next handout, Fourier's Law will be used in the differential energy balance equation to mathematically represent heat transfer by conduction.

Heat Transfer by Radiation

In this section we introduce key concepts of heat transfer by electromagnetic radiation. How can electromagnetic radiation transfer heat? From chemistry or physics we recall that atoms and molecules can absorb a photon of electromagnetic radiation, a process that promotes the atom or molecule into an excited state of a higher energy. In other words, this process makes the atom or molecule "hotter." For example, absorption in the infrared region of the electromagnetic spectrum usually excites a molecule by increasing the vibration (stretching and other motions) of its chemical bonds, while absorption in the visible and ultraviolet is often associated with excitation of d-orbital electrons or of π electrons (e.g. from double C=C bonds). Following absorption of a photon in the visible or ultraviolet region of the spectrum, the energy is then often decayed into the vibrational, rotational, and translational motions of the molecules or atoms. In summary, electromagnetic radiation can transfer heat because photons can be absorbed by matter, and their energy becomes converted to heat in the process.

Following is a summary of key definitions and equations involved in radiative heat transfer.

(i). The **monochromatic absorptivity** a_λ is defined as the fraction of incident radiation, of wavelength λ, absorbed by the surface of an opaque material,

$$a_\lambda = I_{A\lambda} / I_{i\lambda} \tag{34}$$

$I_{A\lambda}$ and $I_{i\lambda}$ are the absorbed and incident intensities of radiation of wavelength λ, respectively. Recall that units of intensity are energy /(area time).

- For a **black body**, $a_\lambda = 1$ at all wavelengths (all incident radiation is absorbed).
- For a **gray body**, $a_\lambda = C$ at all wavelengths where C is a positive constant less than 1 (a fixed fraction of incident radiation is absorbed at all wavelengths).
- For a **real body**, a_λ is less than one and in general varies significantly with wavelength.

(ii). Surfaces also radiate electromagnetic energy. The **monochromatic emissivity** e_λ of a surface is defined as the rate (energy / (area time)) at which electromagnetic energy of wavelength λ is emitted by the surface, divided by the rate at which a black body surface would emit radiation at the same wavelength and *at the same temperature*,

$$e_\lambda = I_{E\lambda} / I_{EB\lambda} \tag{35}$$

$I_{E\lambda}$ and $I_{EB\lambda}$ are intensities emitted by the surface of interest and a black body surface, respectively, at wavelength λ.

- From equation (35), for a black body surface $e_\lambda = 1$. In addition to being best absorbers of radiation, black bodies are also the best emitters.
- For real surfaces that do not fluoresce or luminesce at the wavelength λ, e_λ is less than 1. Real surfaces do not emit as strongly as black body surfaces.

A surface simultaneously emits radiation at many different wavelengths. The **total emissivity** e for a real surface is defined as the ratio of I_E, the total intensity emitted by the real surface over all wavelengths, to I_{EB}, the total intensity emitted by a black body surface over all wavelengths and at the same temperature,

$$e = I_E / I_{EB} \tag{36}$$

Theory and experiment have demonstrated that the emission intensity of a black body surface follows the **Stefan-Boltzmann Law**,

$$I_{EB} = \sigma T_S^4 \tag{37}$$

$\sigma = 1.355 \times 10^{-12}$ cal/(sec cm^2 ^{o}K^4) is the **Stefan-Boltzmann constant** (in English units, $\sigma = 0.1712 \times 10^{-8}$ Btu / (hr ft^2 ^{o}R^4)), and T_S is the absolute temperature of the black body surface. If the temperature of a real surface is known, the rate per area at which it emits electromagnetic radiation can be calculated from

$$I_E = e\sigma T_S^4 \tag{38}$$

Tables of e are available for various materials and surfaces.

For a situation in which a surface is emitting radiation the rate of radiative energy loss by the surface (energy /(area time)) to the surroundings is given directly by equation (38). An example is presented in Fig. 9. Here, we assume a steady state situation (no accumulation of energy) in which a surface has heat conducted to it from one side and then emits this energy to the surroundings on the other side. We define z as the coordinate normal to the surface. For steady state (i.e. no accumulation of energy at the surface), it is then required

$$-k\left(\frac{\mathrm{d}T}{\mathrm{d}z}\right)_{z=0} = e\sigma T_S^4 \tag{39}$$

In equation 39, Fourier's Law was used to represent the conductive heat flux on the left, with the subscript $z = 0$ indicating that the derivative is to be evaluated at the surface. For this steady state situation, equation 39 states that the rate of heat conducted to the surface must equal the rate at which it is removed from the surface by radiative emission.

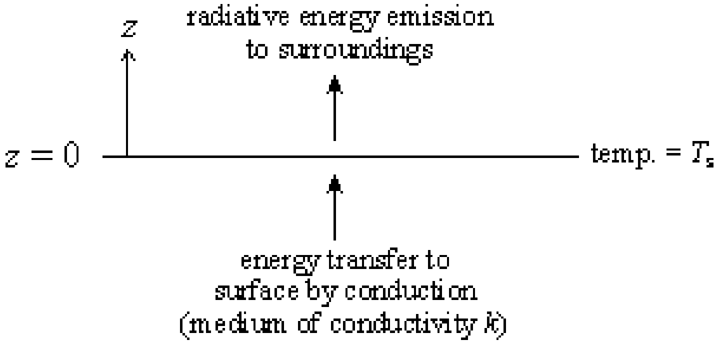

Figure 9

The physical picture gets more complicated when more than one surface is present. The complications arise because all of the surfaces will be emitting to and absorbing radiation from each other. In such cases, geometrical factors are calculated or obtained from tabulated information that provides "**view factors**" between different surfaces; these factors account for how much one surface is exposed to radiation from another. More precisely, the view factor F_{ij} is defined as the fraction of electromagnetic energy that leaves surface i that is subsequently intercepted by surface j. Treatments of heat transfer by radiation for such more complex situations can be found in specialized texts.