Differential Balance Equations

We have previously derived integral balances for mass, momentum, and energy for a control volume. The control volume was assumed to be some large object, such as a pipe. What if the balance equations were instead performed on a differential (infinitesimally small) control volume? As will be shown, the balance equations then assume the form of differential equations. The physical concepts are not new; rather, the differential equations are just a restatement of the three basic balance laws performed on a point in space rather than a macroscopic control volume. The differential balance equations will enable us to determine how velocity, pressure, temperature, composition, and other quantities of interest vary with position and/or time within a fluid flow.

The Divergence Theorem will be used to convert the previously derived integral balance equations to their differential counterparts. The Divergence Theorem is:

$$\oiint_B \mathbf{n}\cdot\mathbf{A}\,dB = \iiint_V \nabla\cdot\mathbf{A}\,dV \qquad (1)$$

As shown by Equation (1), the Divergence Theorem converts a flux integral of **A** over the (closed) surface B into an integral of the divergence of **A** over the volume V, where the volume V is enclosed by the surface B.

1). Differential Equation of Continuity (Conservation of Mass). In the previous set of notes the Integral Equation of Continuity was derived

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V'} \rho\,\mathrm{d}V' = -\iint_A \rho\,\mathbf{n}\cdot\mathbf{v}\,dA \qquad (2)$$

where ρ is the density of the fluid, **v** is the fluid velocity, and **n** is a unit outward normal to the surface A that encloses the control volume V'. The control volume V' is assumed to be fixed, so that its shape or volume do not change. Concretely, this means that the upper and lower limits on the triple integral over V' are constant, and do not change with time. It is then permissible to move the d/dt operator inside the integral on the left hand side of equation (2), resulting in

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V'} \rho\,\mathrm{d}V' = \iiint_{V'} \frac{\partial\rho}{\partial t}\,\mathrm{d}V' \quad (V' \text{ is a } \textit{fixed} \text{ control volume}) \qquad (3)$$

Equation (3) states that the total rate of accumulation of mass in the control volume V' (the left hand side term) equals the sum (i.e. the integral) of the local rates of mass accumulation $\frac{\partial\rho}{\partial t}\mathrm{d}V'$ occurring at all points within V' (the right hand side term). If desired, the control volume V' could be allowed to change with time, but we'll stay away from such complications. So, as stated, V' is a *fixed* control volume whose shape or size are constant.

Using the Divergence Theorem to rewrite the right hand side of equation (2), where the vector $\rho\mathbf{v}$ takes the place of the generic vector **A** in the Divergence Theorem (1), yields

$$-\iint_A \rho\, \mathbf{n}\cdot\mathbf{v}\, dA = -\iiint_{V'} \nabla\cdot\rho\,\mathbf{v}\, dV' \tag{4}$$

Using Equations (3) and (4), the integral equation of continuity (2) becomes

$$\iiint_{V'} \frac{\partial\rho}{\partial t}\, \mathrm{d}V' + \iiint_{V'} \nabla\cdot\rho\,\mathbf{v}\, dV' = 0 \qquad \text{so that}$$

$$\iiint_{V'} \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\,\mathbf{v}\right) \mathrm{d}V' = 0 \tag{5}$$

Since the control volume V' is arbitrary, we cannot choose a "special" V' for which the left hand side of equation (5) would equal zero. Therefore, the only way that the equality to zero can be assured is to require that the following condition hold everywhere within the control volume:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\,\mathbf{v} = 0 \tag{6}$$

In Cartesian coordinates, Equation (6) is (note the use of the summation convention):

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho\, \mathrm{v}_{\mathrm{i}}\right) = 0 \tag{6b}$$

Equation (6) is the **Differential Equation of Continuity**, or, as it is more customarily called, simply the **Equation of Continuity**. The Equation of Continuity expresses the law of mass conservation in the form of a differential equation. The terms in equation (6) have the same physical meaning as they had in the integral mass balance, except that now they apply to a *differential* control volume rather than a macroscopic one: $\partial\rho/\partial t\ \mathrm{d}V$ would be the local rate of mass accumulation in the differential volume dV, while $-\nabla\bullet\rho\mathbf{v}\ \mathrm{d}V$ would be the net rate of mass inflow (convection) into dV. Using the identity $\nabla\cdot\rho\,\mathbf{v} = \rho\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\rho$ (equation (30d) in the handout on vector analysis), the equation of continuity can be written

$$\frac{\partial\rho}{\partial t} + \rho\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\rho = 0 \tag{7}$$

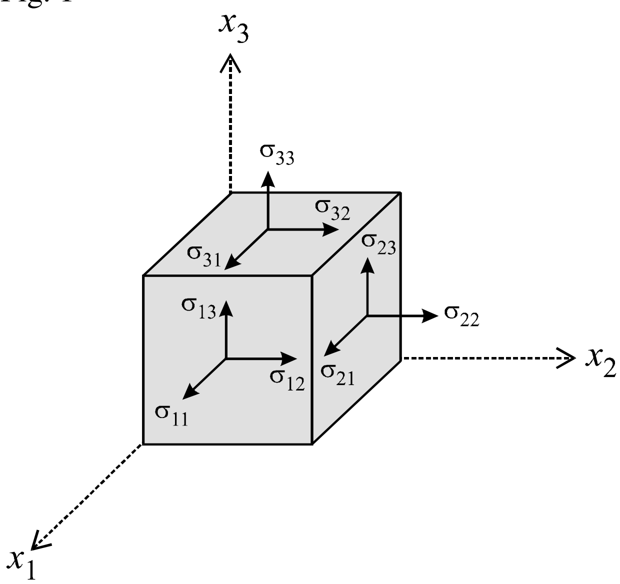
Incompressible Flow. If the flow is incompressible, the density is constant so that the first and last terms in equation (7) become zero. Therefore, for incompressible flow

$$\nabla\cdot\mathbf{v} = 0 \qquad (8).$$

2). Differential Equation of Change of Momentum (also referred to as the "Equation of Motion").

First, we recall the definition of the stress tensor $\boldsymbol{\sigma}$. The i,jth component of the stress tensor, σ_{ij}, is the stress exerted in direction j on a surface that is oriented perpendicular to direction i. Figure 1 illustrates how the various components of $\boldsymbol{\sigma}$ act on a differential cube of fluid.

Fig. 1



The stress $\mathbf{S}$ acting on a surface whose unit normal is $\mathbf{n}$ equals

$$\mathbf{S} = \mathbf{n} \bullet \boldsymbol{\sigma} \tag{9}$$

Equation (9) can be written $S_j = n_i\sigma_{ij}$ (as obtained by applying the definition of the tensor inner product; see handout on tensors for details). For example, if $\mathbf{n} = \boldsymbol{\delta}_1$, then $n_1 = 1$ and $n_2 = n_3 = 0$, so that $S_j = \sigma_{1j}$. In this case, $\mathbf{S}$ is the stress that acts on a surface oriented perpendicular to the x_1 direction. In other words, $\mathbf{S} = \sigma_{11}\boldsymbol{\delta}_1 + \sigma_{12}\boldsymbol{\delta}_2 + \sigma_{13}\boldsymbol{\delta}_3$, as can be verified by direct inspection of Figure 1.

From equation (11) of the previous handout, the integral momentum balance is (note that $\mathbf{n} \bullet \mathbf{v}\rho\mathbf{v}$ is equivalent to $(\mathbf{n} \bullet \mathbf{v})\rho\mathbf{v}$):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\iiint_{V'} \rho\mathbf{v}\,\mathrm{d}V'\right] = -\iint_A \mathbf{n}\cdot\mathbf{v}\rho\mathbf{v}\,\mathrm{d}A + \iiint_{V'} \mathbf{B}\,\mathrm{d}V' + \iint_A \mathrm{d}\mathbf{F}_S \tag{10}$$

The surface force $\mathrm{d}\mathbf{F}_S$ in equation (10) is a differential force acting on an area element on the surface of V'. Therefore, $\mathrm{d}\mathbf{F}_S = \mathbf{n} \bullet \boldsymbol{\sigma}\,\mathrm{d}A$ (since $\mathbf{n} \bullet \boldsymbol{\sigma}$ is the stress on $\mathrm{d}A$; multiplying the stress by the area $\mathrm{d}A$ is the actual force). With this substitution, equation (10) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\iiint_{V'} \rho\mathbf{v}\,\mathrm{d}V'\right] = -\iint_A \mathbf{n}\cdot\mathbf{v}\rho\mathbf{v}\,\mathrm{d}A + \iiint_{V'} \mathbf{B}\,\mathrm{d}V' + \iint_A \mathbf{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}A \tag{11}$$

To proceed, a generalized version of the divergence theorem is needed. In particular, we state without proof the following,

$$\oiint_B \mathbf{n}\cdot\mathbf{G}\,dB = \iiint_V \nabla\cdot\mathbf{G}\,dV \tag{12}$$

where $\mathbf{G}$ is a tensor of rank greater than zero ($\mathbf{G}$ could be a vector, a 2nd rank tensor, a 3rd tensor, etc.). B is a closed surface that encloses the volume V, and $\mathbf{n}$ is the unit surface normal. If $\mathbf{G}$ is a vector, equation (12) becomes the familiar Divergence Theorem. Using equation (12), equation (11) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\iiint_{V'}\rho\mathbf{v}\,\mathrm{d}V'\right] = -\iiint_{V'}\nabla\cdot\mathbf{v}\rho\mathbf{v}\,\mathrm{d}V' + \iiint_{V'}\mathbf{B}\,\mathrm{d}V' + \iiint_{V'}\nabla\cdot\boldsymbol{\sigma}\,\mathrm{d}V' \tag{13}$$

Taking the operator d/d*t* inside the integral on the left side of equation (13) (recall that this is allowed if *V'* is a fixed control volume), and combining all the terms into a single integral over *V'*,

$$\iiint_{V'}\left(\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot\mathbf{v}\rho\mathbf{v} - \mathbf{B} - \nabla\cdot\boldsymbol{\sigma}\right)\mathrm{d}V' = 0 \tag{14}$$

For an arbitrary control volume *V'*, the equality to zero mandates that for all points within the control volume

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot\mathbf{v}\rho\mathbf{v} - \mathbf{B} - \nabla\cdot\boldsymbol{\sigma} = 0 \tag{15}$$

Equation (15) can be expanded, since $\frac{\partial(\rho\mathbf{v})}{\partial t} = \rho\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\rho}{\partial t}$ (product rule of differentiation) and $\nabla\cdot\mathbf{v}\rho\mathbf{v} = \mathbf{v}\nabla\cdot\rho\mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v}$. With these modifications, equation (15) becomes

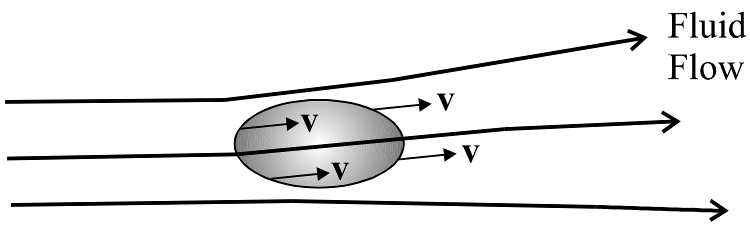
$$\rho\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\rho}{\partial t} + \mathbf{v}\nabla\cdot\rho\mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \mathbf{B} - \nabla\cdot\boldsymbol{\sigma} = 0 \tag{16}$$

The 2nd and 3rd terms in equation (16) can be written as $\mathbf{v}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{v}\right)$, which, according to the Equation of Continuity, must be zero. Therefore the final expression for the differential momentum balance is

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{B} + \nabla\cdot\boldsymbol{\sigma} \tag{17}$$

Often, equation (17) is referred to as the Equation of Change of Momentum, the Momentum Equation, or the Equation of Motion. Note that equation (17) is a vector equation. What do the various terms mean? The $\partial\mathbf{v}/\partial t$ term is the local (i.e. at a point) rate of change of velocity with time. This term is nonzero if the flow pattern is changing with time (i.e. the flow is unsteady). In steady-state, $\partial\mathbf{v}/\partial t$ is equal to zero. The convective term $\mathbf{v} \bullet \nabla\mathbf{v}$ can be thought of as the "projection of the velocity gradient $\nabla\mathbf{v}$ in the direction of $\mathbf{v}$", multiplied by the speed *v* with which the fluid is displaced in that direction. This term came from the convection term in the original integral momentum balance. We will now consider its meaning in more detail.

Picture a small element of fluid. The surface enclosing the element moves with the bulk velocity **v**, so that *no* convection of any quantity (such as mass, momentum, or energy) into the element occurs. **Under these conditions, the fluid element is said to "move with the flow."** We now measure the rate at which a property (ex. density, energy, or the velocity of the element) changes. There will be two contributions to this rate of change. First, a property may change because the flow is unsteady. This change occurs locally at each point in space as time progresses. For example, the pressure or velocity at a point may increase with time. Second, the fluid element may experience a change in a property due to its motion through a gradient of the property. Note that this change would be observed even if the flow was steady state. For example, the fluid element may move from a region where the pressure or velocity is low to one where these properties are high. The sum of the rate of change arising from the unsteady nature of the flow, plus the rate of change by virtue of moving in a gradient, is equal to the total rate of change of the property experienced by the fluid element.



An element of fluid "moving with the flow." The boundary of the element moves with the bulk velocity **v**.

For instance, let's say that the measured property is the velocity, and that the fluid element is part of a *steady state*, incompressible flow in a converging channel as illustrated in Figure 2. Clearly, the velocity must increase as the channel narrows; therefore, a velocity gradient $\nabla\mathbf{v}$ (that is, a change in velocity with position) will exist. As a fluid element moves from left to right, it moves along the velocity gradient and its velocity will therefore increase. This increase will occur even though the flow is steady state. Now, if in addition the flow becomes unsteady, so that the velocity at each point starts to increase with time, then the fluid element will experience this new rate of velocity increase (due to the unsteady nature of the flow) as well as the increase due to the element's motion through a velocity gradient. To sum up: **The total rate of change of a property experienced by a fluid element moving with the flow equals the sum of the local rate of change in the property, arising from the unsteady nature of the flow, plus the rate of change due to moving through a gradient of the property.**

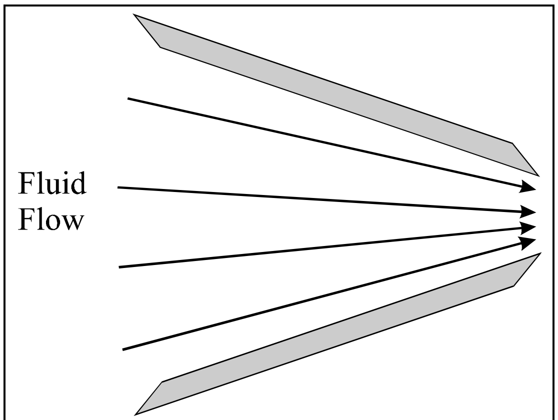


Fig. 2. The velocity of the incompressible fluid must increase as the fluid passes from left to right of the figure since the cross-sectional area of the channel decreases. Therefore, a velocity gradient $\nabla\mathbf{v}$ exists.

In equation (17), the total rate of change in the velocity of a fluid element as it moves with the flow is given by the two terms $\partial\mathbf{v}/\partial t + \mathbf{v} \bullet \nabla\mathbf{v}$. The first term is the local rate of change in velocity, and is zero if the flow is steady state. The second term is the rate of change in velocity by virtue of moving in a velocity gradient. In particular, the fluid element moves through the gradient $\nabla\mathbf{v}$ at a velocity **v**. The term $\mathbf{v} \bullet \nabla\mathbf{v}$ can therefore be thought of as first taking the component of the velocity gradient in the direction of **v** to give the "rate" with which the velocity changes in this direction. This result is then multiplied by the speed v with which the fluid element moves in this direction. The final result, $\mathbf{v} \bullet \nabla\mathbf{v}$, equals the rate of change of velocity, experienced by a fluid element moving with the flow, that is attributable to motion through the velocity gradient $\nabla\mathbf{v}$. Of course, "rate of change of velocity" is the same as acceleration, and the units of both $\mathbf{v} \bullet \nabla\mathbf{v}$ as well as $\partial\mathbf{v}/\partial t$ are length/time^2.

Thinking in terms of the rate at which a property changes for a fluid element moving with the flow is so common in fluid mechanics that a dedicated operator, called the "material derivative" D/Dt, has been defined

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{18}$$

The first term on the right represents rate of change due to unsteady conditions, while the second term represents rate of change due to moving through a gradient. Here is one more example to clarify, this time using the material derivative to operate on a scalar quantity. Imagine that we apply the material derivative to T, the temperature, to yield

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T$$

Then the first term on the right represents rate of change due to unsteady conditions, while the second term represents rate of change due to moving through a gradient of T with a velocity $\mathbf{v}$. Imagine that an observer is located at a point inside a pool of fluid that is being heated (Fig. 3a). Then, at the fixed location at which the observer is located (the velocity $\mathbf{v}$ of the observer is zero), the observer would see the temperature rise as the fluid becomes warmer. In this situation, the temperature of the fluid is not at steady state; i.e. it changes with time. At the observer's fixed position, the rate of rise in temperature corresponds to the unsteady state term $\partial T/\partial t$.

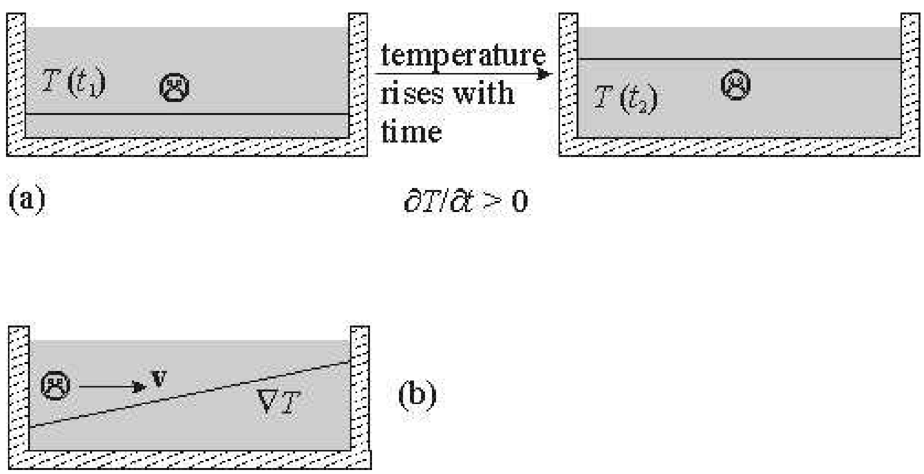


Fig. 3

Now, imagine that an observer is in a pool of fluid in which the temperature is at steady state (i.e. not changing with time) so that, at each point in the fluid, $\partial T/\partial t = 0$. However, imagine also that there is a gradient in the temperature ∇T from one end of the pool to the other (i.e. the fluid is hotter at one end of the pool but cooler at the other; Fig. 3b). If the observer remains stationary, as time goes on the observer would experience the same temperature since $\partial T/\partial t = 0$. However, if the observer begins to move from the cool end of the pool to the warm end, the observer would experience a rate of rise in temperature given by the rate of change of temperature with position (this rate is the gradient ∇T) times the speed at which the observer is moving up this gradient from the cooler to the warmer end. This product of speed times the temperature gradient is the $\mathbf{v} \cdot \nabla T$ term, where $\mathbf{v}$ is the observer's velocity. The dot product ensures that only that component of $\mathbf{v}$ that results in motion in the direction of ∇T

contributes to the observed rate of temperature increase. For example, if the observer were not moving up the gradient ∇T but only across it, meaning that **v** is perpendicular to ∇T, then no change in temperature would be experienced since the observer would move along a curve along which the temperature is constant.

Getting back to the equation of motion, $\frac{D\mathbf{v}}{Dt}$ then is the rate of change of velocity that a fluid element moving with the flow would experience. Using the definition of the material derivative, Equation (17) can be rewritten as

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{B} + \nabla \cdot \boldsymbol{\sigma} \qquad (19)$$

Equation (19) states that the acceleration $D\mathbf{v}/Dt$ of a fluid element moving with the flow equals the sum of the body force **B** and surface force $\nabla \cdot \boldsymbol{\sigma}$ acting on the fluid element (right hand side). Both forces are expressed on a per unit volume basis. Though it may look more complicated, equation (19) is just a restatement of Newton's 2nd law in the form $m\mathbf{a}$/volume = **F**/volume. The reason why this equation assumes this familiar form is that it has been expressed for a fluid element moving with the flow; in other words, for a given "piece" of fluid rather than an open control volume through which fluid flows.

Equation (19) is a vector equation and therefore has three components, one for each coordinate direction. For example, to write the momentum balance for the x_1-direction in a Cartesian coordinate system, those terms in equation (19) are needed that represent the x_1 components of the various vector quantities. Specifically, the x_1-momentum balance is

$$\rho \frac{Dv_1}{Dt} = B_1 + \frac{\partial \sigma_{i1}}{\partial x_i} \qquad (20)$$

In writing equation (20), the fact that $\nabla \cdot \boldsymbol{\sigma} = \partial \sigma_{ij}/\partial x_i$ was used. For the x_1-component j = 1, so that the surface force term becomes $\partial \sigma_{i1}/\partial x_i$. This expression represents a sum of three terms (i = 1, 2, 3), in accordance with the summation convention. The term with i = 1, $\partial \sigma_{11}/\partial x_1$, represents a gradient in the normal surface stress that tends to accelerate the fluid in the x_1 direction (i.e. a pressure gradient). The terms with i = 2 and i = 3, $\partial \sigma_{21}/\partial x_2$ and $\partial \sigma_{31}/\partial x_3$, represent gradients in shear stresses that act to accelerate the fluid in the x_1 direction.

An important point: The stress tensor σ_{ij} is symmetric, so that $\sigma_{ij} = \sigma_{ji}$. If it was not, then a net torque would exist on an infinitesimal fluid element, causing it to accelerate with an infinite angular acceleration (see Figure 4). For a rigorous mathematical proof see, for instance, Dahler and Scriven, *Nature* 192, 36-37, **1961**.

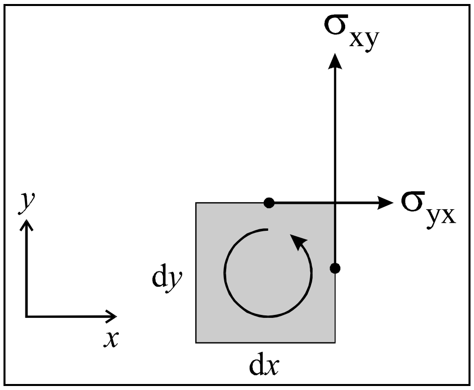


Fig. 4. If the stress tensor is not symmetric, so that $\sigma_{ij} \neq \sigma_{ji}$, then a finite torque on a differential fluid element will be present. In the figure, σ_{xy} is greater than σ_{yx}, causing a counterclockwise torque on the fluid element. In the limit as dx and dy approach zero, the rotational inertia (moment of inertia) of the fluid element vanishes, implying an infinite angular acceleration. Recall the formula: torque = (moment of inertia) (angular acceleration). Infinite angular acceleration would be unphysical, thus justifying the requirement that $\sigma_{ij} = \sigma_{ji}$.

Since the stress tensor $\boldsymbol{\sigma}$ is symmetric, $\sigma_{i1} = \sigma_{1i}$ and equation (20) can be rewritten as

$$\rho \frac{\mathrm{D}v_1}{\mathrm{D}t} = B_1 + \frac{\partial \sigma_{1i}}{\partial x_i} = B_1 + \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \frac{\partial \sigma_{13}}{\partial z} \quad (x\text{-momentum balance}) \qquad (21a)$$

Similarly, for the y and z components,

$$\rho \frac{\mathrm{D}v_2}{\mathrm{D}t} = B_2 + \frac{\partial \sigma_{2i}}{\partial x_i} = B_2 + \frac{\partial \sigma_{21}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} + \frac{\partial \sigma_{23}}{\partial z} \quad (y\text{-momentum balance}) \qquad (21b)$$

$$\rho \frac{\mathrm{D}v_3}{\mathrm{D}t} = B_3 + \frac{\partial \sigma_{3i}}{\partial x_i} = B_3 + \frac{\partial \sigma_{31}}{\partial x} + \frac{\partial \sigma_{32}}{\partial y} + \frac{\partial \sigma_{33}}{\partial z} \quad (z\text{-momentum balance}) \qquad (21c)$$

Equations (21) will be used for determining, for example, velocity distributions in flowing fluids. For now, the application of the equations must wait as we have not yet specified how to calculate the stress tensor $\boldsymbol{\sigma}$.

3. Differential Equation of Conservation of Energy. The integral equation for the conservation of energy, written for a control volume V' enclosed by the surface A, is (equation (16) in the previous handout)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\iiint_{V'} \rho e \, \mathrm{d}V'\right] = -\iint_A \mathbf{n} \cdot \rho e \mathbf{v} \, \mathrm{d}A + \frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{\mathrm{d}W}{\mathrm{d}t} \qquad (22)$$

The rate of heat addition to V', dQ/dt, can be written as the sum of two terms: a surface flux integral of the so-called "heat flux vector" $\mathbf{q}$ over the surface A, and a heat rate generation term per unit volume q'''. The units of the heat flux vector $\mathbf{q}$ are energy/(area time), ex. BTU/ (ft^2 sec). The heat flux vector $\mathbf{q}$ represents the local flow of heat across the control volume surface per unit area per unit time; e.g. due to conduction. The rate of heat flow across the surface of the control volume V' is

$$- \iint_A \mathbf{n} \cdot \mathbf{q} \, \mathrm{d}A \tag{23a}$$

Here, $\mathbf{n} \bullet \mathbf{q}$ is the component of $\mathbf{q}$ perpendicular to the surface A. This is the component of $\mathbf{q}$ that contributes to heat flux across A ($\mathbf{n} \bullet \mathbf{q}$ has units of energy/[area time]). Then $\mathbf{n} \bullet \mathbf{q}$ dA equals the rate of heat flow across the differential area dA ($\mathbf{n} \bullet \mathbf{q}$ dA has units of energy/time). Integration over the surface A results in the total rate of heat flow into V' through A.

In addition to heat flowing across the surface A into V', heat can also be added to V' using some externally coupled mechanism to generate heat within volume V'. For example, the heat generation could be caused by a resistive heating element placed inside V', so that the heating element generates heat at the volumetric rate q''' (q''' has the units energy/[volume time]). The total rate of such heat generation inside V' is equal to the volume integral

$$\iiint_{V'} q''' \mathrm{d}V' \tag{23b}$$

The total rate of heat addition, dQ/dt, to V' is the sum of the heat transport across the surface A (equation (23a)) and the volumetric heat generation contribution (equation (23b)),

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = - \iint_A \mathbf{n} \cdot \mathbf{q} \, \mathrm{d}A + \iiint_{V'} q''' \mathrm{d}V' \tag{23c}$$

Having discussed the rate of heat addition dQ/dt, a few remarks are also in order regarding the rate dW/dt at which the system does work on the surroundings. dW/dt is given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = - \iint_A \mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) \, dA \tag{24}$$

In equation (24), - $\mathbf{n} \bullet \boldsymbol{\sigma}$ is the stress vector -**S** exerted by the system *on* (note the minus sign) the surroundings; therefore, - $\mathbf{n} \bullet \boldsymbol{\sigma}$ dA is the *force* exerted by the system on the surroundings across the differential surface area dA. Dotting this force with the rate of displacement (i.e. the velocity), - ($\mathbf{n} \bullet \boldsymbol{\sigma}$) $\bullet$ $\mathbf{v}$ dA, gives the rate at which work is performed by the system on the surroundings across the area dA. Integrating over the entire surface A of the control volume gives the total rate of work dW/dt, as expressed in equation (24). For the present both shaft work and flow work are included in dW/dt (they will be separated later). Substituting the expressions (23c) and (24) for dQ/dt and dW/dt into equation (22), converting all surface integrals into volume integrals using the generalized Divergence Theorem, moving the operator d/dt on the left hand side of equation (22) inside the integral (since V' is a fixed control volume), and combining all of the resultant volume integrals together yields

$$\iiint_{V'} \left(\frac{\partial \rho e}{\partial t} + \nabla \cdot \rho e \mathbf{v} + \nabla \cdot \mathbf{q} - q''' - \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) \right) \mathrm{d}V' = 0 \tag{25}$$

The volume *V'* is arbitrary, thus it must hold that

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot \rho e \mathbf{v} + \nabla \cdot \mathbf{q} - q''' - \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) = 0 \tag{26}$$

Now $\partial \rho e/\partial t = \rho \partial e/\partial t + e \partial \rho/\partial t$, and $\nabla \bullet \rho e \mathbf{v} = e \nabla \bullet \rho \mathbf{v} + \rho \mathbf{v} \bullet \nabla e$ so equation (26) can be written

$$\rho \frac{\partial e}{\partial t} + e \frac{\partial \rho}{\partial t} + e \nabla \cdot \rho \mathbf{v} + \rho \mathbf{v} \cdot \nabla e + \nabla \cdot \mathbf{q} - q''' - \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) = 0 \tag{27}$$

By equation of continuity, the sum of the 2nd and 3rd terms is zero; furthermore, the sum of the 1st plus 4th terms can be written as $\rho\, (\partial e/\partial t + \mathbf{v} \bullet \nabla e) = \rho\, \mathrm{D}e/\mathrm{D}t$ using the material derivative notation. The final form of the differential law of energy conservation becomes

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -\nabla \cdot \mathbf{q} + q''' + \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) \tag{28}$$

Equation (28) is the last of the three basic laws we wanted to express in differential equation form. It states that the rate of change of the total energy (internal + kinetic + potential) of a fluid element moving with the flow (left hand side) is equal to the rate of heat flow into the element through its surface (1st term on the right), plus the rate of heat generation within the fluid element from externally coupled sources (2nd term on the right), plus the rate at which work is done on the fluid element by the surroundings (last term on the right). The work term can be further broken down into useful work and work that is dissipated to internal energy, but this must wait until after the stress tensor is discussed in greater detail. In Cartesian coordinates, equation (28) can be written as

$$\rho \left(\frac{\partial e}{\partial t} + v_i \frac{\partial e}{\partial x_i} \right) = -\frac{\partial q_i}{\partial x_i} + q''' + \frac{\partial (\sigma_{ij} v_j)}{\partial x_i} \tag{29}$$

According to the summation convention, the last term is summed over both the i and j indices over the dimensions of space (i.e. both i and j range from 1 to 3), and represents a total of nine terms. The double summation resulted from the double use of the dot product in equation (28).

Conclusions. The integral balances for the three basic laws - mass conservation, momentum conservation, and energy conservation - were rewritten as differential equations. The results are:

Equation of Continuity (mass conservation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\, \mathbf{v} = 0, \text{ or equivalently, } \frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \mathbf{v} \tag{30}$$

Equation of Change of Momentum (momentum balance):

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{B} + \nabla\cdot\boldsymbol{\sigma}\text{, or equivalently, } \rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{B} + \nabla\cdot\boldsymbol{\sigma} \qquad (31)$$

Equation of Conservation of Energy:

$$\rho\left(\frac{\partial e}{\partial t} + \mathbf{v}\cdot\nabla e\right) = -\nabla\cdot\mathbf{q} + q''' + \nabla\cdot(\boldsymbol{\sigma}\cdot\mathbf{v})\text{, or equivalently,}$$

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} = -\nabla\cdot\mathbf{q} + q''' + \nabla\cdot(\boldsymbol{\sigma}\cdot\mathbf{v}) \qquad (32)$$

The above equations were derived without recourse to any particular coordinate system description, and are expressed in tensor notation. In the future, whenever these equations are used to solve a problem, an appropriate coordinate system will need to be set up for the problem at hand and the above equations expressed in terms of that coordinate system. Fortunately, many textbooks have tables in which the differential tensor operations are expressed in the Cartesian, cylindrical, and spherical coordinate systems. The Cartesian forms of these equations were already mentioned in this handout (equations (6b), (21a) to (21c), and equation (29)). Other coordinate systems will be examined later in the course. Before these equations can be applied to the solution of actual problems the stress tensor $\boldsymbol{\sigma}$ must be considered in greater detail. After considering how the stress tensor is related to properties of the fluid flow, specialized forms of the basic laws will be formulated that can then be applied to solving problems of interest.