

## Tensor Notation.

Engineers and scientists find it useful to have a general terminology to indicate how many directions are associated with a physical quantity such as temperature or velocity. A scalar, such as temperature, has no direction associated with it. A scalar can then be termed a "tensor" of zero rank to indicate that zero directions are involved. A vector is associated with a single direction; therefore, a vector is a "tensor" of the first rank. Would there ever be interest in defining tensors of rank two and higher? The answer lies in whether quantities associated with two or more directions are useful in representing the behavior of the physical world.

Let's consider the figure below, which depicts three surfaces of different orientation that are acted on by normal and shear stresses (Note: we are *not* dealing with a fluid in static equilibrium, since shear stresses are present). We would like to indicate, unambiguously, one of the stresses. Let's say it is the stress  $\sigma_{21}$ . To refer to the stress, two pieces of information are needed: the direction of the stress, and the orientation of the surface that the stress is acting on. The shear stress  $\sigma_{21}$  could be referred to as "the stress acting in the  $x_1$  direction on a surface that is oriented perpendicular to the  $x_2$  direction." Indeed, stating the direction of the stress and the orientation of the surface on which the stress acts is all that is needed to identify any of the stresses depicted. The above discussion suggests that a "second rank tensor", a quantity that has two directions associated with it, could be used to describe the collection of stresses (ie. the "stress distribution") acting on the three surfaces portrayed in the figure.

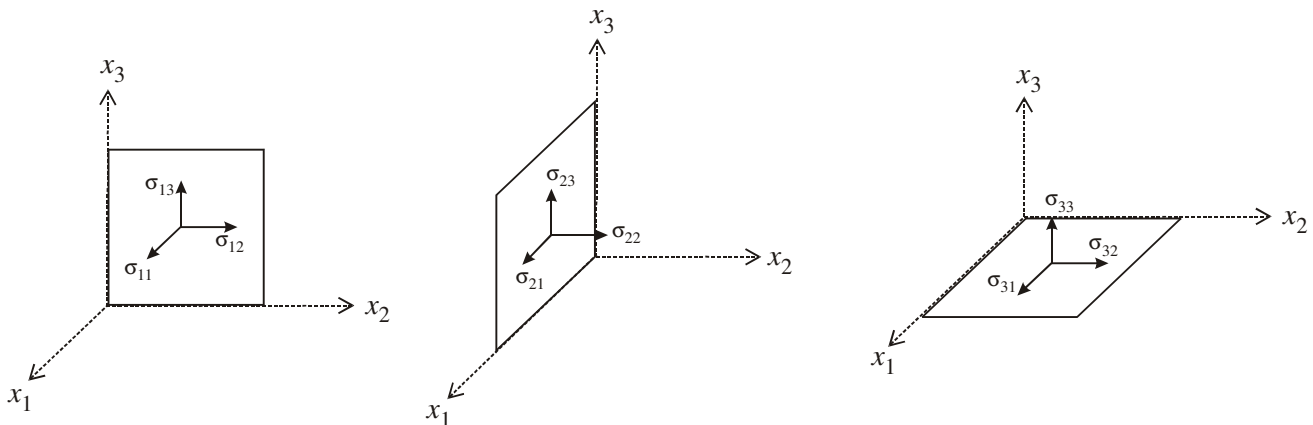


Fig. 1

Before discussing how to setup and use a "second rank tensor" a few general comments are in order. One powerful feature of tensor notation is that it describes physical laws in a manner that is independent of any particular coordinate system (or reference frame) used. Such a requirement is clearly necessary for a mathematical description of a physical law to be valid, since the laws of the universe cannot depend on the reference frame used to describe them. In turn, this requirement defines how the components of a tensor (in the above example, the tensor components would be the  $\sigma_{ij}$ ) transform under a change of reference frame. Coordinate transformations will be discussed later in the course. For now, if we need to "transcribe" an equation from tensor notation to one written for a specific reference frame, that frame will be the CCS.

**Second Rank Tensors:** Second rank tensors are often referred to simply as "tensors." First rank tensors are referred to as vectors, and zero rank tensors as scalars. A component of a second-rank tensor is indicated by two indices. Thus, the  $i,j$  component of tensor  $\mathbf{A}$  is written  $A_{ij}$ . For instance, for

the stress tensor, the component  $\sigma_{ij}$  equals the stress in direction  $j$  acting on a surface that is oriented perpendicular to direction  $i$ . The indices range over the number of dimensions of space; for three-dimensional space, described by coordinates  $x_1$ ,  $x_2$ , and  $x_3$ , the components of a tensor  $\mathbf{A}$  would be  $A_{11}$ ,  $A_{12}$ ,  $A_{13}$ ,  $A_{21}$ ,  $A_{22}$ ,  $A_{23}$ ,  $A_{31}$ ,  $A_{32}$ , and  $A_{33}$  (Aside: what is the physical meaning of the various components of the stress tensor? For instance, how would you describe in words the stresses referred to by the notation  $\sigma_{23}$  and  $\sigma_{33}$ ?).

If  $A_{ij} = A_{ji}$  for all  $i$  and  $j$ , the 2nd rank tensor is said to be "**symmetric**." If  $A_{ij} = -A_{ji}$  for all  $i$  and  $j$ , then the 2nd rank tensor is said to be "**antisymmetric**." For an antisymmetric tensor, components in which  $i$  and  $j$  have the same value (ie.  $A_{11}$ ,  $A_{22}$ , and  $A_{33}$ ) are zero.

There are several fundamental operations involving tensors that will be useful to us.

**The Summation Convention:** The summation convention works as before. In particular, if the same index letter appears twice in a term, summation with respect to that index is implied over the dimensions of the space. This rule *always* applies whenever an index is repeated in a term. For example,

$$\sigma_{jj} = \sigma_{11} + \sigma_{22} + \sigma_{33} \quad (1)$$

$$v_i \sigma_{ji} = v_1 \sigma_{j1} + v_2 \sigma_{j2} + v_3 \sigma_{j3} \quad (2)$$

etc...

**Transpose of a 2nd Rank Tensor:** The transpose of a 2nd rank tensor  $\mathbf{A}$  is a 2nd rank tensor that is written  $\mathbf{A}^\dagger$ . The components of  $\mathbf{A}^\dagger$  are defined by

$$A^\dagger_{ij} = A_{ji} \quad (3)$$

**Trace of a 2nd Rank Tensor:** The trace of a tensor  $\mathbf{A}$  is the scalar  $A_{ii}$ . In other words,

$$\text{trace } \mathbf{A} = A_{ii} = A_{11} + A_{22} + A_{33} \quad (4)$$

**Addition and Subtraction of Tensors:** The sum or difference of two tensors of the same rank is also a tensor of the same rank. Only tensors of the same rank can be added or subtracted. In other words, one cannot add a scalar and a vector, or a vector and a 2nd rank tensor. If  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  are second rank tensors, and

$$\mathbf{A} + \mathbf{B} = \mathbf{C}$$

then

$$A_{ij} + B_{ij} = C_{ij} \quad (5)$$

This is analogous to what happens with tensors of the first rank (vectors), for which  $A_i + B_i = C_i$ . The addition operation works trivially with scalars, as scalars have only a single "component" resulting simply in  $A + B = C$  (no indices are needed).

**Direct Product:** The direct product of two tensors is performed by simply multiplying components from the two tensors together, pair by pair. The two tensors involved in the direct product can be of different ranks. The result of the direct product is a tensor whose rank is the sum of the ranks of the tensors being multiplied. For instance,

i).  $A$  and  $B$  are scalars:

$$A B = C \quad (6a)$$

\*product of two 0th rank tensors is a 0th rank tensor\*

ii).  $A$  is a scalar and  $\mathbf{B}$  is a 2nd rank tensor:

$$A \mathbf{B} = \mathbf{C} \quad \text{where} \quad AB_{ij} = C_{ij} \quad (\text{ex. } AB_{12} = C_{12}) \quad (6b)$$

\*product of a 0th and a 2nd rank tensor is a 2nd rank tensor\*

iii).  $\mathbf{A}$  and  $\mathbf{B}$  are vectors:

$$\mathbf{A} \mathbf{B} = \mathbf{C} \quad \text{where} \quad A_i B_j = C_{ij} \quad (\text{ex. } A_3 B_2 = C_{32}) \quad (6c)$$

\*product of two 1st rank tensors is a 2nd rank tensor\*

iv).  $\mathbf{A}$  is a vector and  $\mathbf{B}$  is a 2nd rank tensor:

$$\mathbf{A} \mathbf{B} = \mathbf{C} \quad \text{where} \quad A_i B_{jk} = C_{ijk} \quad (\text{ex. } A_2 B_{31} = C_{231}) \quad (6d)$$

\*product of a 1st rank and a 2nd rank tensor is a 3rd rank tensor\*

etc.

(Aside: how many components does the  $\mathbf{C}$  tensor have in each of the above examples?

Is the direct product commutative? Answer: not in general. Why?)

When taking the direct product of two tensors, *different* indices are used to indicate the components of the tensors being multiplied together. If the indices were not different, errors would result. For instance, if the same index "i" was used to represent the components of two vectors  $\mathbf{A}$  and  $\mathbf{B}$ , then instead of forming the direct product as in  $A_i B_j = C_{ij}$  (equation (6c), in which different indices "i" and "j" are used), the result would be  $A_i B_i = C_{ii} = C_{11} + C_{22} + C_{33}$  = the *dot* product (a scalar) of  $\mathbf{A}$  and  $\mathbf{B}$ . Therefore, caution must be exercised when assigning component indices in a tensor operation, and attention must be paid to whether indices for the components of the tensors involved are to be the same (as in addition and subtraction) or different (as in the direct product).

**Inner (Dot) Product:** The inner product between two tensors results from first forming the direct product, and then setting the two nearest indices (with one index coming from each tensor) equal to one another and performing the sum according to the summation convention. The dot product between two vectors  $\mathbf{A}$  and  $\mathbf{B}$  is a perfect example of this operation: starting with the direct product of  $\mathbf{A}$  and  $\mathbf{B}$ , represented as  $A_i B_j$ , we set  $i = j$  resulting in  $A_i B_i$ , and finally perform the summation to get the result  $A_1 B_1 + A_2 B_2 + A_3 B_3$ .

Note that the result of the dot product of two vectors is a scalar (a tensor of rank 0). As a rule, the rank of the tensor resulting from an inner product is lower by two than the sum of the ranks of the tensors whose inner product is being calculated. The rank lowering by two occurs because of the "summing over" of two of the indices, thereby eliminating them. Note that the rank of a tensor involved in an inner product must be greater than zero (i.e. each tensor must have an index available to be summed over). Some examples of the inner product are:

i). **A** is a vector and **B** is a 2nd rank tensor (note that nearest indices are set equal by the use of the Kronecker delta):

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{C} \quad \text{where} \quad A_i B_{jk} \delta_{ij} = A_i B_{ik} = C_k \quad (7a)$$

(ex.  $C_1 = A_1 B_{11} + A_2 B_{21} + A_3 B_{31}$ , with similar expressions for  $C_2$  and  $C_3$ )

\*inner product of a 1st and 2nd rank tensors is a 1st rank tensor\*

$$\mathbf{B} \cdot \mathbf{A} = \mathbf{C} \quad \text{where} \quad B_{ij} A_k \delta_{jk} = B_{ij} A_j = C_i \quad (7b)$$

(eg.  $C_1 = B_{11} A_1 + B_{12} A_2 + B_{13} A_3$ , with similar expressions for  $C_2$  and  $C_3$ )

The above two examples demonstrate that, in general, the inner product of two tensors is not commutative (however, the inner product *is* commutative when the two tensors involved are both vectors).

ii). **A** and **B** are both 2nd rank tensors:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{C} \quad \text{where} \quad A_{ij} B_{kl} \delta_{jk} = A_{ij} B_{jl} = C_{il} \quad (7c)$$

(ex.  $C_{11} = A_{11} B_{11} + A_{12} B_{21} + A_{13} B_{31}$ , with similar expressions for  $C_{12}$ ,  $C_{13}$ ,  $C_{21}$ ,  $C_{22}$ ,  $C_{23}$ ,  $C_{31}$ ,  $C_{32}$ , and  $C_{33}$ )

\*inner product of two 2nd rank tensors is a 2nd rank tensor\*

So far, the discussion described the "single" inner product, written as  $\mathbf{A} \cdot \mathbf{B}$  where **A** and **B** are both tensors of rank 1 or greater. A "double" (and higher multiple) inner product can also be defined. To perform a double inner product, we take the result of the single inner product, and then repeat the process of setting the nearest pair of indices equal and summing over them. Only tensors of 2nd rank or higher can participate in double inner products, since at least two indices per tensor must be available to be summed over. An example of a double inner product is:

iii). **A** and **B** are both 2nd rank tensors (note that both the nearest and the next-nearest indices are set equal):

$$\mathbf{A} : \mathbf{B} = C \quad \text{where} \quad A_{ij} B_{ji} = C \quad (8)$$

in detail:  $\mathbf{A} : \mathbf{B} = A_{ij} B_{kl} \delta_{jk} \delta_{il} = A_{ij} B_{jl} \delta_{il} = A_{ij} B_{ji}$  (the Kronecker delta is used to enforce the required equality of nearest, as well as next-nearest, indices)

(therefore,  $C = A_{11} B_{11} + A_{12} B_{21} + A_{13} B_{31} + A_{21} B_{12} + A_{22} B_{22} + A_{23} B_{32} + A_{31} B_{13} + A_{32} B_{23} + A_{33} B_{33}$ )

\*double inner product of two 2nd rank tensors is a 0th rank tensor\*

**Examples of 2nd Rank Tensors:**

The most important example of a 2nd rank tensor in this course is the stress tensor  $\boldsymbol{\sigma}$  described earlier in connection with Figure 1. The components of the stress tensor,  $\sigma_{ij}$ , represent the stress exerted in direction  $j$  on a surface that is oriented perpendicular to direction  $i$ . Recall that the state of stress in an object must be independent of the choice of reference frame in which one chooses to describe it; therefore, it must be true that under a change of reference frame the components of  $\boldsymbol{\sigma}$  will change in a way so that they still represent the same physical distribution of stress.

Here is an illustration of a simple operation involving  $\boldsymbol{\sigma}$ . Let's say we are working in the CCS and we want to calculate the stress  $\mathbf{S}$  on a surface oriented perpendicular to the  $x_1$  direction. How can this information be extracted from  $\boldsymbol{\sigma}$ ? The answer is to simply take the inner product of a unit vector  $\mathbf{n}$  normal to the surface with  $\boldsymbol{\sigma}$ . In other words:

$$\mathbf{S} = \mathbf{n} \cdot \boldsymbol{\sigma} \quad (9)$$

thus,  $S_j = n_i \sigma_{ij}$

For example, if we wish to calculate the stress on a surface oriented perpendicular to the  $x_1$  direction we would use  $\mathbf{n} = \boldsymbol{\delta}_1$ . Therefore,  $n_1 = 1$  and the other components of  $\mathbf{n}$  ( $n_2$  and  $n_3$ ) are zero. Therefore, using equation (9), the components of  $\mathbf{S}$  are  $S_1 = \sigma_{11}$ ,  $S_2 = \sigma_{12}$ , and  $S_3 = \sigma_{13}$ . The total stress is  $\mathbf{S} = \sigma_{11}\boldsymbol{\delta}_1 + \sigma_{12}\boldsymbol{\delta}_2 + \sigma_{13}\boldsymbol{\delta}_3$ . By considering Figure 1, this expression for  $\mathbf{S}$  can be directly verified. Indeed, starting with any arbitrarily oriented surface, a unit normal vector  $\mathbf{n}$  can be derived for it, and the inner product of  $\mathbf{n}$  with the stress tensor will give the stress on the surface following equation 9. Analogous to scalar and vector fields,  $\boldsymbol{\sigma}$  is a "tensor field" that represents the local stress distribution (as captured in the tensor components  $\sigma_{11}, \sigma_{12}, \sigma_{13}, \dots, \sigma_{32}, \sigma_{33}$ ) at every point in space. Note that the stress tensor may vary with position in space, so that  $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x_1, x_2, x_3)$  and the values of the stress tensor components change from point to point.

Another example of a second-rank tensor is the velocity gradient tensor which is written  $\nabla \mathbf{v}$ . In the

CCS,  $\nabla \mathbf{v}$  has the components  $(\nabla \mathbf{v})_{ij} = \frac{\partial v_j}{\partial x_i}$ . The  $\nabla \mathbf{v}$  tensor will be useful in writing down expressions

describing the types of deformation that can be experienced by a fluid element.

**Concluding Comments:** So far, we have referred to tensor components directly, i.e.  $A_{ij}$  was the  $i,j$ th component of tensor  $\mathbf{A}$ . At times, it is more convenient to write out the components of a 2nd rank tensor using a matrix type array. For example, a 2nd rank tensor  $\mathbf{A}$  can be written as

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \quad (10)$$

Alternately,  $\mathbf{A}$  could be written out using a basis of 2nd rank tensors called "**basis dyads**" formed from the direct product of two basis vectors, e.g.

$$\mathbf{A} = A_{11}\boldsymbol{\delta}_1\boldsymbol{\delta}_1 + A_{12}\boldsymbol{\delta}_1\boldsymbol{\delta}_2 + A_{13}\boldsymbol{\delta}_1\boldsymbol{\delta}_3 + A_{21}\boldsymbol{\delta}_2\boldsymbol{\delta}_1 + A_{22}\boldsymbol{\delta}_2\boldsymbol{\delta}_2 + A_{23}\boldsymbol{\delta}_2\boldsymbol{\delta}_3 + A_{31}\boldsymbol{\delta}_3\boldsymbol{\delta}_1 + A_{32}\boldsymbol{\delta}_3\boldsymbol{\delta}_2 + A_{33}\boldsymbol{\delta}_3\boldsymbol{\delta}_3 \quad (11)$$

$\boldsymbol{\delta}_1\boldsymbol{\delta}_1$ ,  $\boldsymbol{\delta}_1\boldsymbol{\delta}_2$  etc. are the basis dyads. The purpose of the dyadic notation is to maintain a semblance of similarity to the notation familiar from vector analysis. The different notational conventions represented by equations (10) and (11) are not necessary for tensor analysis. Their existence reflects the fact that different people have different preferences as to how tensors are represented on paper. Regardless of what tensor notation may be employed, all of the tensor operations remain as defined previously. In this course you may use the various notations interchangeably.