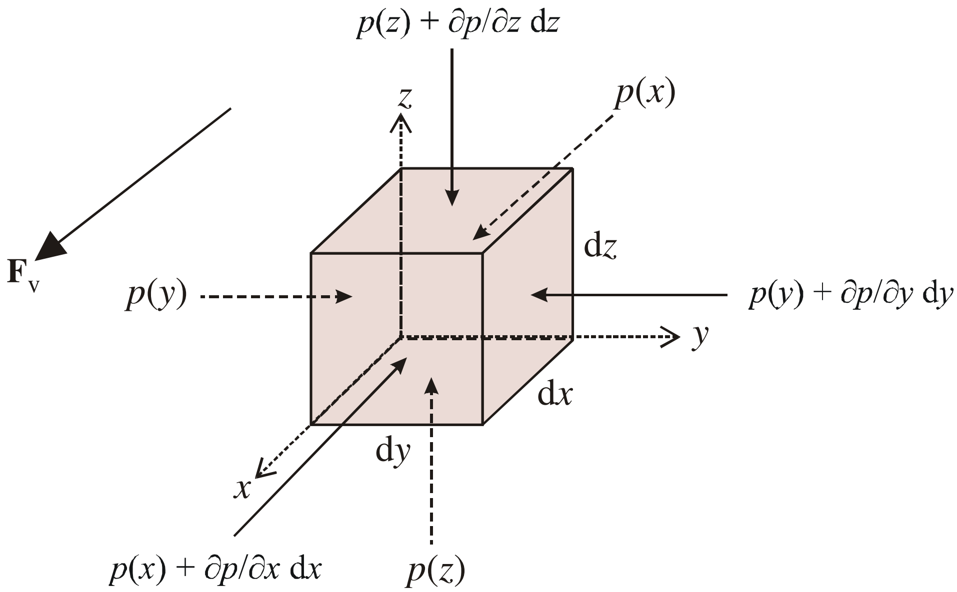
Fluid Statics

Fluid statics deals with situations of "static equilibrium."
In static equilibrium:

- no part of a fluid is in motion relative to another part of the fluid
- no shear stress is present in the fluid
- only normal isotropic stresses (ie. pressure) exist

The fluid could simply be at rest, or it could be moving. If motion of the fluid is involved, the only motions possible in static equilibrium are a solid body translation, a solid body rotation, or a combination of the two (ex. a body of fluid may be rotating about its axis at the same time that it is being translated along a particular direction). The terminology "solid body" implies that the movement of the fluid is equivalent to that of a body in which no part is capable of movement relative to any other part, as in a solid.



Differential Equations of Fluid Statics:

We consider a differential volume element in a fluid that is in static equilibrium. The fluid element is acted on by an arbitrary "body force" *per unit volume* $\mathbf{F}_\mathbf{v} = F_{vi}\boldsymbol{\delta}_i$ (recall the summation convention). A "body force" is a force that acts throughout the volume (body) of a fluid. An example of a body force is gravity since gravity "pulls" simultaneously on all parts of a body. An example of a surface force is pressure or shear stress, since these forces act through direct contact at the surface of a body.

The figure illustrates the forces that act on the fluid element. Also, we chose a CCS coordinate system in which the fluid element is stationary. In this coordinate system, the fluid element is not perceived to accelerate, so that the sum of the forces acting on the fluid element must sum to zero along each of the three coordinate directions x, y and z:

$$p(x)\mathrm{d}y\mathrm{d}z - (p(x) + \partial p/\partial x\ \mathrm{d}x)\mathrm{d}y\mathrm{d}z + F_{vx}\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0 \tag{1a}$$

$$p(y)\mathrm{d}x\mathrm{d}z - (p(y) + \partial p/\partial y\ \mathrm{d}y)\mathrm{d}x\mathrm{d}z + F_{vy}\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0 \tag{1b}$$

$$p(z)\mathrm{d}x\mathrm{d}y - (p(z) + \partial p/\partial z\ \mathrm{d}z)\mathrm{d}x\mathrm{d}y + F_{vz}\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0 \tag{1c}$$

Recall that the force $\mathbf{F}_v$ was defined on a per unit volume basis. Then $\mathbf{F}_v\ \mathrm{d}x\mathrm{d}y\mathrm{d}z$ is the total body force acting on the fluid element. Dividing expressions (1a) through (1c) by $\mathrm{d}x\mathrm{d}y\mathrm{d}z$:

$$\partial p/\partial x = F_{vx} \tag{2a}$$

$$\partial p/\partial y = F_{vy} \tag{2b}$$

$\partial p/\partial z = F_{vz}$ (2c)

or, in vector notation

$\nabla p = \mathbf{F}_v$ (3)

The set of equations (3) comprises the "differential equations of fluid statics."

<u>*Example: Gravitational Force Only:*</u> A container of fluid is at rest. If gravity acts in the - z direction, then the force $\mathbf{F}_v$ becomes $\mathbf{F}_v = -mg/V\ \boldsymbol{\delta}_z = -\rho\, g\ \boldsymbol{\delta}_z$ (here m is the mass of the fluid element, g is the gravitational acceleration, V is the volume of the fluid element, and ρ is the density of the fluid element). Equation (3) then yields

$\partial p/\partial x = 0$
$\partial p/\partial y = 0$
$\partial p/\partial z = -\rho g$

so that

$dp/dz = -\rho g \qquad dp = -\rho g dz$ (4)

If the density ρ of the fluid is constant, it can be taken outside of the integral:

$$\int_{p_1}^{p_2} dp = -\rho g \int_{z_1}^{z_2} dz$$ which evaluates to $(p_2 - p_1) = -\rho g\ (z_2 - z_1)$ (5)

Equation (5) applies for an *incompressible* (ρ = constant) fluid. In general, the density could vary with pressure and temperature, as for an ideal gas. For an ideal gas, $\rho = pM/(RT)$ where M is the molar mass (ex. kg/kg-mol), R is the gas constant, and T is the absolute temperature. Substituting the ideal gas expression for ρ into (4), and assuming isothermal conditions (T = constant):

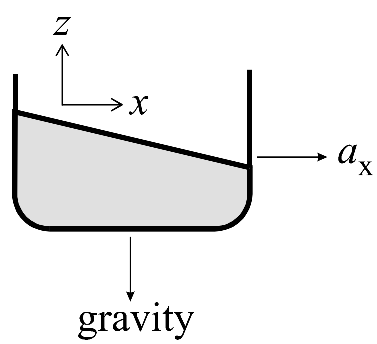
$dp\ /\ p = -g\ M/(RT)\ dz$

$$\int_{p_1}^{p_2} \frac{dp}{p} = -g\frac{M}{RT}\int_{z_1}^{z_2} dz$$

so that

$\ln(p_2/p_1) = -g\,M/(RT)\,(z_2 - z_1)$ $\qquad$ $p_2 = p_1\,e^{-g\,M/(RT)\,(z_2 - z_1)}$ (6)

Example: Linear Acceleration Combined with Gravitational Acceleration: A container of fluid is undergoing constant acceleration a_x along the + x direction. Also, the fluid is in the standard gravitational field which acts in the - z direction. The movement of the fluid corresponds to a rigid body translation, i.e. static equilibrium holds.



The body force per unit volume that acts on an element in the accelerating fluid has the form $\mathbf{F}_v = -\rho a_x\,\boldsymbol{\delta}_x - \rho g\,\boldsymbol{\delta}_z$, where the force component along the x direction, F_{vx}, is equal to $-ma_x/Vol = -\rho a_x$ (note that when the container of fluid is accelerated in the $+x$ direction, the fluid element experiences an inertial body force that acts in the $-x$ direction). Equations (3) become

$\partial p/\partial x = -\rho a_x \Rightarrow \qquad p = -\rho a_x x + f_1(y,z)$ (7a)

$\partial p/\partial y = 0 \qquad \Rightarrow \qquad p = f_2(x,z)$ (7b)

$\partial p/\partial z = -\rho g \Rightarrow \qquad p = -\rho g z + f_3(y,x)$ (7c)

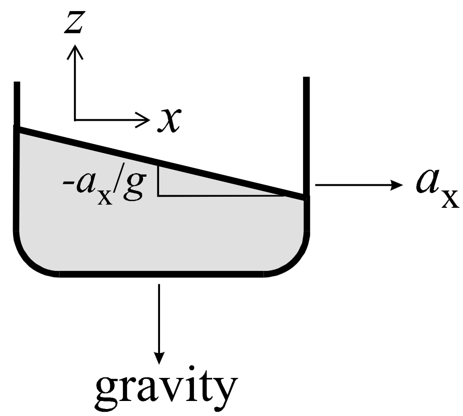
The functions f_1, f_2 and f_3 need to be chosen so that the final expression for p satisfies all three differential equations. By inspection, p cannot vary with y (equation (7b)), and setting $f_1 = -\rho g z + C_1$ and $f_3 = -\rho a_x x + C_2$ will satisfy equations (7a) and (7c) simultaneously. Therefore, the final expression for p is (setting $C_1 + C_2 = C$)

$$p = -\rho a_x x - \rho g z + C \tag{8}$$

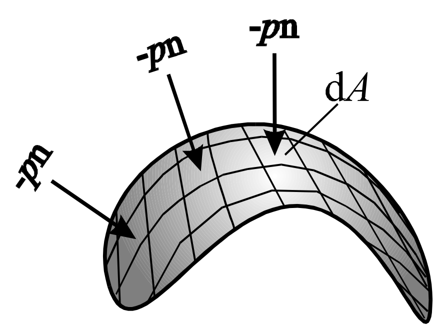
To find the equation for the free surface of the fluid, we set $p = p_o$ where p_o is the external (atmospheric) pressure at the free surface. Rearranging the resultant expression we get a relationship between z and x that specifies the free surface:

$$z = -(a_x/g)x + (C - p_o)/(\rho g) \tag{9}$$

The free surface of the fluid has a negative slope equal to -(a_x/g), as illustrated in the figure below.



Fluid Forces on Submerged Bodies Calculated by Direct Integration:



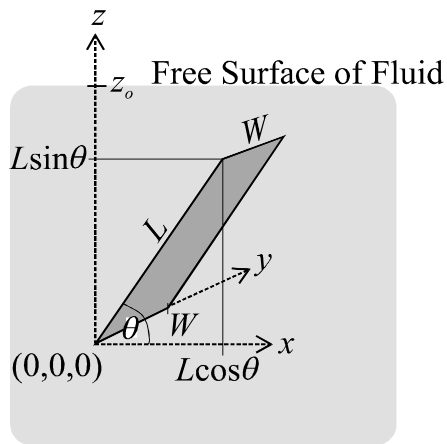
The force of a fluid in static equilibrium on a submerged surface is, at least conceptually, straightforward to calculate. Since the only stress present is pressure, the force on a differential surface element is $-p\mathbf{n}\,dA$ where $\mathbf{n}$ is a surface normal to the surface element dA (see figure below). The minus sign signifies that the stress exerted on a surface by pressure is *compressive* (ie. in the direction of $-\mathbf{n}$). To calculate the total force $\mathbf{F}$ of the fluid on the surface we simply sum the contributions for the entire surface, i.e. we perform the surface integral

$$\mathbf{F} = \iint_A -p\mathbf{n}dA \qquad (10)$$

To perform the integration in (10), the following steps are required:

1). Setup a coordinate system (try to choose one that will make the integration easier).
2). Express p, $\mathbf{n}$ and dA in terms of the system's coordinate variables.
3). Perform the integration according to equation (10).

The following example illustrates the general procedure for a submerged, inclined plane depicted in the figure below. We will calculate the total force due to pressure acting on the *top* surface of the plane. We will assume the liquid is incompressible (i.e. ρ = constant).



1). The coordinate system is setup as shown above. This example will also provide opportunity to practice vector surface integration.

2). The surface is given by $z = x \tan\theta$. Therefore, the surface normal is found from

$$\mathbf{n} = \nabla(z - x\tan\theta) / (\nabla(z - x\tan\theta) \bullet \nabla(z - x\tan\theta))^{1/2}$$
$$\mathbf{n} = (-\tan\theta\,\boldsymbol{\delta}_x + \boldsymbol{\delta}_z) / (\tan^2\theta + 1)^{1/2} = (-\tan\theta\,\boldsymbol{\delta}_x + \boldsymbol{\delta}_z) / \sec\theta$$
$$\mathbf{n} = -\sin\theta\,\boldsymbol{\delta}_x + \boldsymbol{\delta}_z / \sec\theta$$

Using Equation (5), the pressure is (the subscript "o" denotes conditions at the free surface of the fluid):

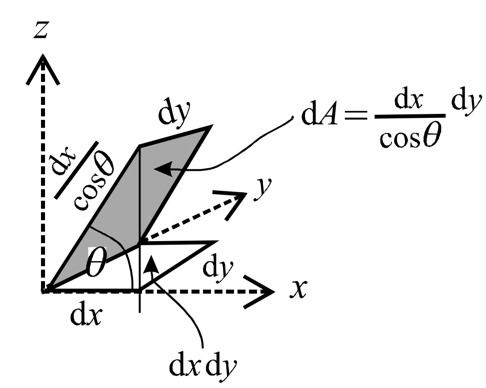
$$p = p_o - \gamma(z - z_o) \qquad \text{where } \gamma = \rho g$$

On the surface of the plane,

$$p = p_o - \gamma(x\tan\theta - z_o)$$

The area element dA, expressed in terms of x and y, is

$$dA = dy\, dx/\cos\theta \quad \text{(see figure at right)}$$



The limits of integration on y are 0 and W, where W is the width of the surface. The limits of integration on x are 0 and $L\cos\theta$, where L is the length of the plate.

3). To evaluate equation (10), we have to integrate:

$$\mathbf{F} = \boldsymbol{\delta}_x \int_0^{L\cos\theta} \int_0^{W} (p_o - \gamma(x\tan\theta - z_o))\sin\theta \, dy \, \frac{dx}{\cos\theta}$$

$$- \boldsymbol{\delta}_z \int_0^{L\cos\theta} \int_0^{W} \frac{(p_o - \gamma(x\tan\theta - z_o))}{\sec\theta} \, dy \, \frac{dx}{\cos\theta}$$

Integrating with respect to y and combining some of the trigonometric terms,

$$\mathbf{F} = \boldsymbol{\delta}_x W\tan\theta \int_0^{L\cos\theta} [p_o - \gamma(x\tan\theta - z_o)]dx - \boldsymbol{\delta}_z W \int_0^{L\cos\theta} [p_o - \gamma(x\tan\theta - z_o)]\, dx$$

Integrating with respect to x

$$\mathbf{F} = \boldsymbol{\delta}_x W\tan\theta \left(p_o x - \gamma \frac{(x\tan\theta - z_o)^2}{2\tan\theta} \right)\Bigg|_0^{L\cos\theta}$$

$$- \boldsymbol{\delta}_z W \left(p_o x - \gamma \frac{(x\tan\theta - z_o)^2}{2\tan\theta} \right)\Bigg|_0^{L\cos\theta}$$

Simplifying,

$$\mathbf{F} = \boldsymbol{\delta}_x WL\sin\theta \left(p_o - \gamma \frac{(L\sin\theta - z_o)^2 - z_o^2}{2L\sin\theta} \right)$$

$$- \boldsymbol{\delta}_z WL\cos\theta \left(p_o - \gamma \frac{(L\sin\theta - z_o)^2 - z_o^2}{2L\sin\theta} \right)$$

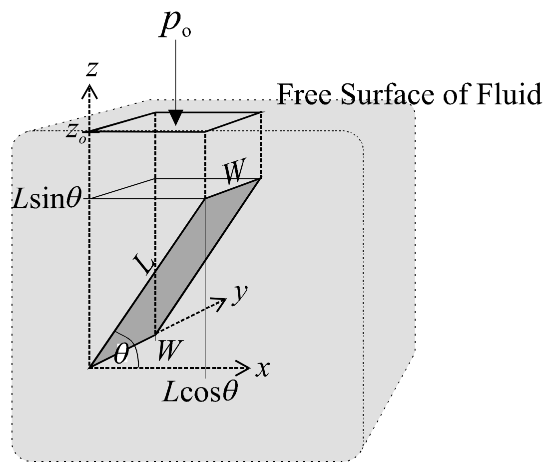
$$\mathbf{F} = \boldsymbol{\delta}_x WL\sin\theta \left(p_o - \gamma \frac{L\sin\theta - 2z_o}{2} \right) - \boldsymbol{\delta}_z WL\cos\theta \left(p_o - \gamma \frac{L\sin\theta - 2z_o}{2} \right) \tag{11}$$

Below we will compare equation (11), which gives the force acting on the top surface of the inclined plane, to other results obtained by simpler routes.

<u>Simplified Approaches for Calculation of Fluid Forces on Submerged Bodies:</u>

(1). Calculation of Vertical Force on a Submerged Surface.

The vertical force on a submerged surface equals the vertical component of the body force acting on the fluid directly above that surface, plus the force due to pressure at the free surface of the fluid. If the only body force is gravitational, then the "vertical component of the body force acting on the fluid" is equal to the weight of the fluid directly above the surface.



To illustrate, we use the inclined plane as an example. The total volume *Vol* of the fluid directly above the plane is (see illustration)

$$Vol = (z_o - L\sin\theta)\, W\, L\cos\theta + (1/2)\, W\, L\sin\theta\; L\cos\theta$$
$$= W\, L\cos\theta\,(z_o - L\sin\theta / 2)$$

The total body force on this fluid volume due to gravity is (the minus sign indicates that the force acts in the - z direction):

$$\mathbf{F}_B = -\,\gamma Vol\, \boldsymbol{\delta}_z = \gamma W\, L\cos\theta\,(L\sin\theta / 2 - z_o)\, \boldsymbol{\delta}_z$$

The pressure on the free surface of the fluid is $p_o\, W\, L\cos\theta$. Therefore, the total downward force $\mathbf{F}_z$ on the inclined plane equals

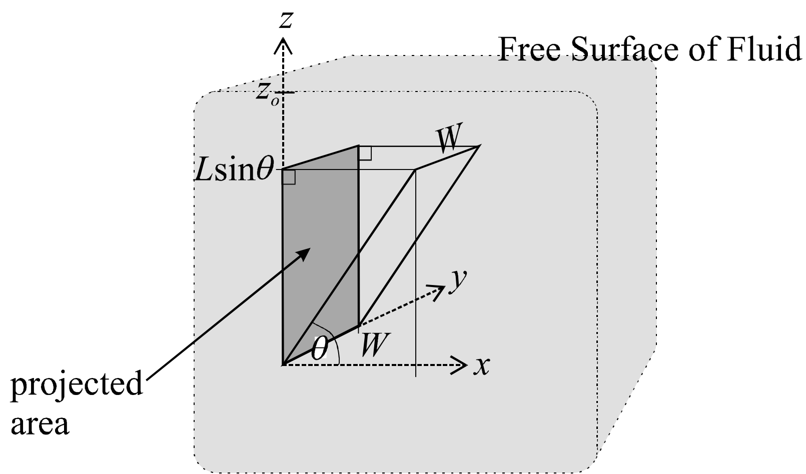
$$\mathbf{F}_z = \mathbf{F}_B - p_o\, W\, L\cos\theta\, \boldsymbol{\delta}_z = -\boldsymbol{\delta}_z\, W\, L\cos\theta\,(p_o - \gamma(L\sin\theta / 2 - z_o))$$

This result is equal to the z-component of the force calculated previously using the surface integral approach (see equation (11)).

(2). Calculation of Horizontal Force on a Submerged Surface.

The horizontal force acting on one side of a submerged surface can be determined by integrating the pressure along a plane vertical surface which is the projected area of the surface of interest.

An example will make the above statement clearer. Again, we refer to the inclined plane. The projected area of the inclined plane onto a vertical plane is illustrated in the figure. The projected area extends from $z = 0$ to $z = L\sin\theta$, and from $y = 0$ to $y = W$. To integrate pressure along the projected area, we evaluate (note that A is now the "projected area", not the actual area of the inclined plane)



$$\mathbf{F}_x = \boldsymbol{\delta}_x \int_A p\,dA = \boldsymbol{\delta}_x \int_0^W \int_0^{L\sin\theta} [p_o - \gamma(z - z_o)]\,dz\,dy$$

$$= \boldsymbol{\delta}_x W \int_0^{L\sin\theta} [p_o - \gamma(z - z_o)]\,dz = \boldsymbol{\delta}_x W \left[p_o z - \frac{\gamma(z - z_o)^2}{2} \right] \Bigg|_0^{L\sin\theta}$$

$$\mathbf{F}_x = \boldsymbol{\delta}_x W \left[p_o L\sin\theta - \gamma \frac{(L\sin\theta - z_o)^2 - z_o^2}{2} \right]$$

$$\mathbf{F}_x = \boldsymbol{\delta}_x W L \sin\theta \, [p_o - \gamma(L\sin\theta/2 - z_o)]$$

This expression is equivalent to the x-component of the force in equation (11).

The above approach allows us to calculate the horizontal force on even highly irregular surfaces, provided we can specify their projected area correctly.

(3). Buoyant Force on a Submerged Object.

The buoyant force on a submerged object is the vertical component of the force exerted by static equilibrium pressure on the object:

$$\text{buoyant force} = \boldsymbol{\delta}_z \bullet \int_A (-p\mathbf{n}dA) = \int_A -p n_z \, dA \qquad (12)$$

where the integral is performed over the entire surface of the object, and the z-axis was taken as denoting the vertical direction. The integral can be evaluated directly, similar to the procedure used earlier for the inclined plane. However, a much easier way is to use "Archimedes' principle," which states that the buoyant force equals

buoyant force = weight of fluid displaced by object (13)

For incompressible fluids (ρ = constant) this principle can be written as

buoyant force = γ(volume of object)

If, in addition to gravity, other body forces act on the fluid along the vertical direction the effect of such forces on the fluid's (apparent) weight must be considered. For example, if the fluid is being accelerated along the z direction instead of being stationary, the resultant inertial body force must be added to the gravitational body force.

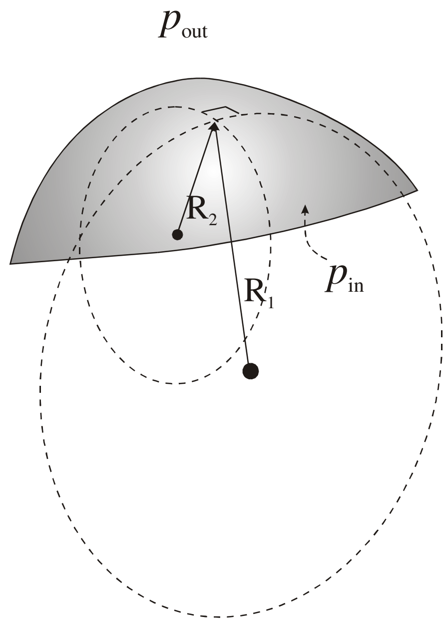
Surface Tension

(1) Introduction.

The interface between two immiscible fluids possesses a tendency to contract. The interface wants to contract because the molecules of each fluid prefer ("prefer" in the sense of minimizing their free energy) being next to molecules from the same fluid rather than molecules from the other fluid. The undesired contact between molecules from one fluid with those from the other fluid is minimized if the area of the interface between the fluids is as small as possible; ie. by contracting the interface as much as possible. The force driving this contraction is termed "surface tension." Surface tension acts along (parallel to) the interface, and under conditions of static equilibrium it is independent of the direction within the interface (i.e. it is isotropic within the interface). If the interface is curved, the presence of the surface tension causes the pressure on the two sides of the interface to be different. On the inside (the concave side) of the interface the pressure will be higher. The expression for the pressure difference is

$$p_{in} - p_{out} = T\,(1/R_1 + 1/R_2) \tag{14}$$

where T is the surface tension (units of force/length, ex. dyne/cm), p_{in} is the pressure on the concave side of the interface, p_{out} is the pressure on the convex side, and R_1 and R_2 are the two radii of curvature of the surface at the point where the pressure difference is $p_{in} - p_{out}$. The radii of curvature are the radii of any two mutually perpendicular circles whose perimeters have the same curvature as the surface at the point of interest. Note that both circles are also perpendicular to the surface. For a planar interface both R_1 and R_2 are infinite, and thus the pressure difference across a planar interface is zero.



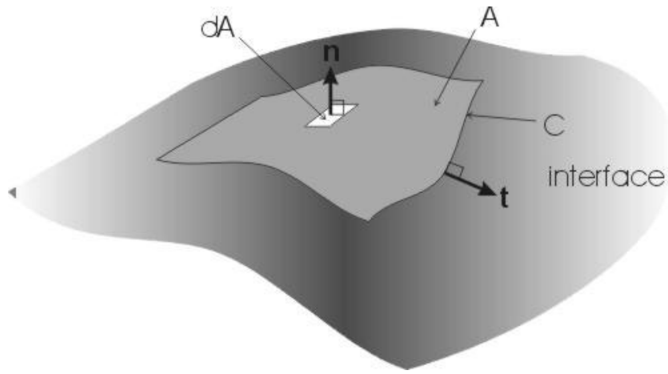
(2) An Interfacial Force Balance

C: a perimeter curve bounding the area *A* of an interface between two fluid phases
t: a unit outward normal vector to *C*, tangential to the interface
n: a unit outward normal vector to A, perpendicular to the interface
T: interfacial tension (units: force/length)
d*l*: a differential displacement (length) along the perimeter *C*

At static equilibrium, the forces on the interfacial region *A* must sum to 0. These forces consist of the pressure acting on *A* from above and below, and of the surface tension force acting on *A* along its perimeter *C*. In mathematical terms, this statement becomes:

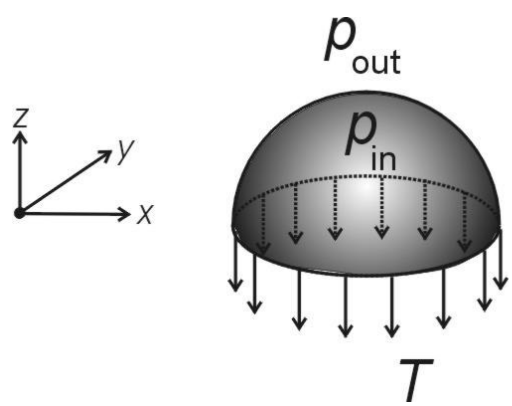
$$\iint_A -p_{out}\mathbf{n}dA + \iint_A p_{in}\mathbf{n}dA + \oint_C T\mathbf{t}dl = 0 \qquad (15)$$

The below figure shows the orientations of vectors **t** and **n**. The first term in the above equation is force on *A* due to pressure from the "outside" fluid (the "outside" fluid is the one pointed at by **n**), the second term is force on *A* from pressure from the "inside" fluid, and the third term is force on *A* due to surface tension which acts on *A* along the perimeter *C*. You can think of the surface region *A* as a control volume to which we applied the force balance given by equation 15.



As a simple example, let's assume we have a perfectly spherical bubble of fluid 1 inside of fluid 2. We cut the bubble in half, so that we are left with two hemispheres, and take the top hemisphere as our surface *A* (see figure). The curve *C* is now the dotted line, with $\mathbf{t} = -\boldsymbol{\delta}_z$. We moreover assume that body forces are negligible, so that the pressure outside the bubble is constant at p_{out}, and the pressure inside the bubble is constant at p_{in} (i.e. no gravitational effects). Taking the dot product of equation (15) with $\boldsymbol{\delta}_z$ leads to

$$-p_{out}\iint_A n_z dA + p_{in}\iint_A n_z dA - T\oint_C dl = 0 \qquad (16)$$



We made use of the fact that *T*, p_{out}, and p_{in}, are constant, $\boldsymbol{\delta}_z \bullet \mathbf{n} = n_z$, and $\boldsymbol{\delta}_z \bullet \mathbf{t} = -1$. The area integrals $\iint_A n_z dA$ are just the projected areas of the hemispherical surface onto the x-y plane; these integrals therefore equal πR^2 where *R* is the bubble radius. The integral $\oint_C dl$ is just the total length of the perimeter, which in this case of a perfect hemisphere is simply $2\pi R$. With these substitutions equation 16 becomes

$$\pi R^2(p_{in} - p_{out}) - 2\pi RT = 0$$

what rearranges to

$$p_{\text{in}} - p_{\text{out}} = 2T/R \tag{17}$$

Equation 17 was derived from the interfacial force balance (15). It shows that, for a spherical interface, the difference in pressures on the two sides is counterbalanced by the surface tension T within the interface multiplied by $2/R$. It is also a specific case of the more general equation (14), in which the two radii of curvature were allowed to be different. Equation 14 can also be derived from 15, through a somewhat more complex process.