Review of Vector Analysis in Cartesian Coordinates

Scalar: A quantity that has magnitude, but no direction. Examples are mass, temperature, pressure, time, distance, and real numbers. Scalars are usually represented by italic letters: for example, T for temperature or p for pressure.

Vector: A quantity that has a magnitude and a direction. Examples are velocity, force, acceleration, and spatial position. You have encountered vectors in calculus and in physics. Graphically, we represent a vector by an arrow starting at an "initial point" and ending at a "terminal point" (Figure 1). The direction of the arrow corresponds to the direction of the vector and the length of the arrow corresponds to the vector's magnitude. Vectors are often represented by bold faced letters: for example, velocity may be written as **v**, and position as **r**.

The "magnitude" of a vector is a scalar. For instance, v could represent speed (speed is the magnitude of velocity) of an object. We will now see how to mathematically represent vectors.

You should be familiar with the three-dimensional Cartesian coordinate system (CCS) depicted in Figure 2. The three CCS axes are labeled x_1, x_2, and x_3 (alternately, it is also common to represent the axes using the letters x, y, and z). Also, three unit vectors are drawn in Figure 2, labeled as $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$ and $\boldsymbol{\delta}_3$. The direction of each vector is along one of the coordinate axes. The vectors are "unit" vectors because their magnitude (length) is equal to 1. The three vectors $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$ and $\boldsymbol{\delta}_3$ are termed the "basis vectors" for the CCS. Note that the basis vectors are mutually orthogonal; in other words, they are at right angles to one another. Although we have drawn the basis vectors at the origin in Figure 2, they are defined at every point of the CCS as shown in Figure 3. Some simple examples: if a particle has a velocity $\mathbf{v} = 1\boldsymbol{\delta}_1$, then the particle moves by one unit length in one unit of time in the positive direction of the x_1 axis. If a particle has a position $\mathbf{r} = 1\boldsymbol{\delta}_1$ relative to the origin, then the particle is located at the point (1,0,0) on the x_1 axis.

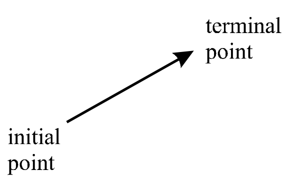


Fig. 1

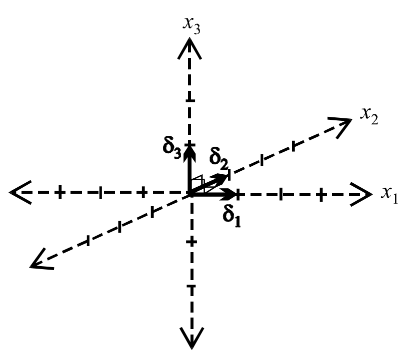


Fig. 2

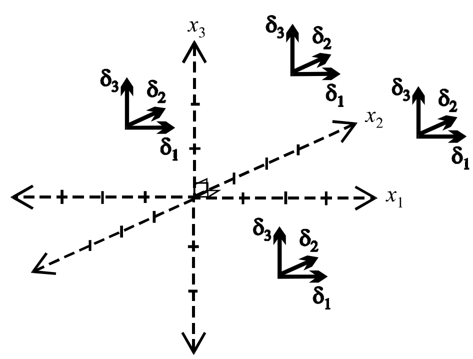


Fig. 3

Vector Algebra

Equality: Two vectors are equal if they have the same magnitude and direction, regardless of the position of their initial points (Figure 4). A vector with a direction exactly opposite to that of vector **A** but with the same magnitude is denoted by **-A** (Figure 5).

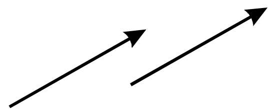


Fig. 4. These vectors are equal.

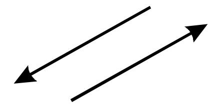


Fig. 5. These vectors are negatives of each other.

Addition and Subtraction: The sum of two vectors **C = A + B** is graphically formed by placing the initial point of **B** on the terminal point of **A** and then joining the initial point of **A** to the terminal point of **B** (Figure 6). The sum can also be formulated in terms of the so-called parallelogram law (Figure 7). Note that vector addition is commutative, ie. **A + B = B + A**. Sums of more than two vectors follow analogous rules; Figure 8 shows the case when three vectors are added: **D = A + B + C**. A subtraction **C = A - B** can always be written as the addition **C = A + (-B)** (Figure 9), and follows the same rules as addition.

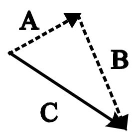


Fig. 6. C = A + B

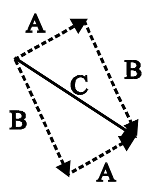


Fig. 7. C = A + B

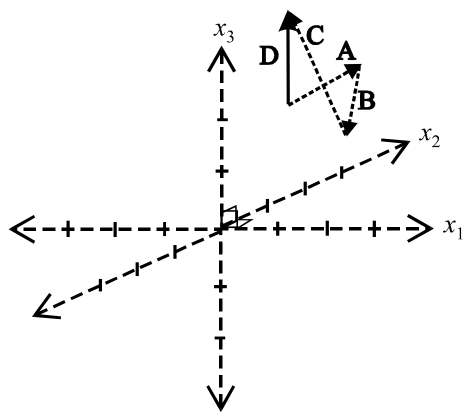


Fig. 8. D = A + B + C

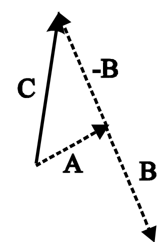


Fig. 9. C = A - B

The CCS basis vectors can be used to represent any arbitrary vector. For example, $\mathbf{A} = 1\boldsymbol{\delta}_1 + 3\boldsymbol{\delta}_2 - 2\boldsymbol{\delta}_3$ is shown graphically in Figure 10. The prefactors 1, 3 and -2 in front of the basis vectors are termed the "components" of the vector **A** along the x_1, x_2 and x_3 directions, respectively. To add (or subtract) two or more vectors, we add (subtract) their components along each of the coordinate directions:

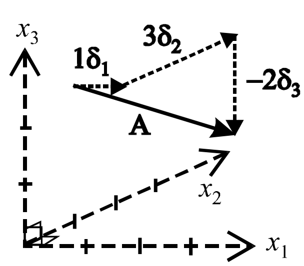


Fig. 10

$$\mathbf{A} = A_1\boldsymbol{\delta}_1 + A_2\boldsymbol{\delta}_2 + A_3\boldsymbol{\delta}_3 \qquad \mathbf{B} = B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \underbrace{(A_1 + B_1)}_{C_1}\boldsymbol{\delta}_1 + \underbrace{(A_2 + B_2)}_{C_2}\boldsymbol{\delta}_2 + \underbrace{(A_3 + B_3)}_{C_3}\boldsymbol{\delta}_3 \qquad (1)$$

$$\mathbf{C} = \mathbf{A} - \mathbf{B} = \overbrace{(A_1 - B_1)}\boldsymbol{\delta}_1 + \overbrace{(A_2 - B_2)}\boldsymbol{\delta}_2 + \overbrace{(A_3 - B_3)}\boldsymbol{\delta}_3 \qquad (2)$$

Also, note that if **C** = **A** + **B**, then **C** - **A** = **B**, **C** + **B** = **A** + 2**B**, etc., same as for ordinary (ie. scalar) algebraic equations involving addition and subtraction.

Multiplication and Division of a Vector by a Scalar: The product $m\mathbf{A}$, where m is a scalar multiplying the vector **A**, increases the magnitude of **A** by a factor of m and either preserves the direction of **A** (if $m > 0$) or reverses it (if $m < 0$) (Figure 11). It is easy to show that (you should verify this graphically if you don't believe it)

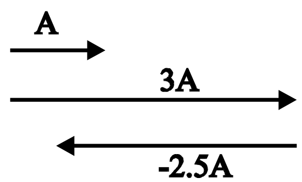


Fig. 11.

$$m\mathbf{A} = \mathbf{A}m = mA_1\boldsymbol{\delta}_1 + mA_2\boldsymbol{\delta}_2 + mA_3\boldsymbol{\delta}_3 \quad (3)$$

Division of a vector **A** by a scalar m is the same as multiplying **A** by $1/m$: $\mathbf{A}/m = (1/m)\mathbf{A}$. Multiplying a vector by 0 gives the "null" vector. The magnitude of the null vector is 0 and its direction is unspecified. Division of a vector by 0 is undefined.

Magnitude of a Vector: The magnitude A of a vector **A** is the square root of the sum of the squares of its components: $A = (A_1^2 + A_2^2 + A_3^2)^{1/2}$. This result comes directly from the Pythagorean theorem. For example, the distance r from the origin for an object whose position is $\mathbf{r} = x_1\boldsymbol{\delta}_1 + x_2\boldsymbol{\delta}_2 + x_3\boldsymbol{\delta}_3$ is simply $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ (Figure 12). The speed v of an object whose velocity is $\mathbf{v} = v_1\boldsymbol{\delta}_1 + v_2\boldsymbol{\delta}_2 + v_3\boldsymbol{\delta}_3$ is $v = (v_1^2 + v_2^2 + v_3^2)^{1/2}$.

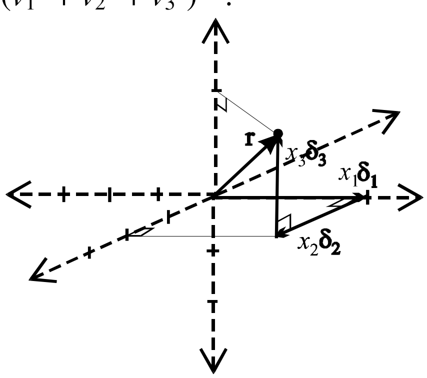


Fig. 12. The magnitude r of the position vector **r** is $(x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Scalar and Vector Fields: In a scalar field, each point in space gets a scalar value. For example, a constant temperature field T can be written $T(x_1,x_2,x_3) = K$, where K is a constant. This expression assigns the same temperature K to every point in space. For a varying temperature field $T(x_1,x_2,x_3)$ would vary with position; for instance, $T(x_1,x_2,x_3) = x_1^3x_2 - x_3^2$.

Writing out (x_1,x_2,x_3) to refer to a point in space can get tiresome, and often it's simpler to just write **r** to denote position. Let's say we have a scalar field S. Then the notations $S(x_1,x_2,x_3)$ and $S(\mathbf{r})$ are equivalent, with the notational convention that the position vector **r** replaces the list of coordinates x_1,x_2,x_3.

In a vector field, each point in space gets a vector assigned to it. The set of position vectors **r** represents a vector field, since each point in space has a vector **r** assigned to it (Figure 12). Similarly, a velocity vector field can be defined. An example of a constant velocity field is $\mathbf{v} = 3.715\boldsymbol{\delta}_1 - 0.5\boldsymbol{\delta}_2 + 1.2\boldsymbol{\delta}_3$. The velocity field is constant, or "uniform", since the components of **v** do not vary with position. An example of a varying velocity field is

$$\mathbf{v}(\mathbf{r}) = \left\{ 1 - \frac{\cos[2\tan^{-1}(x_2/x_1)]}{(x_1^2 + x_2^2)} \right\}\boldsymbol{\delta}_1 - \frac{\sin[2\tan^{-1}(x_2/x_1)]}{(x_1^2 + x_2^2)}\boldsymbol{\delta}_2 \qquad (4)$$

Equation (4) refers to so-called "incompressible potential flow" of fluid around a cylindrical object. In equation (4), the velocity component along the $\boldsymbol{\delta}_3$ direction is zero, and the notation $\mathbf{v}(\mathbf{r})$ instead of $\mathbf{v}(x_1, x_2)$ has been used to indicate the position dependence of **v**.

Vector Dot Product: The dot product of two arbitrary vectors **A** and **B** is written $\mathbf{A} \bullet \mathbf{B}$. Taking the dot product of two vectors produces a scalar. The magnitude of the scalar $S = \mathbf{A} \bullet \mathbf{B}$ equals the product of the magnitudes of **A** and **B** and the cosine of the angle θ between **A** and **B** (Figure 13). In equation form,

$$S = \mathbf{A} \bullet \mathbf{B} = AB\cos\theta \qquad 0 \leq \theta \leq \pi \qquad (5)$$

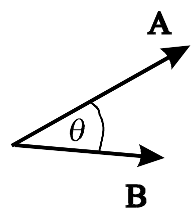


Fig. 13.

From equation (5) it follows that the dot product is commutative, $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$. The dot product is also distributive; ie. $\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$.

The dot product of two vectors that are perpendicular to one another is 0 since $\cos(90^{o}) = 0$. The dot product of two parallel vectors pointing in the same direction is the product of their magnitudes since $\cos(0^{o}) = 1$. The dot product of two parallel vectors pointing in opposite directions is the negative of the product of their magnitudes since $\cos(180^{o}) = -1$. From equation (5) it follows that

$$\boldsymbol{\delta}_1 \bullet \boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 \bullet \boldsymbol{\delta}_2 = \boldsymbol{\delta}_3 \bullet \boldsymbol{\delta}_3 = 1 \qquad (6)$$

(remember that the basis vectors have unit magnitudes) and

$$\boldsymbol{\delta}_1 \bullet \boldsymbol{\delta}_2 = \boldsymbol{\delta}_2 \bullet \boldsymbol{\delta}_3 = \boldsymbol{\delta}_3 \bullet \boldsymbol{\delta}_1 = 0 \qquad (7)$$

since basis vectors pointing along different CCS axes are perpendicular. Using equations (6) and (7) and the distributive law, we can show that for two arbitrary vectors **A** and **B**

$$\mathbf{A} = A_1\boldsymbol{\delta}_1 + A_2\boldsymbol{\delta}_2 + A_3\boldsymbol{\delta}_3 \qquad \mathbf{B} = B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3$$

$$\mathbf{A} \bullet \mathbf{B} = (A_1\boldsymbol{\delta}_1 + A_2\boldsymbol{\delta}_2 + A_3\boldsymbol{\delta}_3) \bullet (B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3) \quad (8a)$$

$$= A_1\boldsymbol{\delta}_1 \bullet (B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3) + A_2\boldsymbol{\delta}_2 \bullet (B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3) + A_3\boldsymbol{\delta}_3 \bullet (B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3) \quad (8b)$$

$$\mathbf{A} \bullet \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 \quad (8c)$$

In arriving at equation (8c), we used the fact that dot products between perpendicular basis vectors are zero. Equation (8c) states that the dot product between two arbitrary vectors can be simply calculated by summing the products of their respective components. The values produced by equations (8c) and (5) are identical, and either equation can be used to calculate the dot product.

Use of the Dot Product: Dot products are convenient for obtaining the magnitude A of a vector **A**:

$$A = (\mathbf{A} \bullet \mathbf{A})^{1/2} = (A_1^2 + A_2^2 + A_3^2)^{1/2} \quad (9)$$

Dot products are also useful for obtaining the component of a vector **A** along an arbitrary direction **a**. This component is often called the "projection of **A** onto the direction **a**". Imagine that an object travels with a velocity

$$\mathbf{v} = v_1\boldsymbol{\delta}_1 + v_2\boldsymbol{\delta}_2 + v_3\boldsymbol{\delta}_3.$$

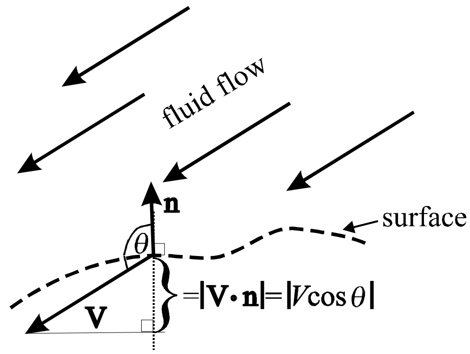
To obtain the projection of **v** along a particular direction, we form the dot product of **v** and a unit vector pointing along the desired direction. For example, if we want the speed with which the object moves along the x_1 direction, we get

$$\boldsymbol{\delta}_1 \bullet \mathbf{v} = v_1$$

Indeed, v_1 is the velocity component that tells us how fast the object advances in the x_1 direction. For a less trivial example, let's say we want the speed of the object in the direction specified by an arbitrary unit vector $\mathbf{n} = n_1\boldsymbol{\delta}_1 + n_2\boldsymbol{\delta}_2 + n_3\boldsymbol{\delta}_3$. Since **n** is a unit vector it will be true that $(n_1^2 + n_2^2 + n_3^2)^{1/2} = 1$. The speed v_n of the object in the direction specified by **n** is $v_n = \mathbf{n} \bullet \mathbf{v} = n_1v_1 + n_2v_2 + n_3v_3$. We could also have used $v_n = v\cos\theta$, where θ is the angle between **v** and **n** (the magnitude n is not explicitly shown in the expression $v\cos\theta$ since it is unity).

The use of the dot product to obtain vector projections onto particular directions is perhaps best illustrated graphically, as in Figure 14. Figure 14 shows how the dot product can be used to calculate the speed of a fluid perpendicular to a surface by dotting the velocity **v** of the fluid with a unit vector **n** that is perpendicular to the surface. Such dot products will be important in calculating the flux of mass, energy, or momentum across a surface due to material transport, for example. Because θ is greater than 90^o in Figure 14, the dot product $v\cos\theta$ will have a negative value. The absolute value of the dot product equals the length of the projection of **v** onto the line along which **n** lies, as drawn in Figure 14. You may have seen similar diagrams in physics, where the component of a force **F** along a particular direction was given by $\mathbf{F} \bullet \mathbf{n} = F\cos\theta$, where **n** pointed along the desired direction and θ was the angle between **F** and **n**.

Fig. 14



Cross Product: The cross product of two arbitrary vectors **A** and **B** is a vector **C** written **C** = **A** × **B.** The magnitude of **C** is the product of the magnitudes of **A** and **B** and the sine of the angle θ between **A** and **B**. In equation form,

$$C = AB\sin\theta \qquad 0 \leq \theta \leq \pi \tag{10}$$

C becomes the null vector when **A** and **B** are parallel (since $\sin 0^{\circ} = 0$). For all other cases, (i.e. nonparallel **A** and **B**), the direction of **C** is perpendicular to the plane formed by **A** and **B** and such that **A**, **B** and **C** form a "right-handed" system. In a right-handed system, the thumb of your right hand points in the direction of the resultant vector (**C**) if the other fingers of your right hand curl from the first vector in the cross product (**A**) toward the second vector of the cross product (**B**) through the angle θ. The right-handedness of the cross product is illustrated in Figure 15. Note that θ is always defined so that it is less than π.

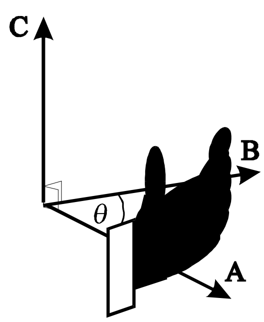


Fig. 15.

Consideration of Figure 15 shows that **A** × **B** = - **B** × **A**, so that the vector product *is not* commutative. The distributive law does hold, ie. **A** × (**B** + **C**) = **A** × **B** + **A** × **C**.

Since basis vectors of the same type are parallel in the CCS, it follows that

$$\boldsymbol{\delta}_1 \times \boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 \times \boldsymbol{\delta}_2 = \boldsymbol{\delta}_3 \times \boldsymbol{\delta}_3 = 0 \tag{11}$$

Furthermore (use the right hand rule and the definition of the cross-product to verify these expressions):

$$\boldsymbol{\delta}_1 \times \boldsymbol{\delta}_2 = \boldsymbol{\delta}_3 \quad \boldsymbol{\delta}_2 \times \boldsymbol{\delta}_1 = -\boldsymbol{\delta}_3 \quad \boldsymbol{\delta}_2 \times \boldsymbol{\delta}_3 = \boldsymbol{\delta}_1 \quad \boldsymbol{\delta}_3 \times \boldsymbol{\delta}_2 = -\boldsymbol{\delta}_1 \quad \boldsymbol{\delta}_3 \times \boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 \quad \boldsymbol{\delta}_1 \times \boldsymbol{\delta}_3 = -\boldsymbol{\delta}_2 \tag{12}$$

The general formula for the cross product of two arbitrary vectors **A** and **B** can be derived using equations (11) and (12):

$$\mathbf{A} = A_1\boldsymbol{\delta}_1 + A_2\boldsymbol{\delta}_2 + A_3\boldsymbol{\delta}_3 \qquad\qquad \mathbf{B} = B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3$$

$$\mathbf{A}\times\mathbf{B} = (A_1\boldsymbol{\delta}_1 + A_2\boldsymbol{\delta}_2 + A_3\boldsymbol{\delta}_3)\times(B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3) \tag{13a}$$

$$= A_1\boldsymbol{\delta}_1\times(B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3) + A_2\boldsymbol{\delta}_2\times(B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3) + A_3\boldsymbol{\delta}_3\times(B_1\boldsymbol{\delta}_1 + B_2\boldsymbol{\delta}_2 + B_3\boldsymbol{\delta}_3) \tag{13b}$$

$$= A_1B_2\boldsymbol{\delta}_3 - A_1B_3\boldsymbol{\delta}_2 - A_2B_1\boldsymbol{\delta}_3 + A_2B_3\boldsymbol{\delta}_1 + A_3B_1\boldsymbol{\delta}_2 - A_3B_2\boldsymbol{\delta}_1$$

$$\mathbf{A}\times\mathbf{B} = (A_2B_3 - A_3B_2)\boldsymbol{\delta}_1 + (A_3B_1 - A_1B_3)\boldsymbol{\delta}_2 + (A_1B_2 - A_2B_1)\boldsymbol{\delta}_3 \tag{13c}$$

In arriving at equation (13c), we used equations (11) and (12) to evaluate the cross products between basis vectors. Equation (13c) can be used to calculate the cross product between any two arbitrary vectors. Note that the formula (13c) can be obtained from the expression

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{14}$$

using standard rules of determinant evaluation. If you are familiar with determinants, Equation (14) can help you remember the formula for calculating the cross product.

Use of the Cross Product: One way cross products are useful in transport phenomena is in determining a quantity called "vorticity". Vorticity will be introduced later on in the course. Also, recall from physics that torque $\boldsymbol{\tau}$ and angular momentum **L** are cross products. For instance, the angular momentum of an object is

$$\mathbf{L} = \mathbf{r}'\times\mathbf{p}$$

where **r'** is the position of the object (relative to the point around which the angular momentum is being calculated) and **p** is the linear momentum of the object.

Vector Division: Division of a vector **A** by another vector **B**, **A/B**, is not defined.

Notational Conventions: Often, "shorthand notations" are employed to make handling vector equations easier and more compact. Here two notational conventions will be introduced: (1) the Kronecker delta and (2) the summation convention. Both are widely employed by engineers and scientists and are routinely used in books that make use of vector analysis.

The Kronecker delta δ_{ij} is defined as follows:

$$\delta_{ij} = 1 \;\text{ if } i = j$$
$$\delta_{ij} = 0 \;\text{ if } i \neq j \tag{15}$$

For instance, using the Kronecker delta we can write equations (6) and (7) as

$$\boldsymbol{\delta}_i \bullet \boldsymbol{\delta}_j = \delta_{ij} \qquad i,j = 1..3 \tag{16}$$

Equation (16) states that the dot product of a basis vector $\boldsymbol{\delta}_i$ with a second basis vector $\boldsymbol{\delta}_j$ is 0 unless i = j, in which case the dot product is unity.

The "summation convention" states that an index is to be summed over whenever it appears twice in a term. Thus the summation convention allows $\mathbf{A} = A_1\boldsymbol{\delta}_1 + A_2\boldsymbol{\delta}_2 + A_3\boldsymbol{\delta}_3$ to be written as $\mathbf{A} = A_i\boldsymbol{\delta}_i$. Because the index i appears twice in the term $A_i\boldsymbol{\delta}_i$, it is understood to be summed over. Many of the earlier equations can be rewritten in a more compact form using the summation convention. For instance, equation (8c) for the dot product becomes $\mathbf{A} \bullet \mathbf{B} = A_iB_i$.

Vector Calculus

Vector Differentiation: Let's first consider the derivative of a scalar S that depends on n variables q_1, $q_2, \ldots, q_n$ (the q's could be coordinates, such as x_1, x_2, and x_3; as well as other variables such as time; temperature; pressure etc.). As you have learned in calculus, the derivative of $S(q_1, q_2, \ldots, q_n)$ with respect to one of the variables, say q_i, is defined as

$$\frac{\partial S}{\partial q_i} = \lim_{\Delta q_i \to 0} \frac{S(q_1, q_2, \ldots, q_i + \Delta q_i, \ldots, q_n) - S(q_1, q_2, \ldots, q_i, \ldots, q_n)}{\Delta q_i} \qquad (17)$$

As an arbitrary example, if we have $S(q_1, q_2, q_3, q_4) = q_1 + \ln(q_2/q_3)/q_4$, then $\frac{\partial S}{\partial q_1} = 1$, $\frac{\partial S}{\partial q_2} = \frac{1}{q_2 q_4}$, etc. The derivative of a vector $\mathbf{A}(q_1, q_2, \ldots, q_n)$ is defined in an analogous fashion, except that each component of the vector is differentiated. In equation form,

$$\frac{\partial \mathbf{A}}{\partial q_i} = \lim_{\Delta q_i \to 0} \frac{\mathbf{A}(q_1, q_2, \ldots, q_i + \Delta q_i, \ldots, q_n) - \mathbf{A}(q_1, q_2, \ldots, q_i, \ldots, q_n)}{\Delta q_i}$$

For $\mathbf{A} = A_1\boldsymbol{\delta}_1 + A_2\boldsymbol{\delta}_2 + A_3\boldsymbol{\delta}_3$, the above equation can be written

$$\frac{\partial \mathbf{A}}{\partial q_i} = \lim_{\Delta q_i \to 0} \frac{A_1(q_1, q_2, \ldots, q_i + \Delta q_i, \ldots, q_n) - A_1(q_1, q_2, \ldots, q_i, \ldots, q_n)}{\Delta q_i}\boldsymbol{\delta}_1 +$$
$$\lim_{\Delta q_i \to 0} \frac{A_2(q_1, q_2, \ldots, q_i + \Delta q_i, \ldots, q_n) - A_2(q_1, q_2, \ldots, q_i, \ldots, q_n)}{\Delta q_i}\boldsymbol{\delta}_2 +$$
$$\lim_{\Delta q_i \to 0} \frac{A_3(q_1, q_2, \ldots, q_i + \Delta q_i, \ldots, q_n) - A_3(q_1, q_2, \ldots, q_i, \ldots, q_n)}{\Delta q_i}\boldsymbol{\delta}_3 +$$

$$\frac{\partial \mathbf{A}}{\partial q_i} = \left(\frac{\partial A_1}{\partial q_i}\right)\boldsymbol{\delta}_1 + \left(\frac{\partial A_2}{\partial q_i}\right)\boldsymbol{\delta}_2 + \left(\frac{\partial A_3}{\partial q_i}\right)\boldsymbol{\delta}_3 \qquad \text{(CCS)} \qquad (18)$$

Using the summation convention, equation (18) is written

$$\frac{\partial \mathbf{A}}{\partial q_i} = \left(\frac{\partial A_j}{\partial q_i}\right)\boldsymbol{\delta}_j \qquad \text{(CCS)} \qquad (19)$$

Note that the derivative of a vector is also a vector. For the present we are only working in the CCS, in which the basis vectors are constant (i.e. their direction and magnitude do not change). For this reason, derivatives of the basis vectors do not appear in equations (18) and (19). **In other coordinate systems, such as cylindrical and spherical coordinate systems, the direction of a unit basis vector may be a function of position, and this dependence will have to be accounted for when vectors are being differentiated with respect to spatial coordinates.** For now, it is important to realize that equation (19) is specialized to the CCS and that later it will have to be generalized when other coordinate systems are considered.

Having defined the derivative of a vector, $\frac{\partial \mathbf{A}}{\partial q_i}$, we can also list some useful formulas for manipulating vector derivatives. These include:

$$\frac{\partial}{\partial q_i}(\mathbf{A}+\mathbf{B}) = \frac{\partial \mathbf{A}}{\partial q_i} + \frac{\partial \mathbf{B}}{\partial q_i} \tag{21a}$$

$$\frac{\partial}{\partial q_i}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial q_i} + \frac{\partial \mathbf{A}}{\partial q_i}\cdot\mathbf{B} \tag{21b}$$

$$\frac{\partial}{\partial q_i}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{\partial \mathbf{B}}{\partial q_i} + \frac{\partial \mathbf{A}}{\partial q_i}\times\mathbf{B} \tag{21c}$$

$$\frac{\partial}{\partial q_i}(S\mathbf{A}) = S\frac{\partial \mathbf{A}}{\partial q_i} + \frac{\partial S}{\partial q_i}\mathbf{A} \tag{21d}$$

In these formulas, **A** and **B** are arbitrary but differentiable vectors, and S is an arbitrary but differentiable scalar. Second and higher order derivatives can be formed by repeated application of equation (19).

As an example of vector differentiation, let's first consider the position vector $\mathbf{r} = x_1\boldsymbol{\delta}_1 + x_2\boldsymbol{\delta}_2 + x_3\boldsymbol{\delta}_3$. Using equation (19), $\frac{\partial \mathbf{r}}{\partial x_1} = \boldsymbol{\delta}_1$. In this case, second and higher order derivatives are zero, since both the magnitude and direction of $\boldsymbol{\delta}_1$ are constant (therefore there is nothing that can be further differentiated). As a second example, let's suppose that a particle in a fluid moves so that its position $\mathbf{r}_p$ varies with time t as $\mathbf{r}_p = t^2\,\boldsymbol{\delta}_1 + 1/t\,\boldsymbol{\delta}_2 + (1-t)\,\boldsymbol{\delta}_3$. The derivative $\frac{d\mathbf{r}_p}{dt} = 2t\boldsymbol{\delta}_1 - 1/t^2\boldsymbol{\delta}_2 - \boldsymbol{\delta}_3$ is the velocity **v** of the particle. Differentiating again with respect to t would give us the acceleration of the particle. Since $\mathbf{r}_p$ depends *only* on t, the ordinary derivative symbol $\frac{d}{dt}$ was used instead of the partial derivative symbol $\frac{\partial}{\partial t}$.

Differentials of Vectors and Scalars: A "differential" is an infinitesimal (arbitrarily small) change in a quantity. The differential dS of a scalar $S(q_1, q_2, \ldots, q_n)$ is (note the use of the summation convention; hereafter, its use will not always be pointed out)

$$dS = \frac{\partial S}{\partial q_i} dq_i \qquad i = 1..n \tag{22}$$

In Equation (22), $dS = S(q_1+dq_1, q_2+dq_2, \ldots, q_n+dq_n) - S(q_1, q_2, \ldots, q_n)$, and dq_i represents an infinitesimally small change in the variable q_i. Equation (22) states that differential changes in the variables q_i (i = 1..n) on which S depends give rise to a differential change dS in S. This resultant change dS equals the sum of n terms, where each term is a product of the rate of change (slope) of S with respect to one of the q_i multiplied by the corresponding change dq_i in that variable.

As an example of equation (22), let's imagine that temperature $T(\mathbf{r})$ depends on position as $T(\mathbf{r}) = x_1^2 + x_2x_3$. Then if an observer walks from **r** to **r** + d**r** - in other words, the observer's position changes from (x_1, x_2, x_3) to $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ - the observer will experience a differential change in temperature equal to $dT = 2x_1dx_1 + x_3dx_2 + x_2dx_3$. Note that this statement is only accurate for truly differential (i.e. infinitesimal) changes.

For a vector $\mathbf{A}(q_1, q_2,\ldots, q_n)$, infinitesimal changes dq_i in the variables q_i (i = 1..n) will give rise to infinitesimal changes in each of the vector components. Therefore, if $\mathbf{A}(q_1, q_2,\ldots, q_n) = A_1(q_1, q_2,\ldots, q_n)\boldsymbol{\delta}_1 + A_2(q_1, q_2,\ldots, q_n)\boldsymbol{\delta}_2 + A_3(q_1, q_2,\ldots, q_n)\boldsymbol{\delta}_3$, then the differential of **A** is

$$d\mathbf{A} = dA_1\,\boldsymbol{\delta}_1 + dA_2\,\boldsymbol{\delta}_2 + dA_3\,\boldsymbol{\delta}_3 \qquad (23)$$

where, for j = 1, 2 or 3

$$dA_j = \frac{\partial A_j}{\partial q_i}dq_i \qquad \text{(CCS)} \qquad (24)$$

Again, caution must be exercised in coordinate systems other than the CCS, since the basis vectors may then also change with the variables q_i (i.e. if some of the q_i are coordinates).

The Vector Differential Operator, "Del": The Del operator is denoted by the symbol ∇, and in CCS it is defined by

$$\nabla \equiv \frac{\partial}{\partial x_i}\boldsymbol{\delta}_i = \frac{\partial}{\partial x_1}\boldsymbol{\delta}_1 + \frac{\partial}{\partial x_2}\boldsymbol{\delta}_2 + \frac{\partial}{\partial x_3}\boldsymbol{\delta}_3 \qquad \text{(CCS)} \qquad (25)$$

From the definition, we see that the ∇ operator takes derivatives with respect to system coordinates, i.e. with respect to position (in fact, some texts prefer the notation $\frac{\partial}{\partial \mathbf{r}}$ instead of ∇, where **r** is the position vector). ∇ has units of inverse length.

The Gradient: Let $S(x_1, x_2, x_3)$ be a differentiable scalar field. Then the gradient of $S(x_1, x_2, x_3)$ is written ∇S (sometimes the notation 'grad S ' is used) and is defined by

$$\nabla S = \frac{\partial S}{\partial x_i}\boldsymbol{\delta}_i = \frac{\partial S}{\partial x_1}\boldsymbol{\delta}_1 + \frac{\partial S}{\partial x_2}\boldsymbol{\delta}_2 + \frac{\partial S}{\partial x_3}\boldsymbol{\delta}_3 \qquad \text{(CCS)} \qquad (26)$$

The gradient ∇S is a vector defined at every point at which S is defined and differentiable. Physically, ∇S is the rate of change of the scalar S with position. For example, if S is the pressure p, then ∇p would be the pressure gradient. If the pressure is measured in pascals (Pa) and distances along the coordinates in meters (m), then ∇p would have the units Pa/m. For instance, if $\nabla p = 3\,\boldsymbol{\delta}_2$ Pa/m, then an observer walking along the $\boldsymbol{\delta}_2$ direction would feel the pressure increase at a rate of 3 Pa/m.

The rate of change of S in a direction specified by a unit vector **a** is given by the dot product $\mathbf{a} \bullet \nabla S$. Because the magnitude of **a** is unity, the dot product simply represents the projection of the vector

∇S onto the direction specified by **a** (cf. Fig. 14). In the pressure example, if the observer walked in the direction of $\mathbf{a} = 3/5\boldsymbol{\delta}_1 + 4/5\boldsymbol{\delta}_2$ (you can verify that **a** has a unit magnitude), then the rate of change in pressure that the observer would experience is $\mathbf{a} \bullet \nabla S = (3/5\boldsymbol{\delta}_1 + 4/5\boldsymbol{\delta}_2) \bullet 3\, \boldsymbol{\delta}_2 = 12/5$ Pa/m (equation (8c) was used to calculate the dot product). Note that the vector **a** has no units and is simply used to specify direction. If the observer walked in the $\boldsymbol{\delta}_1$ (or $\boldsymbol{\delta}_3$) direction he/she would experience no change in pressure. Of course, S can be any scalar quantity, and pressure was simply used as a convenient example to illustrate the physical meaning of the gradient. Since $\mathbf{a} \bullet \nabla S$ is greatest when **a** points in the same direction as ∇S (why? see equation (5)), it follows that the **steepest rate of change in *S* occurs in the direction specified by its gradient.**

Returning to the concept of differentials, the differential $\mathrm{d}S = S(\mathbf{r} + \mathrm{d}\mathbf{r}) - S(\mathbf{r})$ corresponding to an infinitesimal change in position $\mathrm{d}\mathbf{r} = \mathrm{d}x_1\, \boldsymbol{\delta}_1 + \mathrm{d}x_2\, \boldsymbol{\delta}_2 + \mathrm{d}x_3\, \boldsymbol{\delta}_3$ is

$$\mathrm{d}S = \nabla S \bullet \mathrm{d}\mathbf{r} = \frac{\partial S}{\partial x_i}\boldsymbol{\delta}_i \bullet \mathrm{d}x_j\, \boldsymbol{\delta}_j = \frac{\partial S}{\partial x_i}\mathrm{d}x_j\, \delta_{ij} = \frac{\partial S}{\partial x_i}\mathrm{d}x_i \qquad \text{(CCS)} \qquad (27)$$

Let's review what was done in equation (27). Briefly, the summation convention and the Kronecker delta were used to avoid writing out the full expressions for ∇S, d**r**, and their dot product. In detail: after the second '=' sign the summation convention was used to represent ∇S and d**r**. Note that different indices were used for ∇S and d**r** (i was used for ∇S and j was used for d**r**). The indices must be different since each component of ∇S needs to be dotted with each component of d**r**. In other words, taking a specific component i of ∇S, we need to dot it with each of the three components j = 1, 2, 3 of d**r**. If instead i had been used as the index for both ∇S and d**r**, then we would be implying that the component i of ∇S needs to only be dotted with the corresponding component i of d**r**. Fortuitously, in the case of equation (27), this would not produce an error; however, in general incorrect use of indices will produce incorrect results. After the third '=' sign the fact that $\boldsymbol{\delta}_i \bullet \boldsymbol{\delta}_j = \delta_{ij}$ (Equation (16)) was used. Performing a mental sum over the j index and making use of the definition of the Kronecker delta, only terms for which j = i survive (the rest are zero). After the fourth '=' sign, only these nonzero terms are retained (we could also have summed over i instead of j, with the same end result except that j would be the final index instead of i). You can also do the manipulations in equation (27) just as effectively by writing out the ∇S and d**r** vectors in full, but you will have to write much more. Note that the final expression for dS is identical with that obtained using the general equation (22) for the differential of a scalar.

The Divergence: Let $\mathbf{A}(x_1,x_2,x_3) = A_1(x_1,x_2,x_3)\boldsymbol{\delta}_1 + A_2(x_1,x_2,x_3)\boldsymbol{\delta}_2 + A_3(x_1,x_2,x_3)\boldsymbol{\delta}_3$ be a differentiable vector field. The divergence of **A** is written $\nabla\bullet\mathbf{A}$ and is defined by

$$\nabla\bullet\mathbf{A} = \frac{\partial}{\partial x_i}\boldsymbol{\delta}_i \bullet A_j\boldsymbol{\delta}_j = \frac{\partial A_j}{\partial x_i}\delta_{ij} = \frac{\partial A_i}{\partial x_i} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \quad \text{(CCS)} \qquad (28)$$

The divergence is a scalar quantity. In deriving the final result in equation (28), similar manipulations were used as for equation (27) (see above). $\nabla\bullet\mathbf{A} \neq \mathbf{A}\bullet\nabla$ since in $\mathbf{A}\bullet\nabla$ the Del operator does not operate on the vector **A** (instead, $\mathbf{A}\bullet\nabla = A_i\frac{\partial}{\partial x_i}$). The divergence of a vector field is often encountered in transport phenomena (i.e. if $\nabla\bullet\mathbf{v} = 0$ the fluid under study is incompressible, where **v** is the fluid velocity).

The Curl: Let $\mathbf{A}(x_1,x_2,x_3) = A_1(x_1,x_2,x_3)\boldsymbol{\delta}_1 + A_2(x_1,x_2,x_3)\boldsymbol{\delta}_2 + A_3(x_1,x_2,x_3)\boldsymbol{\delta}_3$ be a differentiable vector field. The curl of $\mathbf{A}$ is written $\nabla \times \mathbf{A}$ and is given by the cross product of the Del operator with the vector $\mathbf{A}$

$$\nabla \times \mathbf{A} = \left(\frac{\partial}{\partial x_1}\boldsymbol{\delta}_1 + \frac{\partial}{\partial x_2}\boldsymbol{\delta}_2 + \frac{\partial}{\partial x_3}\boldsymbol{\delta}_3\right) \times (A_1\boldsymbol{\delta}_1 + A_2\boldsymbol{\delta}_2 + A_3\boldsymbol{\delta}_3)$$

$$= \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right)\boldsymbol{\delta}_1 + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right)\boldsymbol{\delta}_2 + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\boldsymbol{\delta}_3 \qquad \text{(CCS)} \qquad (29)$$

Equation (29) can be verified by a procedure analogous to that used in obtaining equation (13c) for the vector cross product. The curl of a vector field produces another vector field. The curl is not commutative so that $\nabla \times \mathbf{A} \neq \mathbf{A} \times \nabla$. Later we will encounter the curl of the fluid velocity, $\nabla \times \mathbf{v}$. When $\nabla \times \mathbf{v} = 0$ the fluid flow is called "irrotational" and simplified approaches can be used to study it. $\nabla \times \mathbf{v}$ is often called the "vorticity."

Formulas Involving ∇: There are many identities involving the ∇ operator. Some of these are listed below.

If S and R are arbitrary but differentiable scalar fields, and $\mathbf{A}$ and $\mathbf{B}$ are arbitrary but differentiable vector fields, then

$$\nabla(S + R) = \nabla S + \nabla R \qquad (30a)$$
$$\nabla \bullet (\mathbf{A} + \mathbf{B}) = \nabla \bullet \mathbf{A} + \nabla \bullet \mathbf{B} \qquad (30b)$$
$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \qquad (30c)$$
$$\nabla \bullet (S\mathbf{A}) = \nabla S \bullet \mathbf{A} + S(\nabla \bullet \mathbf{A}) \qquad (30d)$$
$$\nabla \times (S\mathbf{A}) = \nabla S \times \mathbf{A} + S(\nabla \times \mathbf{A}) \qquad (30e)$$
$$\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\nabla \times \mathbf{A}) - \mathbf{A} \bullet (\nabla \times \mathbf{B}) \qquad (30f)$$
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \bullet \nabla)\mathbf{A} - \mathbf{B}(\nabla \bullet \mathbf{A}) - (\mathbf{A} \bullet \nabla)\mathbf{B} + \mathbf{A}(\nabla \bullet \mathbf{B}) \qquad (30g)$$
$$\nabla (\mathbf{A} \bullet \mathbf{B}) = (\mathbf{B} \bullet \nabla)\mathbf{A} + (\mathbf{A} \bullet \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B}) \qquad (30h)$$
$$\nabla \times (\nabla S) = 0 \qquad (30i)$$
$$\nabla \bullet (\nabla \times \mathbf{A}) = 0 \qquad (30j)$$

For instance, from (30i) we see that the curl of a gradient of an arbitrary scalar field is always zero, and from (30j) that the divergence of a curl of an arbitrary vector field is always zero. The above expressions are general in that they are *not* restricted to the CCS.

The Laplacian Operator: The Laplacian operator is written ∇^2. In cartesian coordinates it is defined by

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \qquad \text{(CCS)} \qquad (31)$$

The Laplacian of a scalar field $S(x_1,x_2,x_3)$ is

$$\nabla^2 S = \frac{\partial^2 S}{\partial x_1^2} + \frac{\partial^2 S}{\partial x_2^2} + \frac{\partial^2 S}{\partial x_3^2} \qquad \text{(CCS)} \qquad (32)$$

The Laplacian of a vector field $\mathbf{A}(x_1,x_2,x_3)$ is (note the implied summation over i)

$$\nabla^2 \mathbf{A} = \left(\frac{\partial^2 A_i}{\partial x_1^2} + \frac{\partial^2 A_i}{\partial x_2^2} + \frac{\partial^2 A_i}{\partial x_3^2} \right) \boldsymbol{\delta}_i \qquad \text{(CCS)} \qquad (33)$$

Again, we need to keep in mind that definitions and operations with the ∇ and ∇^2 as discussed thus far are restricted to the CCS. While extension to other coordinate systems is not difficult, we need to first thoroughly understand these operations in the CCS system.

Vector Integration:

Conservative Vector Fields and Line Integrals. Recall that the definite integral of a scalar $R(q)$, where R is the derivative of a scalar $S(q)$ with respect to q (ie. $R(q) = \mathrm{d}S(q)/\mathrm{d}q$), is

$$\int_{q_L}^{q_H} R(q)\mathrm{d}q = \int_{q_L}^{q_H} \frac{\mathrm{d}S}{\mathrm{d}q}\mathrm{d}q = \int_{S(q_L)}^{S(q_H)} \mathrm{d}S = S(q_H) - S(q_L) \qquad (34)$$

Also recall that graphically the integral has a simple interpretation: it is the area under the curve $R(q)$ in the region $q_L \leq q \leq q_H$ (Figure 16). In Figure 16 there are also drawn three rectangular elements: the sum of the areas of the rectangles can be used to approximate the value of the integral. In particular, the area under the curve can be approximated as the sum

$$\text{Area} = \sum_{i=1}^{N} \Delta S_i \ ,$$

$$\Delta S_i = R(q_i)\, \Delta q_i$$

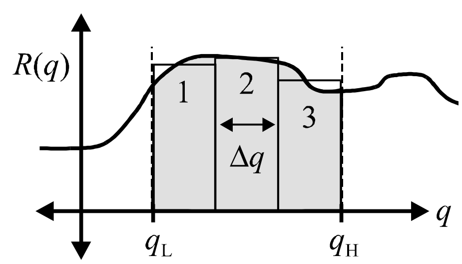


Fig. 16

Here, q_i is the value of q at the midpoint of rectangle i, and Δq_i is the width of rectangle i (in Fig. 16, i ranges from 1 to 3 so that $N = 3$).

In the limit $N \to \infty$ the shaded area gets discretized into an infinite number of rectangles, each possessing a differential (i.e. infinitesimal) width $\mathrm{d}q_i$. Also, the magnitude of each rectangle area ΔS_i approaches a differential value $\mathrm{d}S_i = R(q_i)\, \mathrm{d}q_i$. In this limit, the sum becomes *exactly* equal to the integral,

$$\lim_{N\to\infty}\sum_{i=1}^{N}\Delta S_i = \int_{S(q_L)}^{S(q_H)} dS = S(q_H) - S(q_L) \qquad (35)$$

The main point to recognize is that we can think of the integral in (34) as the sum of an infinite number of infinitesimally small increments dS that, when added together, sum up to the total difference given by $S(q_H) - S(q_L)$.

Now consider the gradient vector field

$\mathbf{A}(x_1,x_2,x_3) = \nabla S(x_1,x_2,x_3)$ ($S(x_1,x_2,x_3)$ is a scalar field that depends on position $\mathbf{r}$)

Then $\mathbf{A} \bullet d\mathbf{r} = \nabla S \bullet d\mathbf{r} = dS$ (see equation (27))

In words: if an observer walked from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$, he/she would observe a change in S from $S(\mathbf{r})$ to $S(\mathbf{r}) + dS$, where $dS = \nabla S \bullet d\mathbf{r}$. Do not let the notation $\mathbf{r}$, d$\mathbf{r}$ etc. confuse you; $\mathbf{r}$ is simply position as specified by the vector $\mathbf{r} = x_1\boldsymbol{\delta}_1 + x_2\boldsymbol{\delta}_2 + x_3\boldsymbol{\delta}_3$ and d$\mathbf{r}$ is an infinitesimal change in position specified by the (differential) vector $d\mathbf{r} = dx_1\boldsymbol{\delta}_1 + dx_2\boldsymbol{\delta}_2 + dx_3\boldsymbol{\delta}_3$.

The integral of $\mathbf{A} \bullet d\mathbf{r} = A_i dx_i$ along a path that starts at an initial position $\mathbf{r}_L$ to a final position $\mathbf{r}_H$ is called a "line integral" and is written

$$\int_{\mathbf{r}_L}^{\mathbf{r}_H} \mathbf{A} \cdot d\mathbf{r} \qquad (36)$$

When $\mathbf{A} = \nabla S$, equation (36) can be manipulated to yield

$$\int_{\mathbf{r}_L}^{\mathbf{r}_H} \mathbf{A} \cdot d\mathbf{r} = \int_{x_{1L},x_{2L},x_{3L}}^{x_{1H},x_{2H},x_{3H}} (A_1 dx_1 + A_2 dx_2 + A_3 dx_3)$$
$$= \int_{S(\mathbf{r}_L)}^{S(\mathbf{r}_H)} dS = S(\mathbf{r}_H) - S(\mathbf{r}_L) = S(x_{1_H}, x_{2_H}, x_{3_H}) - S(x_{1_L}, x_{2_L}, x_{3_L}) \qquad (37)$$

IMPORTANT: When the vector field $\mathbf{A}$ can be expressed as the gradient of a scalar field, $\mathbf{A} = \nabla S$ (as in equation (37)), $\mathbf{A}$ is said to be "conservative" and S is termed the "scalar potential" of $\mathbf{A}$. The line integral of a conservative vector field only depends on the starting and ending points $\mathbf{r}_L$ and $\mathbf{r}_H$, and is *independent* of the particular path along which the integral is performed (Figure 17 left). This independence of path is the reason why we did not have to specify a particular path in equation (37) along which the integral is to be performed. Clearly, the line integral of a conservative vector field $\mathbf{A} = \nabla S$ around a closed path, i.e. when $\mathbf{r}_L = \mathbf{r}_H$, must be zero (Figure 17 right):

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = 0 \qquad (38)$$

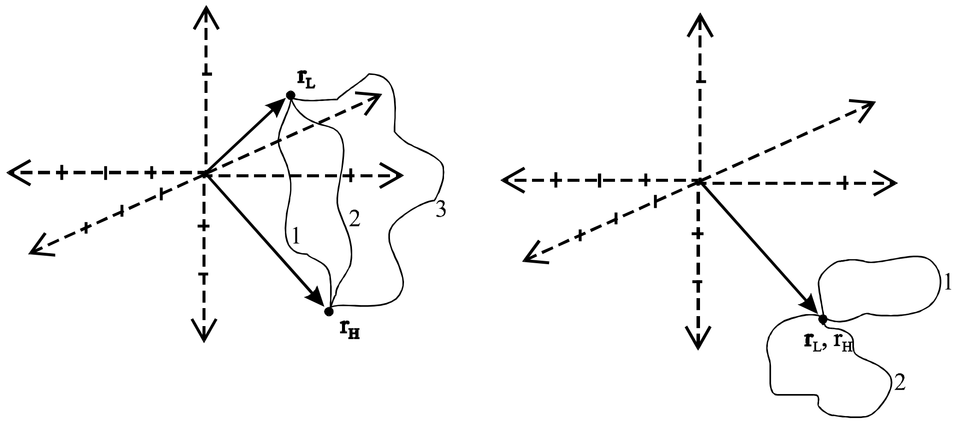


Fig. 17. The line integral of a conservative vector field from $\mathbf{r}_L$ to $\mathbf{r}_H$ will be the same regardless of what path is followed (three possible paths are sketched in the left figure). The line integral of a conservative vector field around any closed path will be zero (two examples of a closed path are sketched in the right figure).

The circle on the integration symbol, $\oint_C$, indicates that the integration is around a closed curve C. Note that $\mathbf{A} = \nabla S$ must be defined for every point on the curve C as well as in the region enclosed by C.

Given a vector field **A**, is there a test to tell whether **A** is conservative? One way to show that **A** is conservative is to show that it can be derived from a scalar field S. Since the component $A_i = \frac{\partial S}{\partial x_i}$, we could integrate A_1 with respect to x_1, A_2 with respect to x_2, and A_3 with respect to x_3 to find S, remembering that the "constants of integration" in each of these cases are functions of the variables with respect to which we are *not* integrating. Alternately, it can be shown (see also identity (30i)) that

$$\mathbf{A} \text{ is conservative } \textit{if and only if } \nabla \times \mathbf{A} = 0$$

The "curl test" is often the easiest way to find out whether **A** is a conservative vector field. For example, the curl test can be used in to determine whether a fluid flow is "irrotational", i.e. whether the condition $\nabla \times \mathbf{v} = 0$ holds ($\mathbf{v}$ is the velocity field of the flow). If it does, then the velocity $\mathbf{v}$ is a conservative vector field and therefore can be derived from a scalar potential Φ, where $\mathbf{v} = \nabla\Phi$. Not surprisingly, Φ is called the "velocity potential."

We can also evaluate line integrals of nonconservative vector fields. Nonconservative vector fields cannot be expressed as a gradient of a scalar field. The value of nonconservative vector line integrals will depend on the path taken between the initial and final positions, and the particular path taken must be explicitly specified when performing the integral. Typically, the path would be parametrized by a variable q (q could be time, for example) so that $\mathbf{r} = \mathbf{r}(q) = x_1(q)\boldsymbol{\delta}_1 + x_2(q)\boldsymbol{\delta}_2 +$

$x_3(q)\boldsymbol{\delta}_3$ and $\mathrm{d}\mathbf{r} = \left(\frac{\mathrm{d}x_1}{\mathrm{d}q}\right)\mathrm{d}q\,\boldsymbol{\delta}_1 + \left(\frac{\mathrm{d}x_2}{\mathrm{d}q}\right)\mathrm{d}q\,\boldsymbol{\delta}_2 + \left(\frac{\mathrm{d}x_3}{\mathrm{d}q}\right)\mathrm{d}q\,\boldsymbol{\delta}_3$. The line integral $\int_{\mathbf{r}_L}^{\mathbf{r}_H} \mathbf{A}\cdot \mathrm{d}\mathbf{r}$ is then written in the form $\int_{q_L}^{q_H}\left(A_i(q)\,\frac{\mathrm{d}\,x_i(q)}{\mathrm{d}q}\right)\mathrm{d}q$ and is performed from q_L to q_H.

Surface Integrals. Let $B(x_1,x_2,x_3)$ be a surface (ex. $x_1^2 + x_2^2 + x_3^2 = \varpi^2$ specifies a spherical surface for which the sphere radius equals ϖ). If the surface is closed, then we will call the outer side of the surface the positive side (Figure 18). If the surface is not closed, then we arbitrarily choose one of the sides to be "positive". Let **n** be a unit vector that is perpendicular to the surface, and that points in the "positive" direction (Figure 18). **n** is called the "positive unit normal."

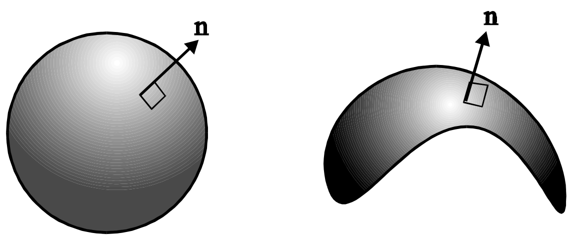


Fig. 18. For a closed surface, such as the sphere on the left, the positive unit normal **n** points outward as drawn. For the open surface on the right, either the top or the bottom side could have been chosen as the positive side. In the drawing, the top side was taken to be positive.

A surface B can be subdivided into arbitrarily small differential surface elements dB. The sum of all the dB elements adds up to the total surface area; i.e. $\iint_B \mathrm{d}B = \text{surface area}$, where the integral is taken over the entire surface B. When deriving some of the basic rules of transport phenomena, we will be particularly interested in surface integrals of the type

$$\iint_B \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}B \qquad (39)$$

This type of integral is often called the "flux" of **A** over the surface B. The reason for this terminology is relatively straightforward. Let's say **A** is the velocity **v** of a fluid, **A** = **v**. Then **v**(**r**) • **n** is the component (projection) of **v** in the direction of **n** at point **r** (Figure 19). In other words, **v**(**r**) • **n** is the speed of fluid particles perpendicular to the surface at point **r** (remember that **n** has a magnitude of unity, and points perpendicular to the surface). For instance, if **v** and **n** are orthogonal to each other at **r**, then **v**(**r**)• **n** = 0 and no fluid particles are passing across the surface at **r** (Figure 20). If **v** and **n** are parallel to each other at **r**, then at point **r** the fluid particles pass across the surface with a speed v.

IMPORTANT: when we multiply **v**(**r**) • **n** by the (infinitesimal) surface area dB, we get the rate of fluid flow through the area element dB in units of volume/time (Figure 19). In other words, **v**(**r**) • **n** dB (you should check that the units are length^3/sec) is the rate at which fluid volume sweeps across the area dB

at point **r**. If all such (infinitesimal) volume flows are summed over the entire surface *B*, what is equivalent to performing the integral $\iint_B \mathbf{v}\cdot\mathbf{n}dB$, the result is the net volumetric flow rate of fluid across the surface *B*. In these types of integrals, **n** d*B* is sometimes simply written as d**B**.

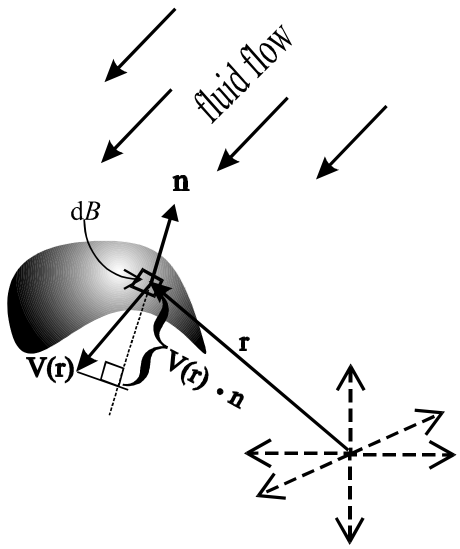


Fig. 19.

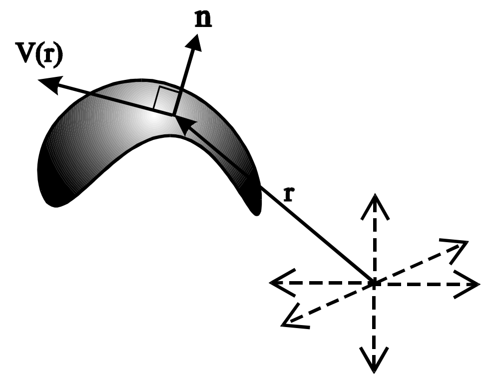


Fig. 20.

Given a differentiable function $f(x_1, x_2, x_3)$ = constant that specifies a surface *B* (ex. a spherical surface would be specified by $f = x_1^2 + x_2^2 + x_3^2 = \varpi^2$), the unit normal **n** can be constructed using the gradient ∇f of f (an example is given below). Note that ∇f points in the direction of **n** since ∇f is perpendicular to the surface. The reason why ∇f is always perpendicular to the surface can be explained as follows. Since f has a constant value on the surface, its rate of change along any direction that is tangential (parallel) to the surface must be zero. Therefore, ∇f, which is the rate of change of f with position, cannot have a component that is tangential (parallel) to the surface. In other words, ∇f must be perpendicular to the surface. To derive **n** from ∇f, ∇f still needs to be divided by its magnitude in order to produce a *unit* vector. In summary, a unit normal vector **n** to a surface can be calculated using

$$\mathbf{n} = \nabla f \,/\, (\nabla f \bullet \nabla f)^{1/2} \qquad (40)$$

As an example of a surface integral, let's take a velocity field $\mathbf{v} = 2\boldsymbol{\delta}_1$ m/s and a surface *B* specified by $x_1 + x_2 = \varpi$ meters, where the constant ϖ is greater than 0. We also confine *B* to the region of space for which $0 \le x_3 \le 1$ m and $x_1, x_2 \ge 0$. The resultant surface and velocity are sketched in Figure 21.

To calculate the unit normal **n** to the surface *B*, we use equation (40)

$$\mathbf{n} = \nabla(x_1 + x_2) \,/\, (\nabla(x_1 + x_2) \bullet \nabla(x_1 + x_2))^{1/2}$$
$$= (\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2) \,/\, ((\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2) \bullet (\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2))^{1/2} = (\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2) \,/\, 2^{1/2}$$

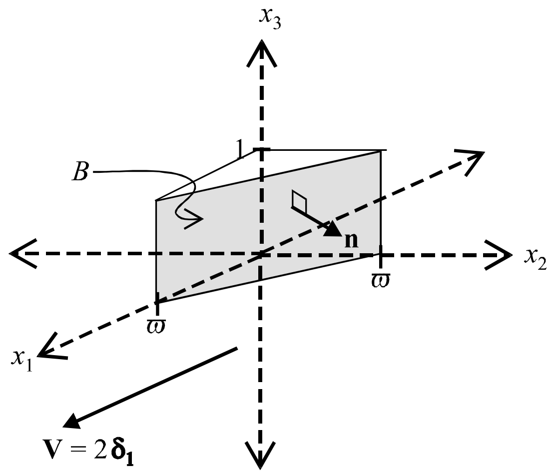


Fig. 21.

Therefore, $\mathbf{n}$, the unit normal to B, is $\mathbf{n} = (\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2) / 2^{1/2}$ (you can check that the magnitude of $\mathbf{n}$ is unity). Using equation (8c), it is straightforward to calculate that $\mathbf{v} \bullet \mathbf{n} = 2/2^{1/2} = 2^{1/2}$ m/s. The volume flux of fluid through the surface B then becomes

$$\iint_B \mathbf{v}\cdot\mathbf{n}dB = 2^{1/2} \iint_B \mathrm{d}B = 2^{1/2}\,(2^{1/2}\,\varpi) = 2\,\varpi \text{ m}^3/\text{s}$$

We used the fact that the area of the surface, given by $\iint_B \mathrm{d}B$, is equal to $2^{1/2}\,\varpi$ m^2. In this case the area can be calculated from simple geometrical considerations - see Figure 21.

Volume Integrals. An example of a vector volume integral is (note the summation convention)

$$\iiint_V \mathbf{A}\mathrm{d}V = \boldsymbol{\delta}_\mathrm{i} \iiint_V A_\mathrm{i}\mathrm{d}V \qquad (41)$$

V is the volume of space over which the integration is carried out, and $\mathbf{A}$ is an arbitrary vector. For example, let's say a force *per unit volume* $\mathbf{F} = 2x_1x_3\,\boldsymbol{\delta}_1 - x_1\,\boldsymbol{\delta}_2 + x_2^2\,\boldsymbol{\delta}_3$ N/m^3 acts on material inside a volume V, where V is enclosed by the surfaces $x_1 = 0$ m, $x_2 = 0$ m, $x_2 = 6$ m, $x_3 = x_1^2$ m, and $x_3 = 4$ m (see Figure 22).

To find the total force $\mathbf{F}_{\mathrm{tot}}$ on the material in V, we need to perform a volume integral,

$$\mathbf{F}_{\mathrm{tot}} = \int_{x_1=0}^{2} \int_{x_2=0}^{6} \int_{x_3=x_1^2}^{4} \mathbf{F}\mathrm{d}V = \int_{x_1=0}^{2} \int_{x_2=0}^{6} \int_{x_3=x_1^2}^{4} (2x_1x_3\boldsymbol{\delta}_1 - x_1\boldsymbol{\delta}_2 + x_2^2\boldsymbol{\delta}_3)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1$$

$$= \int_{x_1=0}^{2} \int_{x_2=0}^{6} [(16x_1 - x_1^5)\boldsymbol{\delta}_1 - (4x_1 - x_1^3)\boldsymbol{\delta}_2 + (4x_2^2 - x_1^2x_2^2)\boldsymbol{\delta}_3]\mathrm{d}x_2\mathrm{d}x_1$$

$$= \int_{x_1=0}^{2} [(96x_1 - 6x_1^5)\boldsymbol{\delta}_1 - (24x_1 - 6x_1^3)\boldsymbol{\delta}_2 + (288 - 72x_1^2)\boldsymbol{\delta}_3]dx_1$$

$\mathbf{F}_{tot} = (192 - 64)\ \boldsymbol{\delta}_1 - (48 - 24)\ \boldsymbol{\delta}_2 + (576 - 192)\ \boldsymbol{\delta}_3 = 128\ \boldsymbol{\delta}_1 - 24\ \boldsymbol{\delta}_2 + 384\ \boldsymbol{\delta}_3$

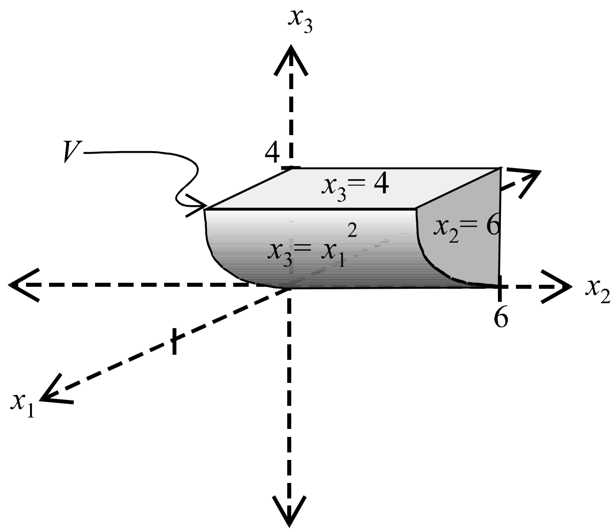


Fig. 22.

The Divergence Theorem:
The Divergence Theorem states that if V is a volume bounded by a closed surface B and $\mathbf{A}(x_1,x_2,x_3)$ is a vector field with continuous derivatives, then

$$\iint_B \mathbf{A}\cdot\mathbf{n}\ dB = \iiint_V \nabla\cdot\mathbf{A}\ dV \tag{42}$$

In equation (42), $\mathbf{n}$ is the positive (outward) unit normal vector to B. Equation (42) states that the net flux of $\mathbf{A}$ through a closed surface B is equivalent to the integral of the divergence of $\mathbf{A}$ over the volume enclosed by that surface. The Divergence Theorem will be used extensively in deriving so-called *differential balances* on mass, energy, and momentum.

Stoke's Theorem:
Stoke's Theorem states that if B is an open surface bounded by a closed, non-intersecting curve C (Figure 23), and if the vector field $\mathbf{A}$ has continuous derivatives, then

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_B (\nabla\times\mathbf{A})\cdot\mathbf{n}\ dB \tag{43}$$

In equation (43), the integral around C is performed in the "positive" direction. If you point the thumb of your *right* hand in the direction of the normal $\mathbf{n}$, then the other fingers of your right hand will curl in the positive direction. As $\nabla\times\mathbf{A}$ is a vector, the method of evaluating the surface integral on the right side of equation (43) is the same as for any other flux integral. First, you would calculate the curl of $\mathbf{A}$, then dot the resultant vector into **n**, and finally integrate the result over the surface *B*. Stokes' Theorem will be useful later in the course when we study potential flow.

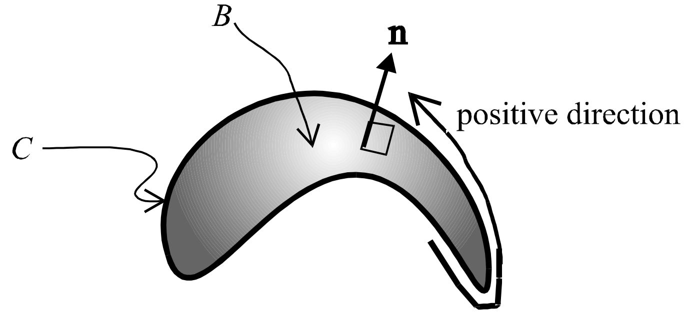


Fig. 23.