Potential Flow

Part I. Theoretical Background.

Potential Flow. Potential flow is irrotational flow. Irrotational flows are often characterized by negligible viscosity effects. Viscous effects become negligible, for example, for flows at high Reynolds number that are dominated by convective transport of momentum. Thus potential flow is often useful for analyzing external flows over solid surfaces or objects at high Re, provided the flows still remain laminar. Moreover, when the flow over a surface is rapid (high Re), the viscous boundary layer region (within which potential flow would be a *bad* assumption) that forms next to the solid body is very thin. Then, to a very good approximation, the presence of the boundary layer can be neglected when analyzing the potential flow region. That is, the potential flow can be assumed to follow the contours of the solid surface, as if the boundary layer was not present.

When the thickness of the boundary layer is small compared to the dimensions of the object over which the potential flow is occurring, we can proceed as follows to analyze the total (potential flow + boundary layer flow) problem:

i). First, determine the velocities and pressure distribution in the potential flow region, assuming that the potential flow extends all the way to any solid surfaces present (ie. neglecting the presence of the boundary layer).

ii). Solve the flow inside the boundary layer using the pressure distribution obtained from the potential flow solution (i) as input. In other words, the potential flow imposes the pressure on the boundary layer (see the earlier discussion of boundary layers). At the edge of the boundary layer, the velocities are matched with those obtained from the potential flow solution (i) through the use of appropriate boundary conditions.

The Velocity Potential. In potential flow the velocity field **v** is irrotational. This means that

$$\text{vorticity} = \boldsymbol{\omega} = \nabla \times \mathbf{v} = 0 \qquad (1)$$

When $\nabla \times \mathbf{v} = 0$ the rate of rotation of an infinitesimal element of fluid is zero. From vector calculus (see the first handout on vector analysis) we know that if a velocity field is irrotational then it can be expressed as the gradient of a "scalar potential" Φ:

$$\mathbf{v} = -\nabla \Phi \qquad \text{(irrotational flow)} \qquad (2)$$

In equation (2), Φ is the "velocity potential." Using the definition of the ∇ operator, in cartesian coordinates:

$$v_1 = -\frac{\partial \Phi}{\partial x_1} \qquad v_2 = -\frac{\partial \Phi}{\partial x_2} \qquad v_3 = -\frac{\partial \Phi}{\partial x_3} \qquad (2a)$$

In cylindrical coordinates:

$$v_r = -\frac{\partial \Phi}{\partial r} \qquad v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} \qquad v_z = -\frac{\partial \Phi}{\partial z} \qquad (2b)$$

In spherical coordinates:

$$v_r = -\frac{\partial \Phi}{\partial r} \qquad v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} \qquad v_\phi = -\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi} \qquad (2c)$$

If, in addition, the flow is incompressible, then

$\nabla\bullet\mathbf{v} = 0$ ➔ $\nabla\bullet\nabla\Phi = 0$ what is equivalent to

$$\nabla^2\Phi = 0 \qquad \text{(irrotational, incompressible flow)} \qquad (3)$$

The Stream Function. The stream function ψ can be defined for any *two-dimensional* flow, whether the flow is irrotational or not, compressible or incompressible. Two-dimensional means that at least one of the velocity components is zero (in other words, at most two of the velocity components are nonzero). Some flow types for which the stream function is useful, and the accompanying definitions of the stream function, are:

Flow in Cartesian coordinates, with $v_3 = 0$:

$$v_1 = -\frac{\partial \psi}{\partial x_2} \qquad v_2 = \frac{\partial \psi}{\partial x_1} \qquad (4a)$$

Flow in cylindrical coordinates with $v_Z = 0$:

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = \frac{\partial \psi}{\partial r} \qquad (4b)$$

Flow in cylindrical coordinates with $v_\theta = 0$:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z} \qquad v_Z = -\frac{1}{r}\frac{\partial \psi}{\partial r} \qquad (4c)$$

Flow in spherical coordinates with $v_\phi = 0$:

$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} \qquad (4d)$$

The definitions (4a) through (4d) are joined by the requirement that the stream function automatically satisfy the equation of continuity for an incompressible fluid, $\nabla\bullet\mathbf{v} = 0$. The divergence $\nabla\bullet\mathbf{v}$ will automatically equal zero because of the equivalence of the mixed second derivatives of the stream function. For instance, for incompressible flow in Cartesian coordinates with $v_3 = 0$

$$\nabla\bullet\mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = -\frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 \psi}{\partial x_2 \partial x_1} = 0 \qquad (5)$$

After the second equal sign in equation (5), the definitions (4a) of the stream function were used. The equality to zero comes about because the mixed second derivatives of ψ are equal.

The stream function is constant along a streamline. The change $d\psi$ in the stream function due to an infinitesimal displacement d**s** along a streamline can be written as

$$d\psi = \nabla\psi \bullet d\mathbf{s} = \nabla\psi \bullet \mathbf{v}dt = \frac{\partial\psi}{\partial x_1}v_1 dt + \frac{\partial\psi}{\partial x_2}v_2 dt \qquad (6)$$

In equation (6), the displacement d**s** along a streamline was expressed as **v** dt (t is time). Because the streamline is everywhere tangent to the velocity field **v**, d**s** and **v** must point in the same direction. Both vectors are infinitesimally small, and it is then justified to take d**s** as equal to **v**dt. Inserting equations (4a) into (6),

$$d\psi = (v_2 v_1 - v_1 v_2)\, dt = 0 \qquad (7)$$

Since the change $d\psi$ in the stream function arising from a displacement along a streamline is zero, the stream function must be constant along a streamline.

The change in the stream function between a pair of streamlines equals the volumetric flowrate between those two streamlines. Consider the flow in Figure 1. *We will define flowrate from right to left as positive*. We wish to calculate the volumetric flowrate Q' (per unit width into the page) that occurs between streamlines 1 and 2. To calculate Q', we will integrate the rate at which fluid flows across the curve that connects points 1 and 2 in the figure. The flowrate Q' is given by

$$Q' = -\int_1^2 \mathbf{v}\cdot\mathbf{n}\, dl = -\int_1^2 (v_1 n_1 + v_2 n_2)\, dl \qquad (8)$$

where dl is an infinitesimal displacement along the curve and **n** is a unit normal to the curve pointing opposite to the fluid flow (see Figure 1). n_1 is the component of **n** along the x_1 direction, v_1 is the velocity component along the x_1 direction, etc.

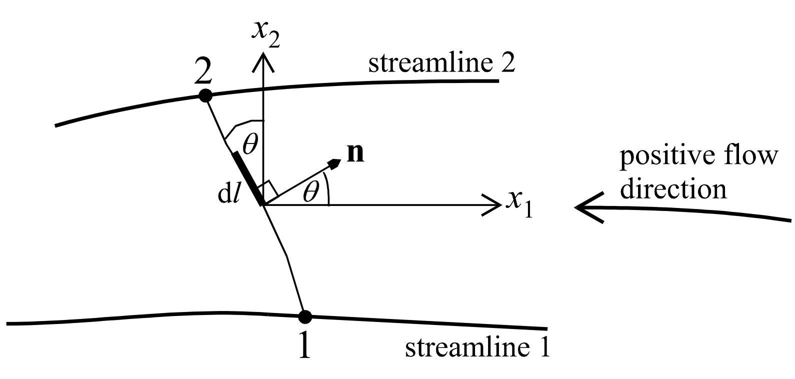

Figure 1

From figure 1,

$$n_1 dl = n\cos\theta\, dl = n\, dx_2 = dx_2 \qquad (9)$$

In equation (9), we first set $n_1 = n \cos\theta$, where n is the magnitude of **n** and θ is as indicated in Figure 1. Second, we observed that $\cos\theta\, dl$ equals the displacement dx_2 that occurs in the x_2 direction when position along the curve changes by dl; thus, we substituted $\cos\theta\, dl = dx_2$. Finally, the magnitude n of **n** was set equal to 1 since **n** is a unit vector. Similarly, it can be shown that

$$n_2 dl = n \sin\theta\, dl = -\, n\, dx_1 = -\, dx_1 \tag{10}$$

In equation (10), we used $dx_1 = -\sin\theta\, dl$. The minus sign must be included because the displacement dx_1 that occurs in the x_1 direction when position along the curve shifts by dl is negative, although $\sin\theta\, dl$ evaluates to a positive number. Inserting equations (9) and (10) into (8),

$$Q' = \int_1^2 (-v_1 dx_2 + v_2 dx_1) = \int_1^2 \left(\frac{\partial\psi}{\partial x_2} dx_2 + \frac{\partial\psi}{\partial x_1} dx_1\right)$$

$$Q' = \int_1^2 d\psi = \psi_2 - \psi_1 \tag{11}$$

In arriving at equation (11), we used the fact that $d\psi = (\partial\psi/\partial x_1)dx_1 + (\partial\psi/\partial x_2)dx_2$. Equation (11) states that the volumetric flowrate Q', per unit width into the page, between streamlines 1 and 2 equals the difference in the stream function. Note that Q' will be positive for flow from right to left, and negative for flow from left to right.

Lines of constant ψ (streamlines) are perpendicular to lines of constant Φ (velocity potential lines). On a streamline, ψ is constant, so that

$$d\psi = \nabla\psi \bullet d\mathbf{s} = \frac{\partial\psi}{\partial x_1} dx_1 + \frac{\partial\psi}{\partial x_2} dx_2 = 0 \tag{12}$$

The stream function does not change (ie. $d\psi = 0$) because the displacement $d\mathbf{s} = \boldsymbol{\delta}_1 dx_1 + \boldsymbol{\delta}_2 dx_2$ is taken along a streamline. Making use of equations (4a), equation (12) becomes

$v_2 dx_1 - v_1 dx_2 = 0$ or, after rearrangement,

$$dx_2 / dx_1 = v_2 / v_1 \tag{13}$$

Equation (13) gives the slope of the streamline, as depicted in Figure 2.

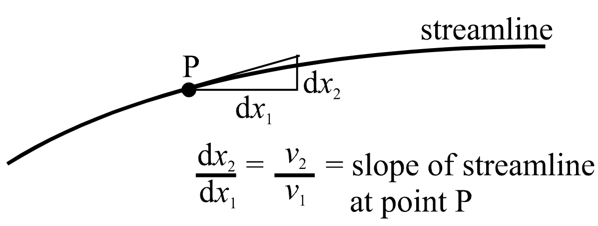

Figure 2.

On a line of constant velocity potential, Φ is constant. Therefore,

$$\mathrm{d}\Phi = \nabla\Phi \bullet \mathrm{d}\mathbf{s} = \frac{\partial \Phi}{\partial x_1}\mathrm{d}x_1 + \frac{\partial \Phi}{\partial x_2}\mathrm{d}x_2 = 0 \qquad (14)$$

In equation (14), the displacement d**s** occurs along a line of constant velocity potential. Inserting equations (2a) into equation (14),

$$- v_1\mathrm{d}x_1 - v_2\mathrm{d}x_2 = 0 \qquad \text{or, after rearrangement,}$$

$$\mathrm{d}x_2 / \mathrm{d}x_1 = - v_1 / v_2 \qquad (15)$$

Equation (15) gives the slope of a line of constant velocity potential. Since the slopes of a streamline (equation (13)) and of a line of constant velocity potential (equation (15)) are negative reciprocals of one another, streamlines and velocity potential lines must be mutually perpendicular.

For irrotational, incompressible, two-dimensional flows, the stream function and the velocity potential obey the same differential equation. From equation (3) for irrotational, incompressible flow,

$$\nabla^2\Phi = 0 \qquad \text{(irrotational, incompressible flow)} \qquad (3)$$

The condition of irrotationality means that

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \boldsymbol{\delta}_3 = \mathbf{0} \qquad (16)$$

In equation (16), for convenience the two-dimensional flow was assumed to take place in the x_1x_2 plane, so that $v_3 = 0$. Equation (16) requires that

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0 \qquad (16b)$$

Inserting the definitions of the velocities in terms of the stream function (equations (4a)) into equation (16b)

$$\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = 0 \qquad \text{which is equivalent to}$$

$$\nabla^2\psi = 0 \qquad \text{(2D, irrotational flow)} \qquad (17)$$

Equations (3) and (17) show that for a two-dimensional, irrotational, incompressible flow, the velocity potential and the stream function both obey "**Laplace's equation**":

$$\nabla^2\Phi = 0 \quad \text{and} \quad \nabla^2\psi = 0 \qquad (18)$$

Functions that obey Laplace's equation are called "**harmonic**" functions. Therefore, under these conditions both the stream function and the velocity potential are harmonic.

Important**: Since Φ and ψ obey the same differential equation (for 2D, irrotational, incompressible flow), a solution to one potential flow problem can be directly used to generate a solution to a second potential flow by interchanging Φ and ψ.** Specifically, if Φ_1 and ψ_1 represent potential flow 1, then the interchange $\Phi_2 = \psi_1$ and $\psi_2 = -\Phi_1$ will represent some other potential flow 2. The minus sign in $\psi_2 = -\Phi_1$ is needed to ensure consistency in the sign of the velocity components derived from Φ and ψ (see examples below).

Different potential flows can be added together to generate new potential flows (the Principle of Superposition). Laplace's equation (equation (18)) is linear. The linear property means that if the stream function and velocity potential are known for two different flows, say flows 1 and 2, then the sum of flows 1 and 2 will also be a solution to Laplace's equation. By "solution to Laplace's equation" we mean that, if

$\nabla^2\psi_1 = 0$ and $\nabla^2\psi_2 = 0$ then

$$\nabla^2(\psi_1 + \psi_2) = \nabla^2\psi_1 + \nabla^2\psi_2 = 0 \tag{19}$$

Here, ψ_1 is the stream function for flow 1 and ψ_2 is the stream function for flow 2. Equation (19) shows that the sum of the stream functions for flows 1 and 2 is also a solution to Laplace's equation. Same comments apply to the sum of the velocity potentials, $\Phi_1 + \Phi_2$, for flows 1 and 2.

Now, if ψ and Φ of a potential flow equal the sum of the stream functions and velocity potentials of two other flows 1 and 2, so that $\psi = \psi_1 + \psi_2$ and $\Phi = \Phi_1 + \Phi_2$, then the velocity field of that flow will equal the sum of the velocity fields of flows 1 and 2. This statement can be easily verified. For example, in terms of the stream functions (equivalently, we could have used the velocity potentials), and using a single prime to indicate velocities for flow 1 and double prime those for flow 2,

Flow 1 (stream function ψ_1): $$v_1' = -\frac{\partial\psi_1}{\partial x_2} \qquad v_2' = \frac{\partial\psi_1}{\partial x_1} \tag{20}$$

Flow 2 (stream function ψ_2): $$v_1'' = -\frac{\partial\psi_2}{\partial x_2} \qquad v_2'' = \frac{\partial\psi_2}{\partial x_1} \tag{21}$$

Flow 1 + Flow 2 (stream function $\psi = \psi_1 + \psi_2$):

$$v_1 = -\frac{\partial\psi}{\partial x_2} = -\frac{\partial(\psi_1+\psi_2)}{\partial x_2} = -\frac{\partial\psi_1}{\partial x_2} - \frac{\partial\psi_2}{\partial x_2} = v_1' + v_1'' \tag{22}$$

$$v_2 = \frac{\partial\psi}{\partial x_1} = \frac{\partial(\psi_1+\psi_2)}{\partial x_1} = \frac{\partial\psi_1}{\partial x_1} + \frac{\partial\psi_2}{\partial x_1} = v_2' + v_2'' \tag{23}$$

Equations (22) and (23) show that if the stream functions from flows 1 and 2 are added to create a new flow, then the velocity field of the new flow will equal the sum of the velocity fields of the two constituent flows. We thus see that the addition of stream functions or velocity potentials physically results in the **superposition** (addition) of the flows represented by those stream functions and velocity potentials. This **superposition principle** is a powerful way to generate solutions to potential flow problems. We will look at an example of superposition in the next section.

Part II. Applications.

Uniform Potential Flow. Figure 3 depicts the case of uniform flow in the x_1 direction. For this uniform flow,

$$v_1 = V_o \qquad v_2 = 0$$

Therefore,

$$-\frac{\partial \Phi}{\partial x_1} = V_o \Rightarrow \qquad \Phi = -V_o x_1 + f(x_2) \tag{24}$$

The derivative of Φ with respect to x_2 is zero, $\partial\Phi / \partial x_2 = - v_2 = 0$. Therefore, the function of integration $f(x_2)$ in equation (24) can at most be a constant. Since a constant of integration would not influence the velocities obtained from Φ, the value of the constant is arbitrary. For simplicity, we set the constant equal to 0 so that equation (24) becomes

$$\Phi = -V_o x_1 \tag{25a}$$

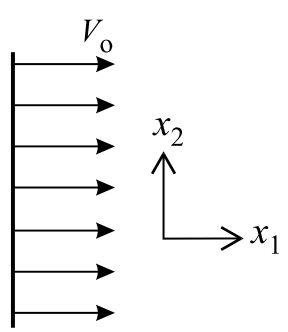

Figure 3.

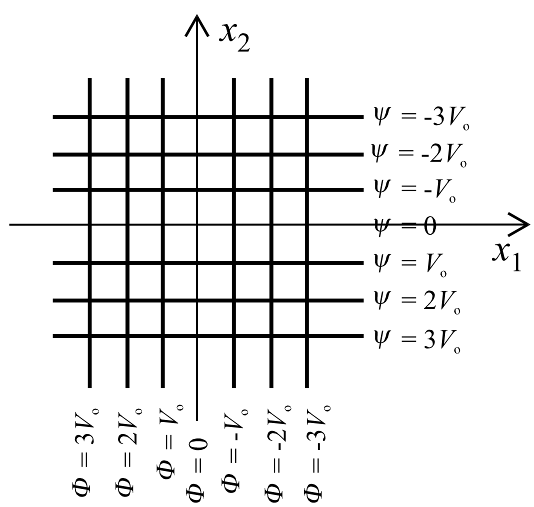

Figure 4.

By using equations (4a), it is straightforward to show that the stream function is

$$\psi = -V_o x_2 \tag{25b}$$

Figure 4 plots the lines of constant ψ and constant Φ. As discussed earlier, since lines of constant ψ are streamlines, they point along the flow direction. Also, as pointed out earlier, the streamlines and lines of constant Φ are mutually orthogonal. If we choose the streamlines for $x_2 = 1$ and $x_2 = 3$ in Figure 4, according to equation (11) the volumetric flowrate between the streamlines is,

$$Q' = \psi_3 - \psi_1 = -3V_o - (-V_o) = -2\, V_o \qquad (26)$$

Recall that Q' is the volumetric flowrate per unit width into the page. Because the flow occurs from *left to right*, according to our earlier convention Q' comes out to a negative number.

If ψ and Φ are interchanged, then

$$\psi = V_o x_1 \qquad (27a)$$

$$\Phi = -V_o x_2 \qquad (27b)$$

Note that, to preserve consistency of sign in the definition of the flow fields derived from Φ and ψ, a minus sign was inserted in (27a). Equations (27) again describe a uniform potential flow, but now the flow field is (you can easily verify it)

$$v_1 = 0 \qquad \text{and} \qquad v_2 = V_o$$

The interchange of ψ and Φ thus resulted in a uniform flow in the x_2 direction. In the context of Figure 4, the streamlines and the lines of constant velocity potential were interchanged.

Source and Sink Flows. In two dimensions, a source is a line (into the page) from which fluid flows outward, and a sink is a line at which fluid flows inward and is removed (Figure 5). For these flows,

$$v_\theta = 0 \qquad \text{and} \qquad v_r = Q'/(2\pi r) \qquad (28)$$

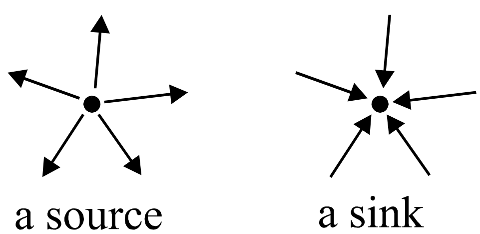

Figure 5.

In equation (28), Q' is the total volumetric flowrate outward from the source, per unit depth into the page. $Q' > 0$ for a source, $Q' < 0$ for a sink. Equation (28) for v_r ensures that the total radial volumetric flowrate is constant. In other words, at each radial distance r from the source or sink the volumetric flowrate per unit depth, given by $2\pi r\, v_r$, equals the constant Q'. This volumetric flowrate is the product of the area per unit depth into the page (ie. $2\pi rW$ would be the area of flow if the extent of the source into the page is W), times the radial flow velocity v_r.

Using equations (2b) and proceeding as for the case of uniform flow, the velocity potential is found to be

$$-\frac{\partial \Phi}{\partial r} = v_r = Q'/(2\pi r) \rightarrow \quad \Phi = -Q'/(2\pi)\ \ln r + f(\theta) \quad \rightarrow \quad \Phi = -Q'/(2\pi)\ \ln r \qquad (29)$$

In equation (29), the function of integration $f(\theta)$ was set to zero since $v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = 0$ implies that at most f can be an arbitrary constant. Similarly, the stream function can be found using equations (4b)

$$-\frac{1}{r}\frac{\partial \psi}{\partial \theta} = v_r = Q'/(2\pi r) \rightarrow \quad \psi = -Q'/(2\pi)\ \theta + f(r) \quad \rightarrow \quad \psi = -Q'/(2\pi)\ \theta$$
(30)

where similar arguments can be made to set $f(r) = 0$. From equation (29), lines of constant Φ are equivalent to lines of constant r, while from equation (30) we see that streamlines (lines of constant ψ) coincide with lines of constant θ. The velocity potential lines and streamlines are illustrated in Figure 6. Again, we see that the curves of Φ and ψ are orthogonal, as required.

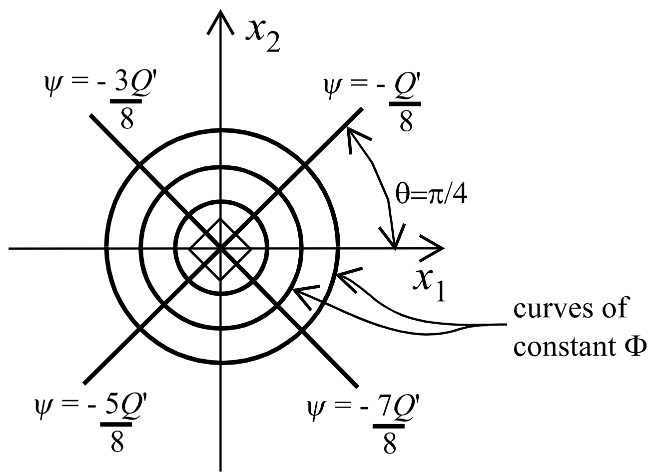

Figure 6.

Potential Vortex Flow. If the velocity potential and stream function for the source/sink flow are interchanged, we get a new flow for which

$$\psi = Q'/(2\pi)\ \ln r \qquad \text{and} \qquad \Phi = -Q'/(2\pi)\ \theta \qquad (31)$$

The velocities for this new flow can be derived from ψ and Φ by using equations (4b) or (2b).The result is

$$v_r = 0 \qquad \text{and} \qquad v_\theta = Q'/(2\pi r) \qquad (32)$$

Equations (32) describe so-called "**potential vortex**" flow, a useful model for phenomena such as tornadoes and whirlpools. In a potential vortex, fluid moves in concentric circles with a velocity v_θ that decreases as $1/r$, where r is the radial distance from the center of the vortex. The diagram of streamlines and velocity potential lines for a potential vortex looks exactly like Figure 6, except that the streamlines and velocity potential lines are interchanged. When referring to potential vortex flow, it is customary to use the symbol Γ instead of Q'. $\Gamma / 2\pi$ is known as the "strength of the vortex." Γ is positive for counterclockwise vortex flow, and negative for clockwise vortex flow. From equation (32) it is clear that the higher $\Gamma / 2\pi$ is, the greater the velocity v_θ. In terms of Γ, the potential vortex equations are

$$\psi = \Gamma/(2\pi)\ \ln r \qquad \Phi = -\ \Gamma/(2\pi)\ \theta \qquad v_\theta = \Gamma/(2\pi r) \tag{33}$$

Superposition of Potential Flows. As an example of the superposition of two potential flows, we will superpose (add) a uniform and a potential vortex flow. The functions ψ and Φ become,

$$\psi = -V_o x_2 + \Gamma/(2\pi)\ \ln r = -V_o x_2 + \Gamma/(4\pi)\ \ln(x_1^2 + x_2^2) \tag{34}$$

$$\Phi = -V_o x_1 - \Gamma/(2\pi)\ \theta = -V_o x_1 - \Gamma/(2\pi)\tan^{-1}(x_2 / x_1) \tag{35}$$

In equations (34) and (35), the first term on the right comes from the uniform flow and the second from the potential vortex flow. ψ and Φ have been expressed in cartesian coordinates, and so can be used to calculate the CCS velocity components v_1 and v_2 directly. For instance, equations (4a) can be applied to the stream function to yield

$$v_1 = V_o - \Gamma x_2 /\{2\pi\ (x_1^2 + x_2^2)\} \tag{36}$$

$$v_2 = \Gamma x_1 /\{2\pi\ (x_1^2 + x_2^2)\} \tag{37}$$

Setting $V_o = 1$ m/s and $\Gamma / 2\pi = 2$ m^2/s, the resultant flow is depicted in Figure 7. At each point an arrow shows the direction of the local velocity field. The length of the arrows is fixed, and so *does not* represent the magnitude of the local velocity. There is a "**stagnation point**" on the top side of the vortex at the position (0,2). At a stagnation point, all velocity components are zero.

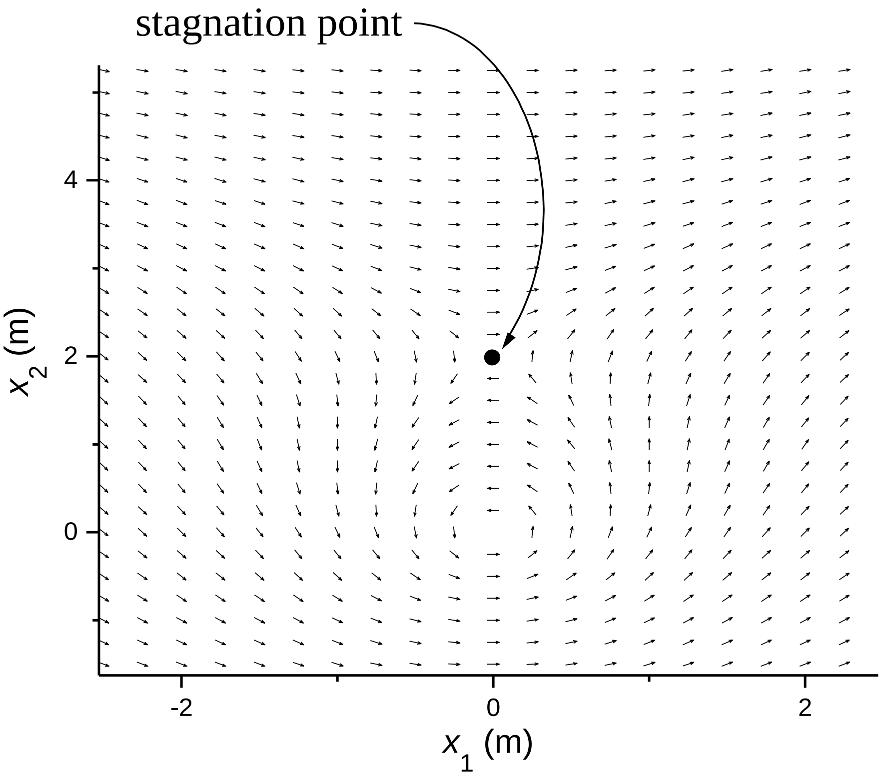

Figure 7.

The Method of Images. At a solid boundary, the magnitude of potential flow perpendicular to the boundary must be zero (i.e. no fluid flow can occur across the solid surface). The previous statement is approximate in that the potential flow only extends to the outer edge of a boundary layer, where a small, perpendicular velocity component is typically present (e.g. as could be obtained from the numerical Blasius solution for the laminar boundary layer over a flat surface). However, if the rate of boundary layer growth along the surface is very slow, as will be true if the flow over the surface is sufficiently rapid, then the perpendicular velocity component will be small and so will the error introduced by neglecting it. Since no flow occurs across a streamline, a streamline meets the same stipulation as a solid boundary. Indeed, **any streamline of a potential flow can be considered as a solid surface, with no resultant change to the flow pattern.** Since fluid flows along a streamline, thinking of a streamline as a solid surface implies flow (i.e. slip) along that surface. This implication does not pose a physical paradox however since in reality potential flow does not extend all the way to a surface (it only extends to the outer edge of the boundary layer) and therefore is *not* subject to a no-slip boundary condition.

The **method of images** is a technique by which streamlines representing solid surfaces can be generated in a systematic fashion. This approach takes advantage of the fact that there can be no flow across lines of symmetry separating identical flows. For instance, the left side of Figure 8 shows two identical sources; due to symmetry, it must be true that on the line of symmetry at $x_2 = 0$ the velocity in the x_2 direction is zero. The line $x_2 = 0$ is a streamline that is coincident with the x_1 axis. According to the previous paragraph, this streamline can be thought of as a solid surface. Thus the flow resulting from the superposition of the two identical sources in Fig. 8 left will be a solution to the problem of a single source next to a solid wall, as illustrated in Fig. 8 right.

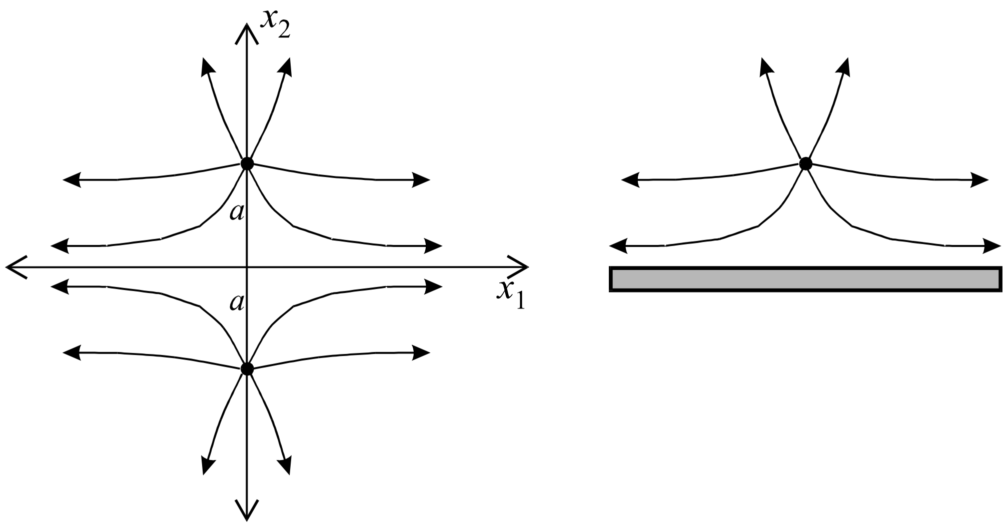

Figure 8. Left: A pair of identical sources, both of strength Q', located at $x_2 = a$ and $x_2 = -a$. The sources are "images" of one another. Right: A single source next to a solid wall.

From equation (29), the velocity potential for a single source centered at the origin is:

$$\Phi = -\,Q'/(2\pi)\ \ln r = -\,Q'/(4\pi)\ \ln(x_1^2 + x_2^2) \tag{38}$$

If the source is centered at $x_2 = a$, then

$$\Phi = -\,Q'/(4\pi)\ \ln(x_1^2 + (x_2 - a)^2) \tag{39}$$

As indicated above, the velocity potential Φ_t for the case of a single source a distance a from a solid wall can be found from the superposition of two identical source flows, one located at $x_2 = a$ and the other at $x_2 = -a$,

$$\Phi_t = -\,Q'/(4\pi)\ \ln(x_1^2 + (x_2 - a)^2)\ -\ Q'/(4\pi)\ \ln(x_1^2 + (x_2 + a)^2) \tag{40}$$

Equation (40) is the velocity potential for both of the flows depicted in Figure 8. Of course, for the single source next to a solid wall only the region $x_2 > 0$ is relevant. The stream function can be obtained by identical procedures, and the velocities can be calculated from either the velocity potential or the stream function.

The method of images works by superposing identical flows in order to create lines of symmetry across which there is no flow. This approach can be helpful in devising solutions for flows near solid surfaces when the location of the surfaces can be made coincident with the lines of symmetry. To repeat, since there is no flow across a streamline, any streamline for any potential flow can be viewed as a solid surface. This principle can be very powerful in analyzing potential flows even for rather complex geometries.

The Use of Bernoulli Equation in Potential Flow. By definition, potential flows are irrotational. If in addition the flow is incompressible, frictionless, and steady state, then the Extended Bernoulli Equation (EBE) becomes (for details, refer to the earlier handout on Bernoulli equation)

$$\frac{V_2^2 - V_1^2}{2} + \frac{p_2 - p_1}{\rho} + g(z_2 - z_1) = -w_s \tag{41}$$

Since potential flow is irrotational, the Bernoulli equation holds between *any* two points 1 and 2, so that the points *do not* have to lie on the same streamline. We have already come across this fact during our previous derivation of the Bernoulli's equation. In contrast, if the vorticity is not zero, then points 1 and 2 must be located on the same streamline. In potential flow, Bernoulli's equation is most often used to compute the pressure distribution. Typically, the shaft work term will be zero. Then once the velocity field is calculated (perhaps using one of the methods described above), the pressure distribution p_2 - p_1 is obtained directly from equation (41).