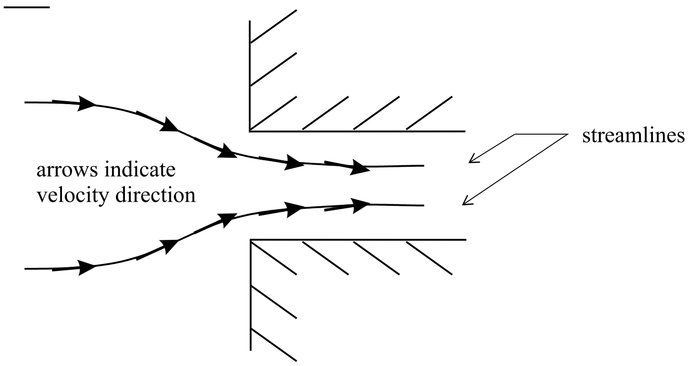
The Bernoulli Equation

Useful Definitions

***Streamline*:** a line that is tangent to velocity **v** at each point at a given instant in time.

Ex:



***Path Lines*:** lines traced out by fluid particles moving with the flow. At steady state, streamlines and path lines are equal.

Recall: differential equation of momentum:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{B} + \nabla\cdot\boldsymbol{\sigma} \qquad (1)$$

Equation 1 can be modified as follows:

i) Use the identity: $$\mathbf{v}\cdot\nabla\mathbf{v} = \nabla\left(\frac{v^2}{2}\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) \qquad (2)$$

(equation 30h in the handout on vector analysis)

ii) Assume that only gravitational body force is present, so that $\mathbf{B} = -\rho\nabla\Psi$ where Ψ is the gravitational potential ($\Psi = gz$ where z is the height in the gravitational field).

iii) Represent $\boldsymbol{\sigma}$ as $\sigma_{ij} = -p\delta_{ij} + \sigma_{ij,other}$, where $-p\delta_{ij}$ is nonfrictional normal stress (i.e. thermodynamic pressure) and $\sigma_{ij,other}$ represents all other contributions to the stress tensor, including viscous and/or other stresses. For example, for a Newtonian fluid $\sigma_{ij,other} = \lambda(\nabla\cdot\mathbf{v})\,\delta_{ij}$

+ $2\mu\, e_{ij}$ (see equations 20 and 24 in Handout 7); note, however, that we are not necessarily assuming a Newtonian fluid.

With modifications (i) through (iii), equation 1 becomes:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\nabla\left(\frac{v^2}{2}\right)-\mathbf{v}\times(\nabla\times\mathbf{v})\right)=-\rho\nabla\Psi-\nabla p+\nabla\cdot\boldsymbol{\sigma}_{other} \tag{3}$$

$$\frac{\partial \mathbf{v}}{\partial t}+\nabla\left(\frac{v^2}{2}\right)-\mathbf{v}\times(\nabla\times\mathbf{v})=-\nabla\Psi-\frac{\nabla p}{\rho}+\frac{1}{\rho}\nabla\cdot\boldsymbol{\sigma}_{other} \tag{4}$$

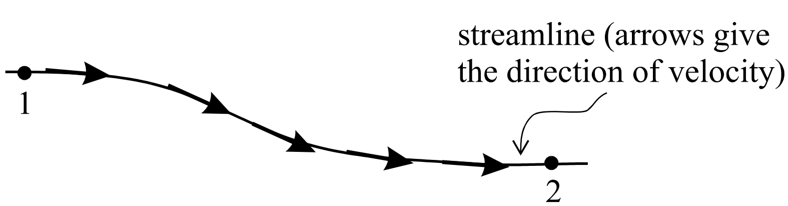
All the terms in equation 4 have the units of force per unit mass (ex. lbf / slug). Now, $\frac{1}{\rho}\nabla\cdot\boldsymbol{\sigma}_{\mathrm{other}}$ represents force acting at a point in space due to stresses other than thermodynamic pressure. We now assume that $\frac{1}{\rho}\nabla\cdot\boldsymbol{\sigma}_{\mathrm{other}}$ only includes viscous frictional forces exerted by the surrounding fluid, $\mathbf{f}_{\mathrm{F}}$ (units force/mass), and "shaft" forces due to moving machinery $\mathbf{f}_{\mathrm{S}}$,

$$\frac{\partial \mathbf{v}}{\partial t}+\nabla\left(\frac{v^2}{2}\right)-\mathbf{v}\times(\nabla\times\mathbf{v})=-\nabla\Psi-\frac{\nabla p}{\rho}+\mathbf{f}_{\mathrm{F}}+\mathbf{f}_{\mathrm{S}} \tag{5}$$

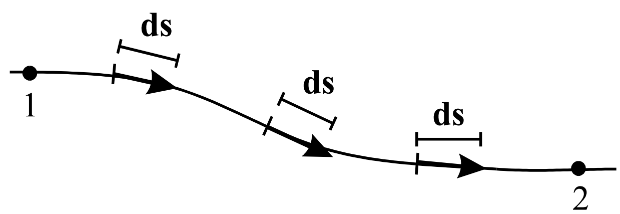
Equation 5 will be used to derive the so-called "Extended Bernoulli Equation."

The Extended Bernoulli Equation ("EBE")

The EBE is derived by integrating equation 5 between points 1 and 2 on a streamline:



Along the streamline, **ds** is used to represent a differential displacement:



Taking the dot product of equation 5 with **ds** and integrating between 1 and 2:

$$\int_1^2 \frac{\partial \mathbf{v}}{\partial t}\cdot \mathbf{ds} + \int_1^2 \nabla\left(\frac{v^2}{2}\right)\cdot \mathbf{ds} - \int_1^2 \left[\mathbf{v}\times(\nabla\times\mathbf{v})\right]\cdot \mathbf{ds} = -\int_1^2 \nabla\Psi\cdot\mathbf{ds} - \int_1^2 \frac{\nabla p}{\rho}\cdot\mathbf{ds} + \int_1^2 \mathbf{f}_\mathrm{F}\cdot\mathbf{ds} + \int_1^2 \mathbf{f}_\mathrm{S}\cdot\mathbf{ds} \quad (6)$$

The following facts will be useful:

i). From our review of vector analysis, the dot product of a gradient of a scalar field with a differential displacement is equal to the differential change in the scalar (i.e. $\nabla S \bullet \mathbf{dr} = dS$). In equation 6, the differential displacement **dr** is represented by **ds**.

ii). The vector $\mathbf{v}\times(\nabla\times\mathbf{v})$ is perpendicular to **v** and therefore to **ds**, since **ds** points along the streamline. Thus, $[\mathbf{v}\times(\nabla\times\mathbf{v})]\cdot\mathbf{ds} = 0$.

iii). The term $\int_1^2 \mathbf{f}_\mathrm{F}\cdot\mathbf{ds}$ is force times displacement and represents the work performed, per unit mass of fluid, by frictional forces acting between fluid particles as the fluid flows from point 1 to 2. This term can therefore be written as the "frictional work" w_F:

$$\int_1^2 \mathbf{f}_\mathrm{F}\cdot\mathbf{ds} = -w_\mathrm{F}$$ (w_F is frictional work performed by the streamline fluid on the surrounding fluid)

iv). Similarly, the term $\int_1^2 \mathbf{f}_\mathrm{S}\cdot\mathbf{ds}$ is shaft work performed per unit mass of fluid flowing from point 1 to point 2. This term will be written as w_S:

$$\int_1^2 \mathbf{f}_\mathrm{S}\cdot\mathbf{ds} = -w_\mathrm{S}$$ (w_S is shaft work performed by fluid on the surroundings)

Applying considerations (i) – (iv), equation 6 becomes:

$$\int_1^2 \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{ds} + \int_1^2 \mathrm{d}\left(\frac{v^2}{2}\right) = -\int_1^2 \mathrm{d}\Psi - \int_1^2 \frac{\mathrm{d}p}{\rho} - w_\mathrm{F} - w_S \tag{7}$$

Assuming steady ($\frac{\partial \mathbf{v}}{\partial t} = 0$), incompressible ($\rho$ = constant) flow, equation 7 can be integrated to give

$$\frac{v_2^{\;2}}{2} - \frac{v_1^{\;2}}{2} + \Psi_2 - \Psi_1 + \frac{p_2 - p_1}{\rho} = -w_\mathrm{F} - w_S \tag{8}$$

Since Ψ is the gravitational potential gz,

$$\frac{v_2^{\;2} - v_1^{\;2}}{2} + g(z_2 - z_1) + \frac{p_2 - p_1}{\rho} = -w_\mathrm{F} - w_S \tag{9}$$

Equation 9 is the Extended Bernoulli Equation, and holds for steady, incompressible flow along a streamline.

Notes regarding the EBE:

i). ALL the terms are in units of energy or work per unit mass of flowing fluid.

ii). The terms on the left of equation 9 are the change in kinetic energy, gravitational potential energy, and flow work performed per unit mass of fluid, as fluid flows from point 1 to point 2.

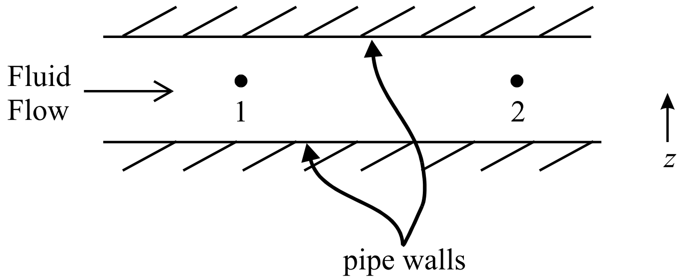
iii). w_F is work done by the flowing fluid against retarding frictional forces between points 1 and 2. $w_\mathrm{F} \geq 0$ for flows of real fluids. $w_\mathrm{F} < 0$ is unphysical.

w_F is often rewritten as:

$$w_\mathrm{F} = gH_\mathrm{L} \tag{10}$$

g: gravitational acceleration constant
H_L: called "head loss", with units of length.

Equation 10 defines H_L. For example, consider the following situation where fluid flows from point 1 to point 2 inside a pipe:



If the flow is steady and incompressible, and $z_1 = z_2$, $v_1 = v_2$, and $w_S = 0$, then equation 9 becomes:

$$\Delta p = p_2 - p_1 = -\rho g H_L$$

iv). The shaft work w_S is positive, $w_S > 0$, if work is done by the fluid (on a turbine, for instance). $w_S < 0$ if work is done ON the fluid (ex. by a pump).

In terms of head loss, the EBE is written:

$$\frac{v_2^2 - v_1^2}{2} + g(z_2 - z_1) + \frac{p_2 - p_1}{\rho} = -gH_L - w_S \tag{11}$$

(steady, incompressible flow, points 1 and 2 are on the same streamline)

Special Cases of the EBE:

1. Frictionless flow with no shaft work. Then $H_L = 0$, $w_S = 0$, and the EBE becomes:

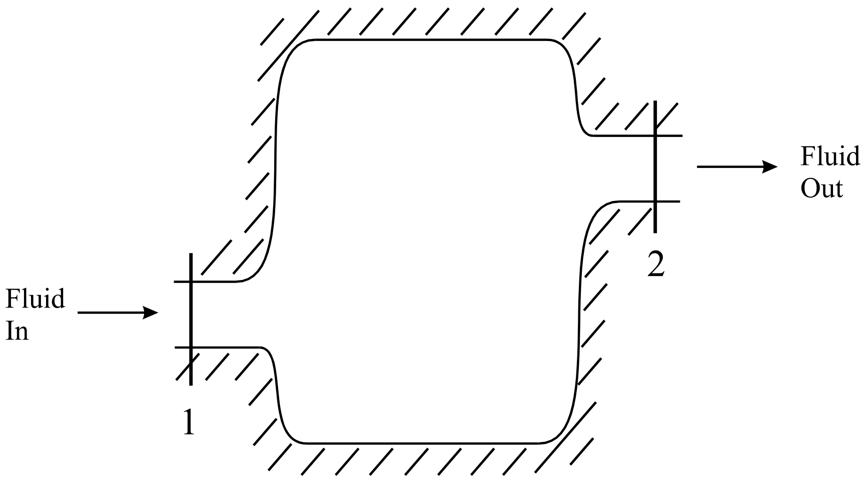
$$\frac{v_2^2 - v_1^2}{2} + g(z_2 - z_1) + \frac{p_2 - p_1}{\rho} = 0 \tag{12}$$

Equation 12 applies to steady, incompressible, frictionless flow with zero shaft work, where points 1 and 2 are on the same streamline. Equation 12 is known simply as the **Bernoulli Equation**.

2. Irrotational Flow: Then $\nabla \times \mathbf{v} = 0$. Therefore, if the derivation of the Extended Bernoulli Equation was repeated from equation 6, for irrotational flow we would not need to invoke the condition that points 1 and 2 are on the same streamline in order to drop the $\int_1^2 [\mathbf{v} \times (\nabla \times \mathbf{v})] \cdot \mathbf{ds}$ term (see equation 6). Thus, for irrotational flow, the EBE (equation 9) holds between any two

points 1 and 2 in the flow, where 1 and 2 do not have to lie on the same streamline. The flow still has to be steady and incompressible, however.

3. The EBE (equation 9) can be integrated over the cross-section of a pipe or other control volume. For instance, we could have a control volume like that shown below:



Designating the entry port as port "1" and the exit port as port "2", we want to sum equation 9 for all streamlines passing through these ports. The contribution of each streamline to the sum must be weighed by the rate at which fluid mass passes along it through the control volume. This rate of mass flow is $\rho \mathbf{v} \cdot \mathbf{n}$, where $\mathbf{n}$ is the unit normal to the surface of the control volume. Integrating each term in equation 9 over all streamlines (i.e. over the areas of the ports), equation 9 becomes

$$\iint_{A_2} \left(\frac{v^2}{2} + gz + \frac{p}{\rho} \right) \rho \mathbf{v} \cdot \mathbf{n} \mathrm{d}A + \iint_{A_1} \left(\frac{v^2}{2} + gz + \frac{p}{\rho} \right) \rho \mathbf{v} \cdot \mathbf{n} \mathrm{d}A = -\frac{\mathrm{d}W_\mathrm{F}}{\mathrm{d}t} - \frac{\mathrm{d}W\mathrm{s}}{\mathrm{d}t} \qquad (13)$$

(steady, incompressible flow)

In equation 13, $\mathrm{d}W_\mathrm{F}/\mathrm{d}t = \iint_{A_2} w_f (\rho \mathbf{v} \cdot \mathbf{n}) \mathrm{d}A$ is the rate of work against retarding frictional forces in the control volume. This term includes work done along all streamlines, not just along a particular streamline. Similarly, $\mathrm{d}W_\mathrm{S}/\mathrm{d}t = \iint_{A_2} w_s (\rho \mathbf{v} \cdot \mathbf{n}) \mathrm{d}A$ is the rate of shaft work performed in the control volume, inclusive of contributions from all streamlines. If $\mathbf{v}$, z, and p can be regarded as uniform over the areas A_1 and A_2, then equation 13 simplifies to:

$$m^* \left(\frac{v_2{}^2}{2} + gz_2 + \frac{p_2}{\rho} - \frac{v_1{}^2}{2} - gz_1 - \frac{p_1}{\rho} \right) = -\frac{\mathrm{d}W_\mathrm{F}}{\mathrm{d}t} - \frac{\mathrm{d}W\mathrm{s}}{\mathrm{d}t} \qquad (14)$$

(steady, incompressible flow; uniform properties over A_1 and A_2)

Here $m^* = -\rho \iint_{A_1} (\mathbf{v}\cdot\mathbf{n})\mathrm{d}A = \rho \iint_{A_2} (\mathbf{v}\cdot\mathbf{n})\mathrm{d}A$ is the mass flowrate of fluid through the ports. Since steady state was assumed, mass flow rate "in" equals mass flow rate "out."

Equation 13 can be compared with the integral version of the total energy balance. The total energy balance was derived in the handout on integral balance equations. For steady, incompressible flow through a control volume containing an entry port 1 and an exit port 2, the total energy balance becomes

$$\iint_{A_1+A_2} \left(\frac{v^2}{2} + gz + \underbrace{u}_{\text{internal energy per unit mass of fluid}} + \frac{p}{\rho}\right)\rho(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}A = \frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{\mathrm{d}W\mathrm{s}}{\mathrm{d}t} \tag{15}$$

Subtracting equation 13 from equation 15 yields

$$\iint_{A_1+A_2} u\rho(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}A = \frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{\mathrm{d}W_\mathrm{F}}{\mathrm{d}t} \tag{16}$$

or

$$\overbrace{\rho\iint_{A_2} u(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}A}^{\text{rate of } U \text{ out}} = \overbrace{-\iint_{A_1} \rho u(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}A}^{\text{rate of } U \text{ in}} + \overbrace{\frac{\mathrm{d}Q}{\mathrm{d}t}}^{\text{rate of heat addition}} + \overbrace{\frac{\mathrm{d}W_\mathrm{F}}{\mathrm{d}t}}^{\text{rate of frictional work}} \tag{17}$$

Here, U means "internal energy." Equation 17 states that the internal energy of the fluid flowing out of the control volume at port 2 equals the internal energy flowing into the control volume at port 1, plus the heat added and work done against frictional resistances as the fluid passes through the control volume. In particular, we see that the work of friction W_F is dissipated to internal energy. Equation 17 is a specialized form of the 1st Law of Thermodynamics. Note that a term due to compression/expansion of the system (often referred to as "*PV work*") is not present since steady state conditions, assumed in equations 13 and 15, imply that the control volume does not change.

Calculating Power:

Since the shaft work w_S is work performed per unit mass of flowing fluid, the power P, which is shaft work performed per unit time, is

$$P = m^* w_S \tag{18}$$

where m^* is the mass flowrate of fluid (mass/time).

Common units of power are: 1 watt (W) = 1 Joule/s

$$1\,\frac{\text{lbf ft}}{\text{sec}} = \frac{1}{550}\,\text{horsepower (hp)} = 1.356\ \text{watt}$$