

Orthogonal Curvilinear Coordinates

Introduction. Rectangular Cartesian coordinates are convenient when solving problems in which the geometry of a problem is well described by the coordinates x_1 , x_2 , and x_3 . For example, in Fig. 1, the geometry of the flow is easily described by saying that it occurs in the x_1 direction, with the velocity $\mathbf{v} = v_1 \boldsymbol{\delta}_1$ being zero at the boundaries $x_2 = 0$ and $x_2 = d$ (i.e. no-slip boundary condition). But what about the flow in Fig. 2? Here, the flow could be described by stating that it occurs in the θ direction, with v_θ equal to zero at $r = r_1$ and to ωr_2 at $r = r_2$. Such a description uses cylindrical, not Cartesian, coordinates. It is not necessary to use cylindrical coordinates, but their use does simplify the description (and mathematical solution) of a problem like the one in Fig. 2. If instead the flow in Figure 2 is described in Cartesian coordinates, the description becomes more convoluted and may go as follows: the components v_1 and v_2 of the velocity $v_1 \boldsymbol{\delta}_1 + v_2 \boldsymbol{\delta}_2$ add so as to make a fluid element travel in a circle, with both components equal to zero at $(x_1^2 + x_2^2)^{1/2} = r_1$ and obeying the condition $v_1^2 + v_2^2 = \omega^2 r_2^2$ at $(x_1^2 + x_2^2)^{1/2} = r_2$. This description is correct, but solving a problem possessing a cylindrical geometry using Cartesian coordinates will be much more cumbersome.

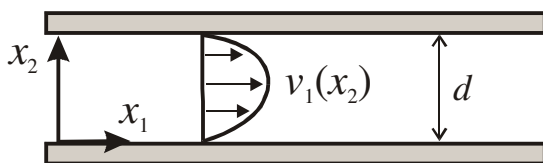


Figure 1. Flow between two flat plates.

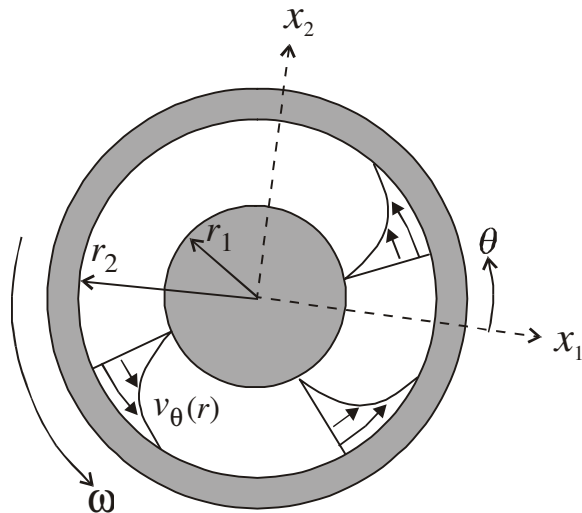


Figure 2. Circular flow in an annulus. The outer wall of the annulus is rotating with an angular velocity ω .

For geometries as in Fig. 2, proper use of "orthogonal curvilinear coordinates" can simplify the solution to the problem. What are orthogonal curvilinear coordinates? The most familiar examples (there are many others) are cylindrical and spherical coordinates as illustrated in Figures 3 and 4. The cylindrical and spherical coordinate systems are termed "curvilinear" because some of the coordinates change along *curves*. For instance, in cylindrical coordinates, θ changes along a curve that can be thought of as forming a circle about the origin. The Cartesian coordinate system is not curvilinear since all of the coordinates change along straight lines. The curve along which one coordinate changes while the other coordinates remain fixed is a **coordinate curve** for that coordinate. Coordinate curves for θ in the cylindrical coordinate system describe circles around the origin, while those for r are lines that radiate outward from the origin. Two coordinate curves for θ and one for r are shown in Figure 3. Coordinate curves for θ in the spherical geometry (Figure 4) describe semicircles ($0 \leq \theta \leq 180^\circ$), while those for ϕ describe circles about the origin. In addition to being curvilinear, the cylindrical and spherical coordinate systems are also "orthogonal" because coordinate curves corresponding to different coordinates are perpendicular to one another. For example, in cylindrical coordinates, the coordinate

curves for r are perpendicular to those for θ and x_3 . The coordinate curves for θ are perpendicular to those for r and x_3 , etc. Clearly, the Cartesian coordinate system is also orthogonal.

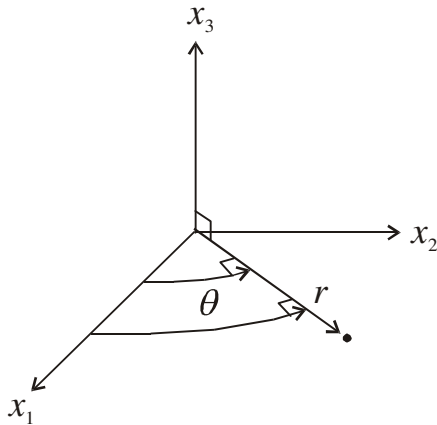


Figure 3. Cylindrical coordinates. The coordinates are r , θ and x_3 .

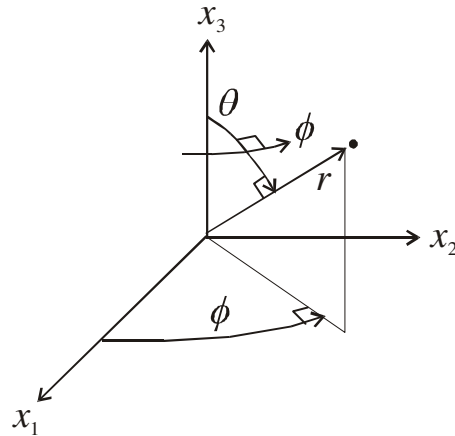


Figure 4. Spherical coordinates. The coordinates are r , θ and ϕ .

Unit basis vectors are defined at each point in the coordinate system as sketched in Figures 5 and 6. In general, the direction of a basis vector changes with position. In cylindrical coordinates, both δ_r and δ_θ change direction with position as illustrated in Fig 5, while the direction of δ_{x_3} is independent of position. In the spherical coordinate system, all three basis vectors δ_r , δ_θ and δ_ϕ change direction with position. The magnitudes of the basis vectors do not change, since by definition the length of a basis vector is always unity.

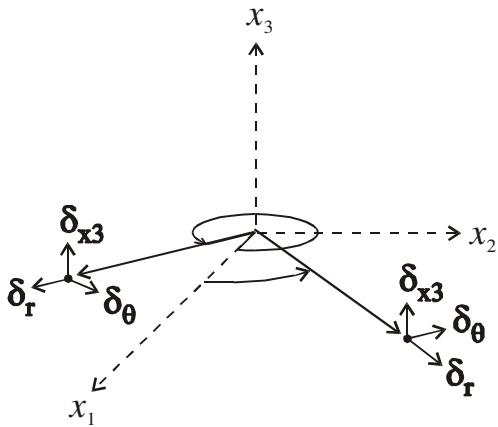


Figure 5. Basis vectors for the cylindrical coordinate system.

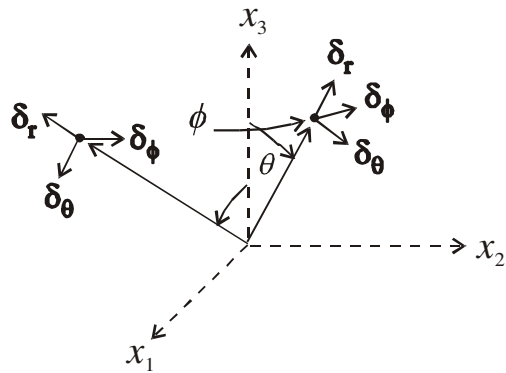


Figure 6. Basis vectors for the spherical coordinate system.

An important fact to recall: the length s of an arc on a circle is given by the product of the circle's radius r and the angle ζ (ζ is in **radians**) through which the arc sweeps; $s = r\zeta$, see Figure 7. This relation, especially in its differential form $ds = r d\zeta$, will be useful in the following discussion.

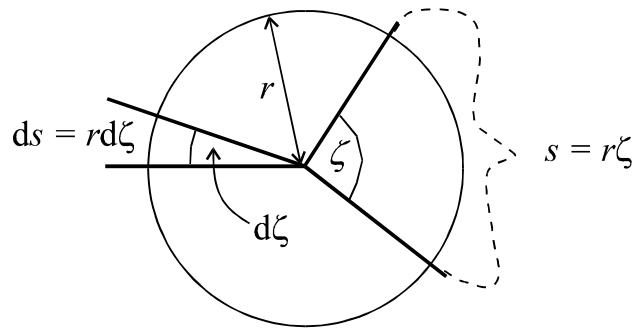


Figure 7.

Transformation of Coordinates.

The coordinates of a point in space, expressed in terms of two different coordinate systems, can be related by **transformation of coordinates** equations. Let's say the first coordinate system employs q_1 , q_2 , and q_3 as coordinates, and the second employs x_1 , x_2 , and x_3 . Then the transformation of coordinates equations have the general form

$$x_1 = x_1(q_1, q_2, q_3) \quad x_2 = x_2(q_1, q_2, q_3) \quad x_3 = x_3(q_1, q_2, q_3) \quad (1)$$

and

$$q_1 = q_1(x_1, x_2, x_3) \quad q_2 = q_2(x_1, x_2, x_3) \quad q_3 = q_3(x_1, x_2, x_3) \quad (2)$$

For example, if x_1 , x_2 , and x_3 are the Cartesian coordinates and q_1 , q_2 , and q_3 are the cylindrical coordinates r , θ , and z , then

$$x_1 = r \cos \theta \quad x_2 = r \sin \theta \quad x_3 = x_3 \quad (3)$$

and

$$r = (x_1^2 + x_2^2)^{1/2} \quad \theta = \tan^{-1}(x_2/x_1) \quad x_3 = x_3 \quad (4)$$

Equations 3 and 4 are the *transformation of coordinates* equations between the Cartesian and the cylindrical coordinate systems. For spherical coordinates, the transformation equations are

$$x_1 = r \sin \theta \cos \phi \quad x_2 = r \sin \theta \sin \phi \quad x_3 = r \cos \theta \quad (5)$$

and

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2} \quad \theta = \cos^{-1}(x_3 / (x_1^2 + x_2^2 + x_3^2)^{1/2}) \quad \phi = \tan^{-1}(x_2/x_1) \quad (6)$$

Basis Vectors. The basis vectors possess unit magnitude and point along the direction of coordinate curves as illustrated in Figures 5 and 6. Because the Cartesian basis vectors are constant in direction as well as magnitude, expressing curvilinear basis vectors in terms of the Cartesian ones can simplify mathematical derivations in some instances. Therefore, it is desired to derive expressions for the

curvilinear basis vectors in terms of the Cartesian basis. To begin, we first define the position \mathbf{p} of a point in space in terms of a Cartesian coordinate system:

$$\mathbf{p} = x_1(q_1, q_2, q_3) \boldsymbol{\delta}_1 + x_2(q_1, q_2, q_3) \boldsymbol{\delta}_2 + x_3(q_1, q_2, q_3) \boldsymbol{\delta}_3 \quad (6b)$$

Expressions for the Cartesian coordinates x_i in expression 6b in terms of the cylindrical coordinates are given by equations 3; for the spherical coordinate system, by equations 5. A unit basis vector $\boldsymbol{\delta}_i$ in the direction of the coordinate q_i can be expressed as

$$\boldsymbol{\delta}_i = \frac{\partial \mathbf{p}}{\partial q_i} / \left| \frac{\partial \mathbf{p}}{\partial q_i} \right| \quad (7)$$

where $\left| \frac{\partial \mathbf{p}}{\partial q_i} \right| = \left(\frac{\partial \mathbf{p}}{\partial q_i} \cdot \frac{\partial \mathbf{p}}{\partial q_i} \right)^{1/2}$ is the magnitude of the vector $\frac{\partial \mathbf{p}}{\partial q_i}$. Note that the vector $\frac{\partial \mathbf{p}}{\partial q_i}$ is tangent to (i.e. points along) the coordinate curve for q_i (Figure 8). The division by the magnitude $\left| \frac{\partial \mathbf{p}}{\partial q_i} \right|$ scales the vector $\frac{\partial \mathbf{p}}{\partial q_i}$ to unit magnitude; thus, $\left| \frac{\partial \mathbf{p}}{\partial q_i} \right|$ is often referred to as the "scale factor" h_i . The result of equation (7) is a unit basis vector whose direction points along the coordinate curve for q_i and whose length is normalized to unity by dividing by $\left| \frac{\partial \mathbf{p}}{\partial q_i} \right|$; in other words, this is the unit basis vector corresponding to the coordinate q_i .

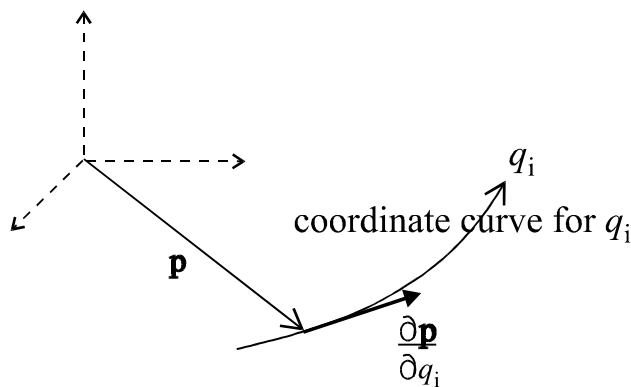


Figure 8

Equation (7) works trivially for the Cartesian coordinate system. In the Cartesian system, $\mathbf{p} = x_1 \boldsymbol{\delta}_1 + x_2 \boldsymbol{\delta}_2 + x_3 \boldsymbol{\delta}_3$ so that, for example, $\boldsymbol{\delta}_1 = (\partial \mathbf{p} / \partial x_1) / |(\partial \mathbf{p} / \partial x_1)| = \boldsymbol{\delta}_1 / (\boldsymbol{\delta}_1 \cdot \boldsymbol{\delta}_1)^{1/2} = \boldsymbol{\delta}_1 / 1 = \boldsymbol{\delta}_1$. Here, the scale factor $h_1 = (\boldsymbol{\delta}_1 \cdot \boldsymbol{\delta}_1)^{1/2} = 1$. What about the cylindrical coordinate system? In the cylindrical system, using equations (3)

$$\mathbf{p} = r \cos \theta \boldsymbol{\delta}_1 + r \sin \theta \boldsymbol{\delta}_2 + x_3 \boldsymbol{\delta}_3 \quad (8)$$

Then,

$$\begin{aligned}
\delta_r &= (\partial \mathbf{p} / \partial r) / |(\partial \mathbf{p} / \partial r)| = (\cos \theta \delta_1 + \sin \theta \delta_2) / [(\cos \theta \delta_1 + \sin \theta \delta_2) \cdot (\cos \theta \delta_1 + \sin \theta \delta_2)]^{1/2} \\
&= (\cos \theta \delta_1 + \sin \theta \delta_2) / [\cos^2 \theta + \sin^2 \theta]^{1/2} \\
\delta_r &= \cos \theta \delta_1 + \sin \theta \delta_2
\end{aligned} \tag{9}$$

Note that the scale factor $h_r = [(\cos \theta \delta_x + \sin \theta \delta_y) \cdot (\cos \theta \delta_x + \sin \theta \delta_y)]^{1/2} = [\cos^2 \theta + \sin^2 \theta]^{1/2} = 1$. Similarly,

$$\begin{aligned}
\delta_\theta &= (\partial \mathbf{p} / \partial \theta) / |(\partial \mathbf{p} / \partial \theta)| \\
&= (-r \sin \theta \delta_1 + r \cos \theta \delta_2) / [(-r \sin \theta \delta_1 + r \cos \theta \delta_2) \cdot (-r \sin \theta \delta_1 + r \cos \theta \delta_2)]^{1/2} \\
&= (-r \sin \theta \delta_1 + r \cos \theta \delta_2) / [r^2 \sin^2 \theta + r^2 \cos^2 \theta]^{1/2} \\
&= (-r \sin \theta \delta_1 + r \cos \theta \delta_2) / [r^2 (\sin^2 \theta + \cos^2 \theta)]^{1/2} \\
&= (-r \sin \theta \delta_1 + r \cos \theta \delta_2) / r \\
\delta_\theta &= -\sin \theta \delta_1 + \cos \theta \delta_2
\end{aligned} \tag{10}$$

Note that the scale factor $h_\theta = |(\partial \mathbf{p} / \partial \theta)| = r$. To summarize, for the cylindrical coordinate system

$$\delta_r = \cos \theta \delta_1 + \sin \theta \delta_2 \quad \delta_\theta = -\sin \theta \delta_1 + \cos \theta \delta_2 \quad \delta_{x3} = \delta_3 \tag{11}$$

$$h_r = 1 \quad h_\theta = r \quad h_{x3} = 1 \tag{12}$$

By similar procedures it can be shown that for the spherical coordinate system,

$$\begin{aligned}
\delta_r &= \sin \theta \cos \phi \delta_1 + \sin \theta \sin \phi \delta_2 + \cos \theta \delta_3 & \delta_\theta &= \cos \theta \cos \phi \delta_1 + \cos \theta \sin \phi \delta_2 - \sin \theta \delta_3 \\
\delta_\phi &= -\sin \phi \delta_1 + \cos \phi \delta_2
\end{aligned} \tag{13}$$

$$h_r = 1 \quad h_\theta = r \quad h_\phi = r \sin \theta \tag{14}$$

From expressions 11 and 13 it should be clear that the direction of δ_r and δ_θ in the cylindrical system, and the direction of all three basis vectors in the spherical coordinate system, changes with position. Expressions 11 and 13 are the desired equations that express the curvilinear basis vectors in terms of their Cartesian counterparts. How do we apply these equations? For instance, an angular velocity $\mathbf{v} = v_\theta \delta_\theta$ expressed in the cylindrical coordinate system can be converted to $\mathbf{v} = -v_\theta \sin \theta \delta_1 + v_\theta \cos \theta \delta_2$ by using equation 11 for δ_θ . If the expression $\theta = \tan^{-1}(x_2/x_1)$ is also substituted (see equations 4), the velocity will then be purely expressed in terms of Cartesian coordinate variables and basis vectors.

Length, Area and Volume Elements. For a set of orthogonal curvilinear coordinates q_1, q_2 and q_3 , with corresponding scaling factors h_1, h_2 , and h_3 , a differential displacement in position $d\mathbf{p}$ is given by (note the use of the summation convention)

$$d\mathbf{p} = h_1 dq_1 \boldsymbol{\delta}_1 + h_2 dq_2 \boldsymbol{\delta}_2 + h_3 dq_3 \boldsymbol{\delta}_3 = h_i dq_i \boldsymbol{\delta}_i \quad (15)$$

For the Cartesian coordinate system, all of the scale factors equal one, and equation (15) becomes

$$d\mathbf{p} = dx_1 \boldsymbol{\delta}_1 + dx_2 \boldsymbol{\delta}_2 + dx_3 \boldsymbol{\delta}_3 \quad (16)$$

as seen previously. For the cylindrical coordinate system, using equations (12) for the scale factors,

$$d\mathbf{p} = dr \boldsymbol{\delta}_r + r d\theta \boldsymbol{\delta}_\theta + dx_3 \boldsymbol{\delta}_{x_3} \quad (17)$$

Why are the scale factors h_i needed in equation 15 for $d\mathbf{p}$? Earlier, it was mentioned (Figure 7) that the *length* of an arc on the circumference of a circle is equal to the product of the central angle that spans the arc times the radius of the circle on whose circumference the arc lies. Therefore, the *distance* that is traversed when θ changes by an infinitesimal amount $d\theta$ is equal to $r d\theta$ (Figure 9). Since $d\mathbf{p}$ expresses a change in *distance*, $r d\theta$ must be used in equation 17 rather than just simply $d\theta$ ($d\theta$ is a change in angular position, and is not a distance). The distance corresponding to differential changes in the various coordinates is:

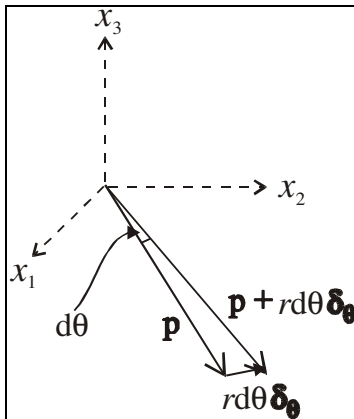


Figure 9. An object initially at position \mathbf{p} is displaced by $rd\theta \boldsymbol{\delta}_\theta$ to a final position $\mathbf{p} + rd\theta \boldsymbol{\delta}_\theta$.

Table I.

Coordinate System	Coordinate	Distance corresponding to an infinitesimal change in coordinate
Cartesian	x_1	dx_1
	x_2	dx_2
	x_3	dx_3
Cylindrical	r	dr
	θ	$rd\theta$
	x_3	dx_3

Spherical	r	dr
	θ	$r d\theta$
	ϕ	$r \sin\theta d\phi$

The volume dV of an infinitesimal volume element is obtained as usual; i.e. it is given by the product of the lengths of the three sides defining the length, depth and width of the element. Each of the three sides of the volume element is taken to lie along one of the coordinate directions. Note that the sides are assured to be mutually orthogonal since only orthogonal coordinate systems are being considered. In general

$$dV = h_1 h_2 h_3 dq_1 dq_2 dq_3 \tag{18}$$

For the Cartesian, cylindrical and spherical systems, equation 18 evaluates to

Table II.

Coordinate System	dV	
Cartesian	$dx_1 dx_2 dx_3$	(19a)
Cylindrical	$dr rd\theta dx_3 = r dr d\theta dx_3$	(19b)
Spherical	$dr rd\theta r\sin\theta d\phi = r^2 \sin\theta dr d\theta d\phi$	(19c)

By similar reasoning, the area dA of an infinitesimal area element is given by the product of the lengths of its sides. For instance, a differential area element in the Cartesian x_1 - x_2 plane is $dA = dx_1 dx_2$. A differential area element in the θ - x_3 cylindrical surface is $r d\theta dx_3$ (Figure 10). A differential area element in the θ - ϕ spherical surface is $r^2 \sin\theta d\theta d\phi$, etc.

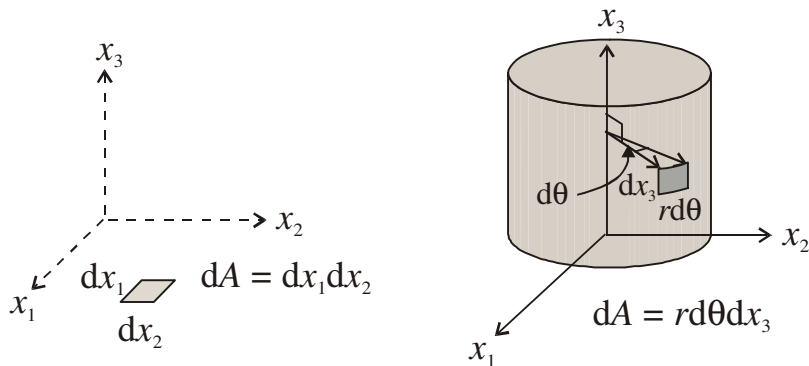


Figure 10.

Derivatives of Basis Vectors with Respect to Coordinate Variables.

As pointed out earlier (Figures 5 and 6), in general the direction of a basis vector can change from point to point. If a change in coordinate position can cause a change in the direction of a basis vector, that means that the basis vector must have a nonzero derivative with respect to at least some of the

coordinate variables q_i . For example, in the cylindrical coordinate system the basis vectors can be written (equations 11)

$$\delta_r = \cos\theta\delta_1 + \sin\theta\delta_2 \quad \delta_\theta = -\sin\theta\delta_1 + \cos\theta\delta_2 \quad \delta_3 = \delta_3 \quad (11)$$

The derivative of δ_r with respect to θ then becomes

$$\partial\delta_r/\partial\theta = -\sin\theta\delta_1 + \cos\theta\delta_2 = \delta_\theta.$$

Therefore, the derivative $\partial\delta_r/\partial\theta$ is nonzero. By similar calculations, it can be shown (you should verify these equations by directly considering expressions 11 above):

$$\begin{aligned} \partial\delta_r/\partial r &= 0 & \partial\delta_r/\partial\theta &= \delta_\theta & \partial\delta_r/\partial x_3 &= 0 \\ \partial\delta_\theta/\partial r &= 0 & \partial\delta_\theta/\partial\theta &= -\delta_r & \partial\delta_\theta/\partial x_3 &= 0 \\ \partial\delta_{x_3}/\partial r &= 0 & \partial\delta_{x_3}/\partial\theta &= 0 & \partial\delta_{x_3}/\partial x_3 &= 0 \end{aligned} \quad (20)$$

For the spherical coordinate system, the basis vectors are given by equations 13

$$\begin{aligned} \delta_r &= \sin\theta\cos\phi\delta_1 + \sin\theta\sin\phi\delta_2 + \cos\theta\delta_3 & \delta_\theta &= \cos\theta\cos\phi\delta_1 + \cos\theta\sin\phi\delta_2 - \sin\theta\delta_3 \\ \delta_\phi &= -\sin\phi\delta_1 + \cos\phi\delta_2 \end{aligned} \quad (13)$$

The derivatives of the basis vectors for the spherical coordinate system become

$$\begin{aligned} \partial\delta_r/\partial r &= 0 & \partial\delta_r/\partial\theta &= \delta_\theta & \partial\delta_r/\partial\phi &= \sin\theta\delta_\phi \\ \partial\delta_\theta/\partial r &= 0 & \partial\delta_\theta/\partial\theta &= -\delta_r & \partial\delta_\theta/\partial\phi &= \cos\theta\delta_\phi \\ \partial\delta_\phi/\partial r &= 0 & \partial\delta_\phi/\partial\theta &= 0 & \partial\delta_\phi/\partial\phi &= -\sin\theta\delta_r - \cos\theta\delta_\theta \end{aligned} \quad (21)$$

Each of these expressions can be derived from equations 13.

Gradient, Divergence and Curl.

The gradient operator ∇ takes the derivative of a quantity with respect to a change in position. In orthogonal curvilinear coordinates q_1 , q_2 and q_3 , the ∇ operator is (note that the scaling factors h_i in the denominator ensure that a derivative is being taken with respect to *distance*)

$$\nabla = \delta_1 \frac{1}{h_1} \frac{\partial}{\partial q_1} + \delta_2 \frac{1}{h_2} \frac{\partial}{\partial q_2} + \delta_3 \frac{1}{h_3} \frac{\partial}{\partial q_3} \quad (22)$$

$$\text{Cartesian} \quad \nabla = \delta_1 \frac{1}{1} \frac{\partial}{\partial x_1} + \delta_2 \frac{1}{1} \frac{\partial}{\partial x_2} + \delta_3 \frac{1}{1} \frac{\partial}{\partial x_3} \quad (23a)$$

$$\text{Cylindrical} \quad \nabla = \delta_r \frac{1}{r} \frac{\partial}{\partial r} + \delta_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \delta_{x_3} \frac{1}{r} \frac{\partial}{\partial x_3} \quad (23b)$$

$$\text{Spherical} \quad \nabla = \delta_r \frac{1}{r} \frac{\partial}{\partial r} + \delta_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \delta_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \quad (23c)$$

The divergence of a vector \mathbf{A} , written $\nabla \cdot \mathbf{A}$, can be calculated using the definition of the gradient. For the cylindrical coordinate system,

$$\mathbf{A} = A_r(r, \theta, x_3) \delta_r + A_\theta(r, \theta, x_3) \delta_\theta + A_{x_3}(r, \theta, x_3) \delta_{x_3}$$

$$\begin{aligned} \text{and} \quad \nabla \cdot \mathbf{A} &= \left(\delta_r \frac{\partial}{\partial r} + \delta_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \delta_{x_3} \frac{\partial}{\partial x_3} \right) \cdot [A_r(r, \theta, x_3) \delta_r + A_\theta(r, \theta, x_3) \delta_\theta + A_{x_3}(r, \theta, x_3) \delta_{x_3}] \\ &= \delta_r \cdot (\partial/\partial r) [A_r(r, \theta, x_3) \delta_r + A_\theta(r, \theta, x_3) \delta_\theta + A_{x_3}(r, \theta, x_3) \delta_{x_3}] + \\ &\quad \delta_\theta \cdot 1/r (\partial/\partial \theta) [A_r(r, \theta, x_3) \delta_r + A_\theta(r, \theta, x_3) \delta_\theta + A_{x_3}(r, \theta, x_3) \delta_{x_3}] + \\ &\quad \delta_{x_3} \cdot (\partial/\partial x_3) [A_r(r, \theta, x_3) \delta_r + A_\theta(r, \theta, x_3) \delta_\theta + A_{x_3}(r, \theta, x_3) \delta_{x_3}] \end{aligned}$$

Using the product rule for differentiation

$$\begin{aligned} &= \delta_r \cdot (\delta_r \partial A_r/\partial r + A_r \partial \delta_r/\partial r + \delta_\theta \partial A_\theta/\partial r + A_\theta \partial \delta_\theta/\partial r + \delta_{x_3} \partial A_{x_3}/\partial r + A_{x_3} \partial \delta_{x_3}/\partial r) + \\ &\quad \delta_\theta \cdot 1/r (\delta_r \partial A_r/\partial \theta + A_r \partial \delta_r/\partial \theta + \delta_\theta \partial A_\theta/\partial \theta + A_\theta \partial \delta_\theta/\partial \theta + \delta_{x_3} \partial A_{x_3}/\partial \theta + A_{x_3} \partial \delta_{x_3}/\partial \theta) + \\ &\quad \delta_{x_3} \cdot (\delta_r \partial A_r/\partial x_3 + A_r \partial \delta_r/\partial x_3 + \delta_\theta \partial A_\theta/\partial x_3 + A_\theta \partial \delta_\theta/\partial x_3 + \delta_{x_3} \partial A_{x_3}/\partial x_3 + A_{x_3} \partial \delta_{x_3}/\partial x_3) \end{aligned}$$

Substituting equations 20 for the derivatives of the basis vectors yields

$$\begin{aligned} &= \delta_r \cdot (\delta_r \partial A_r/\partial r + \delta_\theta \partial A_\theta/\partial r + \delta_{x_3} \partial A_{x_3}/\partial r) + \\ &\quad \delta_\theta \cdot 1/r (\delta_r \partial A_r/\partial \theta + A_r \delta_\theta + \delta_\theta \partial A_\theta/\partial \theta - A_\theta \delta_r + \delta_{x_3} \partial A_{x_3}/\partial \theta) + \\ &\quad \delta_{x_3} \cdot (\delta_r \partial A_r/\partial x_3 + \delta_\theta \partial A_\theta/\partial x_3 + \delta_{x_3} \partial A_{x_3}/\partial x_3) \end{aligned}$$

Performing the dot products (remembering that $\delta_i \cdot \delta_j = \delta_{ij}$ since the basis vectors are mutually orthogonal)

$$\nabla \cdot \mathbf{A} = \partial A_r/\partial r + A_r/r + 1/r \partial A_\theta/\partial \theta + \partial A_{x_3}/\partial x_3 \quad (24)$$

Expression 24 is the divergence of an arbitrary vector \mathbf{A} in cylindrical coordinates. An analogous approach could be used to derive $\nabla \cdot \mathbf{A}$ in spherical or Cartesian coordinates. The results are

$$\nabla \cdot \mathbf{A} = \partial A_1/\partial x_1 + \partial A_2/\partial x_2 + \partial A_3/\partial x_3 \quad (\text{Cartesian}) \quad (25a)$$

$$\nabla \cdot \mathbf{A} = \partial A_r / \partial r + A_r / r + 1/r \partial A_\theta / \partial \theta + \partial A_{x_3} / \partial x_3 \quad (\text{cylindrical}) \quad (25b)$$

$$\nabla \cdot \mathbf{A} = \partial A_r / \partial r + 2A_r / r + 1/r \partial A_\theta / \partial \theta + \cos \theta A_\theta / (r \sin \theta) + 1/(r \sin \theta) \partial A_\phi / \partial \phi \quad (\text{spherical}) \quad (25c)$$

It can be shown that, for an orthogonal curvilinear system, the divergence can be written as

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (h_2 h_3 A_1) + \frac{\partial}{\partial q_2} (h_1 h_3 A_2) + \frac{\partial}{\partial q_3} (h_1 h_2 A_3) \right] \quad (26)$$

where the h_i are the scale factors for the chosen coordinate system (i.e. equations 12 and 14) and q_i are the coordinate variables. Equation 26 will work for any orthogonal curvilinear coordinate system, and will reproduce equations 25a through 25c if the appropriate h_i and q_i are substituted into it.

Expressions for the curl $\nabla \times \mathbf{A}$ of a vector \mathbf{A} could be derived using a similar approach to that used in deriving equation 24 for $\nabla \cdot \mathbf{A}$. Here, we simply write the final general formula for $\nabla \times \mathbf{A}$,

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[h_1 \delta_1 \left(\frac{\partial}{\partial q_2} (h_3 A_3) - \frac{\partial}{\partial q_3} (h_2 A_2) \right) + h_2 \delta_2 \left(\frac{\partial}{\partial q_3} (h_1 A_1) - \frac{\partial}{\partial q_1} (h_3 A_3) \right) + h_3 \delta_3 \left(\frac{\partial}{\partial q_1} (h_2 A_2) - \frac{\partial}{\partial q_2} (h_1 A_1) \right) \right] \quad (27)$$

Equation 27 can be used to write out $\nabla \times \mathbf{A}$ in the coordinate system of interest, provided that the coordinate system is orthogonal curvilinear.

Concluding Remarks.

Some of the above expressions, even if written in simplified form, are rather cumbersome. Fortunately, most of the equations needed for fluid dynamics have already been written down for the coordinate systems of greatest interest, i.e. the Cartesian, cylindrical, and spherical systems. Therefore, it will not be necessary to apply the above equations to express the Navier-Stokes equations in spherical coordinates, for example. The desired expressions can be found in virtually any text on transport phenomena.