Interpreting Differential Equations of Transport Phenomena

There are a number of techniques generally useful in interpreting and simplifying the mathematical description of physical problems. Here we introduce several of them that we will encounter when solving transport problems.

1). Estimating magnitudes of terms in a differential equation. Transport problems can be mathematically challenging, being described by nonlinear, coupled partial differential equations. In these instances, it may be possible to simplify the description by using physical reasoning and approximate estimates to decide whether some of the terms in a differential equation are small in magnitude compared to the others. If so, it may be possible to neglect those terms altogether and work with the simplified equation which, although an approximation, still captures the physical behavior sufficiently accurately. Most often one encounters the need to estimate the magnitude of first or second order derivatives.

Taking a dependent variable g and an independent variable x, an order of magnitude estimate of a first order derivative can be obtained as follows:

$$\frac{\partial g}{\partial x} \sim \frac{\Delta g}{\Delta x} \tag{1}$$

where Δg is the maximum difference in g expected over the range Δx of x.

For example, imagine you are given a flow in which fluid flows through a cylindrical pipe of radius $R = 0.2$ m with an average velocity of $V = 5$ m/s. You want to estimate the derivative of the velocity v with respect to the radial coordinate r, $\partial v/\partial r$. You use your physical intuition by recognizing that, at the pipe wall, the fluid would be expected to have zero velocity if the no-slip boundary condition holds, $v(r = R) = 0$. Therefore, you may approximate that

$$\frac{\partial v}{\partial r} \sim \frac{\Delta v}{\Delta r} = \frac{5-0}{0.2-0} = 25\ \text{s}^{-1}$$

Next, let's estimate the second order derivative $\partial^2 g/\partial x^2$. To do so, it is often customary to assume that the first order derivative ranges from 0 to $\partial g/\partial x \sim \Delta g/\Delta x$. With this assumption we get,

$$\frac{\partial^2 g}{\partial x^2} \sim \frac{\frac{\Delta g}{\Delta x} - 0}{\Delta x} = \frac{\Delta g}{(\Delta x)^2} \tag{2}$$

In the preceding example of flow through a pipe, we would estimate $\frac{\partial^2 v}{\partial r^2} \sim \frac{\Delta v}{(\Delta r)^2} = \frac{5-0}{(0.2-0)^2} =$ 125 $\text{m}^{-1}\text{s}^{-1}$. The assumption of 0 as the lower limit on $\partial g/\partial x$ will usually overestimate the second derivative. The overestimate, however, is not a serious problem if the goal is to decide whether a second order derivative term is sufficiently small to be dropped from an equation, since in this case the overestimation simply imposes a stricter criteria for the elimination of the term.

Estimates based on equations 1 and 2 can be tremendously useful in simplifying an intractable problem by deciding which terms may be dropped from an equation. However, it is important to note that the above approximations work best when the change in g with x is monotonic and smooth. For example, the estimates will be unreliable when dealing with problems in which the derivative of g with x changes in sign over the x-domain of interest, or in which sudden changes in g with x occur over parts of the domain while elsewhere g depends only weakly on x.

2). Pseudo steady-state approximation. Often, the boundary conditions in a transport problem are independent of time. This simple situation, however, does not always apply. In fact, there are many interesting cases in which the boundary conditions themselves change with time. In such a scenario we have **time-dependent boundary conditions**. The additional time dependence can greatly complicate the mathematics, making it difficult to solve the problem. Fortunately, in some instances, it may be possible to avoid this complexity by solving the problem as if the boundary conditions were constant; i.e. independent of time. Such a simplification is possible when the boundary conditions change "sufficiently slowly." The boundary conditions are said to change "sufficiently slowly" if their rate of change is much more gradual than the response of the system. To put this another way, if the system responds very quickly to a perturbation in boundary conditions, so fast that during this time the boundary conditions hardly change, then the problem can be solved as if the boundary conditions were fixed.

For example, imagine that heat is conducted down a metal rod one end of which, at an initial time t_0, is placed into a reservoir at a temperature of 30 ^oC and the other end is placed into a reservoir at a temperature of 20 ^oC. After 1 minute, the temperature distribution in the rod is calculated to reach steady state; i.e. after 1 min, further changes in the T profile in the rod are negligible. The T distribution in the rod is determined by the energy balance and the reservoir temperatures. Now, imagine that the temperature of the hotter reservoir is instead slowly increased so that, over 1 year, it rises from 30 ^oC to 31 ^oC. The rate at which this boundary condition changes is much slower than the 1 minute required to establish a steady-state T distribution. In this example, therefore, we may suspect that it is acceptable to solve for the temperature profile in the rod as if the hotter reservoir was at a constant temperature T_H. Once this **pseudo steady-state** solution is obtained *under the assumption* of a constant T_H, the temperature distribution in the rod at any time t during the year can be found by substituting into the solution the value of T_H at that time. Thus, when we make a pseudo steady-state approximation, we are treating time-dependent boundary conditions as constant, because we suspect that they will not change significantly during the time needed by the system to reach steady state.

How can we tell when a pseudo steady-state approximation is justified? We need two pieces of information: one that represents the time required by the system to reach steady state (assuming constant boundary conditions), and the second that compares this time to how fast the boundary conditions are changing. We illustrate this for the example of flow between a moving and a stationary plate, depicted in Fig. 1. Here, the upper plate is uniformly accelerating with time such that its velocity V in the x_1 direction is given by $V(t) = at$, where a is the acceleration and t is time. Moreover, the fluid is assumed to be Newtonian, to be incompressible with a constant density ρ, and to possess a constant viscosity μ. The plates are assumed to be infinite in size. In solving for the unknown velocity v_1 between the plates we want to decide under what conditions we can assume that V is constant; i.e. when is it acceptable to invoke the pseudo steady-state approximation.

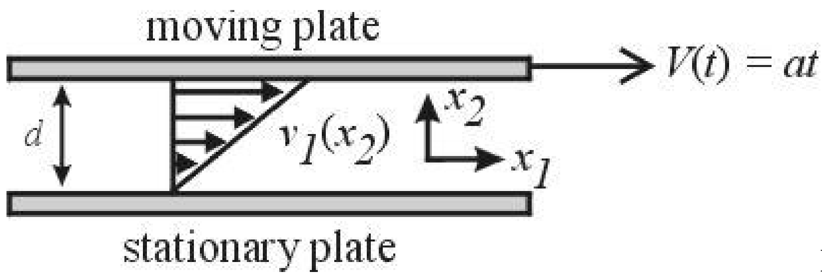

Figure 1

The flow of Newtonian fluids with constant ρ and μ is governed by the momentum balance as given in equations 14 and 15 of Handout 8,

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{B} - \nabla p + \mu \nabla^2 \mathbf{v} \qquad \text{(incompressible, constant } \mu \text{ Newtonian fluids)} \qquad (3)$$

The forces applied by the plates act along the x_1 direction; therefore, to analyze their influence on the flow we need to use the x_1 component of the momentum balance. Taking the x_1 component and expanding the material derivative and Laplacian terms leads to

$$\rho\left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3}\right) = B_1 - \frac{\partial p}{\partial x_1} + \mu\left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2}\right) \quad (4)$$

The flow is two-dimensional in x_1 and x_2. There is no flow along x_3 so that $v_3 = 0$; also, all derivatives with respect to x_3 must vanish,

$$\rho\left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2}\right) = B_1 - \frac{\partial p}{\partial x_1} + \mu\left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2}\right) \quad (5)$$

Moreover, we take gravity to point downward along the x_2 direction, thus $B_1 = 0$, and assume that there are no pressure gradients so that $\partial p/\partial x_1 = 0$ (with these assumptions the flow is purely driven by the forces imposed by the plates). We also recognize that $v_2 = 0$ since we do not expect any flow to arise perpendicular to the plates. Finally, we stipulate that v_1 depends on t and x_2 but not on x_1, noting that two fluid elements at the same x_2 position but at different x_1 positions will experience the same forces, thus they will accelerate at the same rate, and have the same velocity. These considerations simplify equation 5 to

$$\rho\left(\frac{\partial v_1}{\partial t}\right) = \mu\left(\frac{\partial^2 v_1}{\partial x_2^2}\right) \quad (6)$$

Next, we want to estimate the time τ required for the velocity v_1 to reach steady state. We can obtain this estimate as follows. A velocity disturbance Δv_1 (say of magnitude w) imposed at the moving plate will have propagated, after a time Δt, a distance δ into the fluid. The propagation occurs because fluid particles accelerated by the moving plate accelerate fluid particles further away from the plate, which in turn accelerate particles even further into the gap. Approximating the differential terms in equation 6 according to equations 1 and 2 yields

$$\rho\left(\frac{w}{\Delta t}\right) \sim \mu \, \frac{w}{\delta^2} \qquad \text{or} \qquad \Delta t \sim (\rho/\mu)\, \delta^2 \tag{7}$$

In equation 7, Δt is the time required for the velocity disturbance to propagate a distance δ (note $\delta < d$) into the gap. Steady state will be reached once the velocity disturbance had sufficient time to cross the entire gap between the plates, i.e. once δ becomes equal to d. Therefore, using equation 7, the time τ required to reach steady state is

$$\tau \sim (\rho/\mu)\, d^2 \tag{8}$$

We now have a first important piece of information: an estimate for the time required by the system to reach steady state. For a psedo steady-state approximation to be valid, it is further required that the plate velocity V does not change significantly during the period τ. In other words, we want $\Delta V << V$, where ΔV is the change in V over the duration τ. If the change ΔV is much smaller than V itself, then it is acceptable to regard V as nearly constant. This requirement leads to

$$\Delta V << V$$

$$V(\text{time} = t + \tau) - V(\text{time} = t) << V(\text{time} = t)$$

$$at + a\tau - at << at$$

$$a\tau << at$$

Thus, pseudo steady-state approximation will be valid if

$$t >> \tau \qquad \text{or} \qquad t >> (\rho/\mu)\, d^2 \tag{9}$$

The result in equation 9 is informative. Initially, when the upper plate has just started to accelerate from rest and t is less than or comparable to $(\rho/\mu)\, d^2$, invoking the pseudo steady-state approximation is expected to produce errors in the calculated v_1 profile because inequality 9 is not satisfied. However, for times $t >> (\rho/\mu)\, d^2$, the pseudo steady-state approximation is justified. For these times, solving the problem under the assumption that V is constant should produce accurate results for v_1.

Can you physically motivate the above conclusions? How would you mathematically setup this problem for short and long times?

The specification of conditions under which a pseudo steady-state approximation may be invoked depends on using physical intuition to analyze the problem being considered. However, in general it is necessary to compare the rapidity of response of the system, i.e. the time scale needed by the system to realize steady state, to the rate at which the boundary conditions are changing.

"Similarity"

Another powerful tool in mathematical and engineering analysis is based on the concept of "similarity." Consider two fluid flows that are similar in geometry. For example, they may both involve a fluid flowing through a cylindrical pipe. While the flow geometry is similar, the values of parameters such as viscosity and density, pipe size, flow velocity etc. may be different. Is it possible to measure properties of interest for just one of the pipe flows, and then use those results to predict the same properties for the other flow without remeasuring them? Likewise, can a small model of a chemical reactor be used to determine performance of a more expensive large reactor before building it? In this handout we will see that, under certain conditions, experiments or measurements on a model can indeed be used to predict how well the real machine or device will work.

Derivation of Dimensionless Similarity Parameters From Fundamental Equations.
If the behavior of one system is to be used for predicting the behavior of a second system, it must be true that the physical laws and the mathematical descriptions governing the two systems are closely related. Therefore, in establishing whether two systems are indeed "similar," what we are really asking is whether the physical statements describing them are similar. These statements may include conservation laws, constitutive relations, boundary conditions, and other information necessary to specifying the system behavior.

Such a comparison of similarity is best accomplished using a "dimensionless" description, reasons for which will become clear below. Although the concept of similarity is general and applicable to a wide variety of problems, for illustrative purposes we will initially specialize to an incompressible, constant viscosity, Newtonian flow. The only body force present will be assumed to be gravitational, $\mathbf{B} = -\rho\nabla\psi$, where ψ is the gravitational potential gh with h the height in the gravitational field. In Cartesian coordinates, the equation of continuity then becomes

$$\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}=0 \qquad (10)$$

and the i*th* component of the Navier-Stokes equations is

$$\frac{\partial v_i}{\partial t} + v_1 \frac{\partial v_i}{\partial x_1} + v_2 \frac{\partial v_i}{\partial x_2} + v_3 \frac{\partial v_i}{\partial x_3} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_i}{\partial x_1^2} + \frac{\partial^2 v_i}{\partial x_2^2} + \frac{\partial^2 v_i}{\partial x_3^2} \right) - \frac{\partial \psi}{\partial x_i} \qquad (11)$$

The above equations are made dimensionless by dividing each variable (i.e. quantities that vary with position or time) that possesses units by a reference, *constant* value with the same units. For instance, we divide all coordinates by a constant reference length, all pressures by a constant reference pressure, all velocities by a constant reference velocity, etc. We will denote the reference quantities by a subscript "o". The choice of the reference quantities is to some extent arbitrary. However, the reference quantity must be related to the problem at hand. For example, for pipe flow, the reference velocity V_o could be the average velocity of the fluid in that pipe (V_o = volumetric flowrate / pipe cross-sectional area), or it could be the velocity of the fluid in the center of the pipe, or some other choice as long as it describes the pipe flow. It cannot be a velocity that has nothing to do with the flow; e.g. the velocity of the Moon orbiting the Earth. This is because the reference quantity serves to normalize the scale of the system; this can only happen if the quantity itself is derived from the system. Also, historically, certain conventions have been adopted. For example, for pipe flow V_o is usually the average velocity, while for flow around a sphere V_o is the free stream velocity.

In this manner, a set of reference quantities is chosen so that we can write

$$v_i^* = v_i / V_o \qquad x_i^* = x_i / L_o \qquad p^* = p / p_o \qquad t^* = t \ (V_o / L_o) \qquad \psi^* = \psi / gL_o \qquad (12)$$

Here L_o is a reference length, p_o a reference pressure, and L_o/V_o a reference time (L_o / V_o represents the time it takes to traverse distance L_o when moving with a speed V_o). The resultant dimensionless variables are denoted with an asterisk. To convert back to dimensioned (regular) variables, equations 12 can be rearranged to

$$v_i = v_i^* \, V_o \qquad x_i = x_i^* \, L_o \qquad p = p^* \, p_o \qquad t = t^* \, (L_o / V_o) \qquad \psi = \psi^* \; gL_o \qquad (13)$$

Inserting expressions 13 into the equations of continuity and the Navier-Stokes equations and slightly rearranging yields

$$\frac{\partial v_1^*}{\partial x_1^*} + \frac{\partial v_2^*}{\partial x_2^*} + \frac{\partial v_3^*}{\partial x_3^*} = 0 \qquad (14)$$

and

$$\frac{\partial v_i^*}{\partial t^*} + v_1^* \frac{\partial v_i^*}{\partial x_1^*} + v_2^* \frac{\partial v_i^*}{\partial x_2^*} + v_3^* \frac{\partial v_i^*}{\partial x_3^*} = -\frac{p_o}{\rho V_o^2} \frac{\partial p^*}{\partial x_i^*} + \frac{\mu}{\rho V_o L_o}\left(\frac{\partial^2 v_i^*}{\partial x_1^{*2}} + \frac{\partial^2 v_i^*}{\partial x_2^{*2}} + \frac{\partial^2 v_i^*}{\partial x_3^{*2}} \right) - \frac{gL_o}{V_o^2} \frac{\partial \psi^*}{\partial x_i^*}$$

(15)

Every term in equations 14 and 15 is dimensionless. Furthermore, three dimensionless combinations (also called **dimensionless groups** or **dimensionless numbers**) have appeared in equation 15, each of which has a unique name:

Reynolds Number: $Re = \rho V_o L_o / \mu$ (16)

Froude Number: $Fr = V_o^2 / (gL_o)$ (17)

Euler Number: $Eu = p_o / (\rho V_o^2)$ (18)

Note that some texts define the Froude number as $V_o / (gL_o)^{1/2}$. *All three of these numbers are dimensionless.* Using equations 16 to 18, the momentum balance can be rewritten

$$\frac{\partial v_i^*}{\partial t^*} + v_1^* \frac{\partial v_i^*}{\partial x_1^*} + v_2^* \frac{\partial v_i^*}{\partial x_2^*} + v_3^* \frac{\partial v_i^*}{\partial x_3^*} = -Eu \; \frac{\partial p^*}{\partial x_i^*} + \frac{1}{Re}\left(\frac{\partial^2 v_i^*}{\partial x_1^{*2}} + \frac{\partial^2 v_i^*}{\partial x_2^{*2}} + \frac{\partial^2 v_i^*}{\partial x_3^{*2}} \right) - \frac{1}{Fr}\frac{\partial \psi^*}{\partial x_i^*} \quad (19)$$

In tensor form, the dimensionless Navier-Stokes equations are

$$\frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{v}^* \cdot \nabla^* \mathbf{v}^* = -Eu \; \nabla^* p^* + \frac{1}{Re} \nabla^{*2} \mathbf{v}^* - \frac{1}{Fr} \nabla^* \psi^* \quad (20)$$

When using differential equations to solve a problem, we recall from previous discussion that integration constants need to be specified using boundary conditions. For example, we may have

$$v_i = v_{ib} \quad \text{at} \quad f_1(x_1, x_2, x_3) = 0$$
$$p = p_b \quad \text{at} \quad f_2(x_1, x_2, x_3) = 0 \quad (21)$$

where the subscript "b" indicates the value of a variable at a boundary, and the boundary is specified by the function $f(x_1, x_2, x_3) = 0$ (ex. for a surface located at $x_1 = 5$, f would be $x_1 - 5 = 0$). Other boundary conditions could involve the derivatives of velocity and pressure, or employ temperature if the differential internal energy balance is being solved. If dimensionless differential equations are used to solve a problem, the boundary conditions also need to be rewritten in a dimensionless form by dividing all quantities with units by their reference values. For example, equations 21 would become

$$v_i^* = v_{ib}^* \quad \text{at} \quad f_1(x_1^*, x_2^*, x_3^*) = 0$$
$$p^* = p_b^* \quad \text{at} \quad f_2(x_1^*, x_2^*, x_3^*) = 0 \quad (22)$$

and so on.

We are now ready to define similarity more concretely, in terms of **geometric** and **dynamic similarity**.

Geometric Similarity ➔ Two systems are geometrically similar if they have identical boundary (and initial) conditions and obey the exactly same differential equations when expressed in dimensionless form (ex. two flows in different pipes, two flows around different sized spheres, etc.)

Dynamic Similarity ➔ Two systems are dynamically similar if, in addition to geometric similarity, they are characterized by identical values of all applicable dimensionless numbers (*Re*, *Fr*, *Eu*, etc.). Note that the choices for the reference quantities in the two systems must be consistent; for example, if pipe diameter is used as the reference length L_o for one of the systems, then it must also be used as the reference length in the other system.

Geometric and Dynamic Similarity ➔ It two systems, call them Flow #1 and Flow # 2, are geometrically and dynamically similar, then the dimensionless differential balance equations and boundary conditions that govern the two problems are identical. Therefore, the solutions to the dimensionless problems will be identical for Flow #1 and Flow #2. These solutions present *dependent dimensionless* variables such as v_i^*, p^*, T^* as functions of *independent dimensionless* variables (the independent variables are typically the position variables x_i^* and time t^*) and dimensionless groups (*Re*, *Fr*, *Eu*, etc.). Although these dimensionless solutions will be in terms of dimensionless variables and parameters, they can be readily converted to *dimensioned* solutions by using equations 13

$$v_i = v_i^* V_o \qquad x_i = x_i^* L_o \qquad p = p^* p_o \qquad t = t^* (L_o / V_o) \qquad \psi = \psi^* \; gL_o \qquad (13)$$

In equations 13, the values of the reference quantities (V_o, L_o etc.) are for the specific problem under consideration - that is, Flow #1 or Flow #2; i.e. using the values of these parameters for Flow #1 will produce the dimensioned solution for Flow #1.

It is important to recognize that the solutions that describe a flow problem do not have to be calculated, but can be also obtained experimentally. Indeed, this is more often the approach for complex, real world situations. For instance, rather than calculating velocity and pressure profiles, these quantities could be directly measured in an experiment on a model system. An experimental procedure might go as follows:

i). measurement of dependent experimental variables (velocities, pressures, etc.) on a "model" system under a variety of conditions
ii). presentation of the measured dependent variables in dimensionless form as functions of dimensionless independent variables and dimensionless groups that characterize the system
iii). application of the measurements to predict the behavior of geometrically and dynamically similar systems.

Complications. It may not be possible to ensure dynamic similarity between two systems. We illustrate this for the specific case of the Navier Stokes equations, as expressed in equation 19. Because gravitational field cannot be easily adjusted, achieving dynamic similarity for two systems in the Froude number would require that V_o^2 / L_o is same for both. In turn, fixing this ratio limits the ways in which the Reynolds number can be adjusted. Still, it is usually possible to ensure dynamic similarity in at least one of the dimensionless quantities. Often, such "less than perfect similarity" is sufficient. For example, in many problems the *absolute value* of pressure does not strongly influence the flow and only pressure gradients, or differences in pressure, are important. In such flows, it is common to simply set $p_o = \rho V_o^2$ so that the Euler number becomes unity. If Euler number is set to one, then only two dimensionless numbers are left in the Navier-Stokes equation 19: *Re* and *Fr*. However, there are circumstances when the reference pressure p_o must be chosen carefully and the Euler number cannot be set to unity. Such situations occur when the absolute pressure of the flow is important. One example is that of "**cavitation**." Cavitation refers to the formation of vapor cavities (bubbles) that occurs when the absolute pressure in a liquid falls below the liquid's vapor pressure. We will not go into details regarding such complications, but nevertheless should be aware of their existence.

It can also be shown that if the problem does not possess a boundary condition influenced by gravity (such a boundary could be a free surface of a liquid, such as the surface of a river or the ocean, for example) then the requirement of dynamic similarity in the Froude number can be usually neglected. Therefore, for flows that do not possess a free surface (ex. flow in a pipe when the pipe is fully filled with fluid; or the flow of an infinite body of fluid past an object) the Reynolds number is the only similarity parameter of interest in equation 19. If a free surface is present and its shape depends on gravity, then in general the Froude number must also be considered. Some problems for which the Froude number becomes important include flows in open channels, propagation of waves, drainage of tanks under the action of gravity, and design of marine vessels.

In summary: To achieve dynamic similarity for two geometrically similar, incompressible, constant viscosity Newtonian flows:

- In general *Re*, *Fr*, and *Eu* must be same for both flows
- If absolute pressure does not matter, $Eu = 1$. *Re* and *Fr* must be same for both flows.
- If there are no free surfaces whose shape is subject to gravitational action, *Fr* may be disregarded. *Re* is the only relevant parameter for dynamic similarity.

Finally, it should be emphasized that all of the above discussion presumed that the fluid density and viscosity were *constant*. Furthermore, forces due to additional possible effects such as surface tension or electromagnetic fields are not included in the Navier-Stokes equations. If such forces were present, they would have to be added to the momentum balance or its boundary conditions and these terms would give rise to additional dimensionless groups. Briefly, some other dimensionless groups one may encounter are:

- If surface tension exerts a strong influence on the flow: *Weber number*: $We = \rho V_o^2 L_o / T$ (here T is surface tension, not temperature)
- If the flow is compressible: *Mach number*: $Ma = V_o / V_S$ (V_S is the speed of sound under specified reference conditions)
- If the differential energy balance is made dimensionless: *Prandtl number*: $Pr = \mu C_p / \kappa$ (C_p is the heat capacity at constant pressure, and κ is the heat conductivity)
- If the flow is subject to "free convection" due to thermally-induced density gradients: *Grashof number*: $Gr = g\, \rho_o^2\, \beta\, (T_1 - T_o)\, L_o^3 / \mu^2$ (β is called the temperature coefficient of volume expansion, and $(T_1 - T_o)$ is a characteristic temperature difference for the problem)

There are many, many additional dimensionless groups that have been defined in transport phenomena, and the above are just a few examples.

Dynamic Similarity When the Governing Equations are Unknown. The safest way to derive the dimensionless groups governing dynamic similarity is from the differential balance equations and boundary conditions. The dimensionless numbers arise naturally during the non-dimensionalization of these equations, as seen previously in the context of the Navier Stokes equations. However, what if the problem is so complex that it is not clear what equations should be used or what boundary conditions apply? For example, the stirring of an open, baffled tank becomes a challenging problem to describe mathematically. In such situations, we can still apply the concepts of similarity, but the approach is a little different. For example, imagine we want to determine the power needed to propel a ship. We can built an exact prototype of the ship on a smaller scale, to enforce geometric similarity. Next, we need to identify the dimensionless groups that govern dynamic similarity for ship design. For example, say that we (somehow…see below) determine that equivalency of Re and Fr numbers is all that is needed to ensure dynamic similarity between measurements with the prototype and what would be observed using the full size ship. We can then measure the power needed for the prototype under conditions of dynamic similarity; i.e. for the same values of Re and Fr that will apply to the operation of the full size vessel. The power, once determined on the prototype setup, can then be scaled using the appropriate reference quantities to estimate the power needed to drive the full scale ship.

In order to apply the above method we must first identify the groups that govern dynamic similarity. When the mathematical description is too complex to allow this, it is common to rely on the "Buckingham pi theorem."

Buckingham Pi Theorem**:** "From a set of P variables and parameters that involve a total of U fundamental units (i.e. mass, length, time, temperature, charge), the total number of **independent** dimensionless groups that can be formed is P - U." A dimensionless group is said to be "independent" if it cannot be expressed in terms of the other dimensionless groups.

How do we generate dimensionless groups using the Buckingham pi theorem?

(1). Make a list of all dependent variables (ex. velocity, pressure), independent variables (ex. time, position), and parameters (ex. density, viscosity) that you think are relevant to a problem. This is the most tricky step - if an important variable or parameter is missed it will not be possible to deduce all the relevant dimensionless groups. On the other hand, including irrelevant parameters or variables can lead to extra dimensionless groups that are not needed to characterize the problem.

Example. Steady state, isothermal pipe flow perpendicular to a gravitational field (Figure 1):

(i) Dependent variable of interest: shear stress σ^o_{RZ} at the pipe wall
(ii) Independent variables on which the dependent variable depends (an educated guess): none

We expect σ^o_{RZ} to be the same everywhere on the pipe surface, therefore it does not depend on position. Also, if the pipe surface and fluid flow are not changing with time, then σ^o_{RZ} will not change with time.
(iii) Other parameters on which the dependent variable depends (another educated guess; if we wish, we could also have lumped these with the independent variables): D (pipe diameter), μ (fluid viscosity), ρ (fluid density), e (length characterizing average roughness of pipe surface), V_o (average fluid velocity through the pipe).

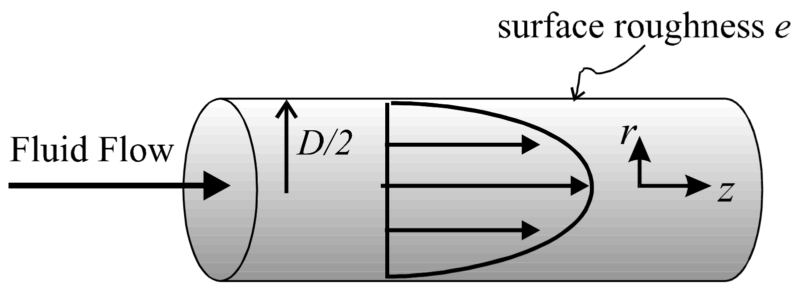

Figure 1

(2). Identify the units of all the variables and parameters in terms of fundamental dimensions (mass M, length L, time t, temperature T, charge C):

σ^o_{RZ} → $M\,L^{-1}\,t^{-2}$
D → L
μ → $M\,L^{-1}\,t^{-1}$
ρ → $M\,L^{-3}$
e → L
V_o → $L\,t^{-1}$

We have 6 variables and parameters in total, 3 fundamental units (M, L, t). According to the Buckingham Pi Theorem, there will be 6 - 3 = 3 independent dimensionless groups.

(3). Choose *U* parameters (preferably not including dependent variables) from the list that can represent the *U* different fundamental units in the problem. These are called the "**repeating parameters.**"

Could choose:
D → length
ρ → mass
V_o → time

(4). Construct dimensionless versions of all the non-repeating variables and parameters by grouping each with the *U* repeating variables, so that all units cancel. Below, "*" signifies that a quantity is dimensionless, and the exponents *a*, *b* and *c* are constants to be determined.

i). $\sigma^o_{RZ}{}^* = \sigma^o_{RZ}\, D^a \rho^b\, V_o^c$ ====> $M^0L^0t^0 = M^1\, L^{-1}\, t^{-2}\;\; L^a\;\; M^b\, L^{-3b}\, L^c\, t^{-c}$
mass: $0 = 1 + b$ → $b = -1$
time: $0 = -2 - c$ → $c = -2$

length: $0 = -1 + a - 3b + c$ → $a = 0$

Therefore, $\sigma^{o}_{RZ}^{*} = \sigma^{o}_{RZ} / (\rho V_o^2)$ is one way to make σ^{o}_{RZ} dimensionless. Note that, if different parameters had been selected, this dimensionless shear stress would be defined differently.

ii). $\mu^{*} = \mu D^a \rho^b V_o^c$ ====> $M^0 L^0 t^0 = M^1 L^{-1} t^{-1} L^a \; M^b L^{-3b} L^c t^{-c}$

mass: $0 = 1 + b$ → $b = -1$

time: $0 = -1 - c$ → $c = -1$

length: $0 = -1 + a - 3b + c$ → $a = -1$

therefore, $\mu^{*} = \mu / (\rho V_o D)$ is one way to make μ dimensionless (note that this is just the Reynolds number).

iii). $e^{*} = e D^a \rho^b V_o^c$ ====> $M^0 L^0 t^0 = L^1 \; L^a \; M^b L^{-3b} L^c t^{-c}$

mass: $0 = b$ → $b = 0$

time: $0 = c$ → $c = 0$

length: $0 = 1 + a - 3b + c$ → $a = -1$

therefore, $e^{*} = e / D$ is one way to make the surface roughness dimensionless.

(5). The dimensionless dependent variables are functions of the other dimensionless groups that have been determined (these other groups should not involve dependent variables). In our example, the only dimensionless dependent variable is $\sigma^{o}_{RZ}^{*}$, and there are two other dimensionless groups - the Reynolds number $Re = \rho V_o D/\mu$ and $e^{*} = e/D$. Therefore,

$$\sigma^{o}_{RZ}^{*} = f(Re, e^{*}) \qquad (23)$$

where f is a function to be determined. f could, in principle, be calculated by solving differential dimensionless equations supplemented with appropriate boundary conditions. In practice, that is usually too difficult and it is easier to simply measure f by experimentally obtaining $\sigma^{o}_{RZ}^{*}$ as a function of Re and e^{*}.

Usefulness of Dimensional Analysis

(1). Dimensionless representation avoids repeating mathematical solutions or experimental measurements for geometrically and dynamically similar systems. In other words, having obtained a mathematical or an experimental solution for a model system, the results can then be applied to other systems that are geometrically and dynamically similar. The solutions for the model system will be in dimensionless form - for instance, presenting $\sigma^{o}_{RZ}^{*}$ as a function of Re and e^{*}. However, these solutions can be easily converted to dimensioned form by multiplying by the appropriate reference quantities as in equations 13. For example, let's say that we have made measurements on pipe flow on a model system, which we call system # 1, and as a result we determined the function

$$\sigma^o{}_{RZ}^* = f(Re_1, e^*_1) \tag{24}$$

where

$$\sigma^o{}_{RZ}^* = \sigma^o{}_{RZ1} / (\rho_1 V_{o1}^2)$$

$$Re_1 = \rho_1 V_{o1} D_1 / \mu_1$$

$$e^*_1 = e_1 / D_1$$

The subscript "1" signifies that the above quantities refer to model system #1. Then for any other geometrically similar pipe flow, call this flow #2, $\sigma^o{}_{RZ2}$ can be obtained from the model data as follows. First, calculate Re_2 and e^*_2 for flow #2 (the subscript "2" signifies quantities pertaining to this other pipe flow):

$$Re_2 = \rho_2 V_{o2} D_2 / \mu_2$$

$$e^*_2 = e_2 / D_2$$

Next, determine the dimensionless shear stress at the wall, using the known function *f*, equation 24, evaluated at Re_2 and e^*_2:

$$\sigma^o{}_{RZ}^* = f(Re_2, e^*_2)$$

Finally, convert the dimensionless value to the actual, dimensioned shear stress $\sigma^o{}_{RZ2}$:

$$\sigma^o{}_{RZ2} = \sigma^o{}_{RZ}^* \ \rho_2 V_{o2}^2$$

In summary,

$$\sigma^o{}_{RZ2} = f(Re_2, e^*_2)\, \rho_2 V_{o2}^2$$

where *f* is the function determined from experiments on the model system #1.

(2). Another advantage of working in dimensionless form is that the total number of parameters describing a system is reduced. For instance, for the example of pipe flow (see the above discussion of the Buckingham pi theorem), we initially guessed that a total of 5 parameters would be important in determining the shear stress at the interface between the pipe wall and the fluid:

$$\sigma^o{}_{RZ} = \sigma^o{}_{RZ}(D, \mu, \rho, e, V_o) \tag{25}$$

In dimensionless form, we showed that the total number of relevant parameters is just two:

$$\sigma^o{}_{RZ}^* = \sigma^o{}_{RZ}^*(Re, e^*) \tag{26}$$

Therefore, the dimensionless form equation 26 shows that only *Re* and e^* are needed to characterize how σ^o_{RZ} changes with flow conditions and pipe roughness. If we have to perform experiments or numerical calculations for each set of conditions to determine the corresponding value of σ^o_{RZ}, working with equation 26, in which only two parameters need to be varied to fully determine how σ^o_{RZ} depends on experimental conditions, is much more efficient than having to vary all five parameters as in equation 25. The reduction in the number of degrees of freedom that have to be varied comes simply from imposing the constraint that the solution must be dimensionally consistent.

(3). Dimensionless analysis also helps us identify how experimental results, or the results of calculations, should be presented. In other words, it tells us how to correlate data. For instance, in the pipe flow example we would present experimental measurements of $\sigma^o_{RZ}{}^*$ as a function of *Re* and e^*.

(4). The values of dimensionless groups that govern a problem can also help identify how the mathematical description of that problem can be simplified. For example, let's consider the dimensionless Navier-Stokes equation that was derived earlier,

$$\frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{v}^* \cdot \nabla^* \mathbf{v}^* = -Eu\ \nabla^* p^* + \frac{1}{Re} \nabla^{*2} \mathbf{v}^* - \frac{1}{Fr} \nabla^* \psi^* \tag{20}$$

We will compare the relative magnitude of two of the terms in equation 20 to see if one or the other could be neglected and dropped from the differential equation. First, we recall that the term $\mathbf{v}^* \cdot \nabla^* \mathbf{v}^*$ on the left hand side represents transport of momentum by convection, while the term $(1/Re)\nabla^{*2}\,\mathbf{v}^*$ on the right represents transport of momentum by viscous forces (i.e. "friction" between fluid elements). Therefore, the ratio of convective momentum transport to that due to viscous forces is

$$\text{Convective transport / viscous forces} = Re\left(\mathbf{v}^* \cdot \nabla^* \mathbf{v}^*\right)/\nabla^{*2}\ \mathbf{v}^* \tag{27}$$

Equation 27 shows that, when *Re* is large, convective (sometimes called inertial) transport is dominant over transport of momentum due to viscous forces. For this reason, the Reynolds number *Re* can be thought of as representing the relative importance of convective to viscous modes of momentum transport. When *Re* is very large, it will be a good approximation to neglect the $(1/Re)\nabla^{*2}\,\mathbf{v}^*$ term, representing momentum transport by viscous forces, in the dimensionless Navier Stokes equation. This is because this term becomes negligible in the limit $Re \rightarrow \infty$ in comparison to the other terms. Thus, if we are only interested in the solution for high *Re* number flows, we could solve a simplified equation 20, without the viscous term.

As we have seen, by taking the differential equation and/or boundary conditions for a problem and making them dimensionless, as done above for the Navier Stokes equation, dimensionless groups will be generated. An analysis can then be made to deduce the physical interpretation of the dimensionless groups, as was just done above for the case of *Re*. Being familiar with such interpretations can be very helpful in using the magnitude of a dimensionless group to decide which physical mechanisms (e.g. convective vs. viscous transport of momentum in the example involving *Re*) are dominant. In turn, this information can be used to simplify the mathematical modeling of the problem.