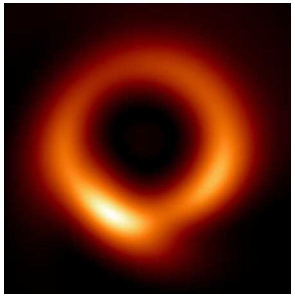
14. Black Hole Entropy as TD Entropy

- Prunkl & Timpson (2019)



1. Motivations

$$S_{\mathrm{BH}} = A/4$$

The surface area A of the event horizon of the black hole.

$$S_{\mathrm{TD}}(\sigma_2) \equiv \int_{\sigma_1}^{\sigma_2} \frac{\delta Q_R}{T} + S_0$$

The ratio of the change in heat to temperature of a reversible process that connects an initial state to a final state.

Is $S_{\mathrm{BH}} = S_{\mathrm{TD}}$?

Is the thermodynamic entropy of a black hole proportional to the surface area of its event horizon?

- *Bekenstein's motivations*:
  - The laws of black hole mechanics are formally similar to the laws of thermodynamics.

    Only establishes an analogy.
  - If we don't identify the TD entropy of a black hole with its event horizon, then the 2^{nd} Law is violated.

    Only persuasive for reversible processes.

What's needed:

An explicit demonstration that a black hole can undergo a reversible process.

2. Problem: Negative Heat Capacity

- *Recall*: A reversible process is a process that takes an initial equilibrium state to a final equilibrium state.
- *So*: If a black hole can undergo a reversible process (and thus be ascribed S_{TD}), it must be able to be in an equilibrium state with its environment.

Problem: A black hole has a negative heat capacity!

Heat capacity $C =$ amount of absorbed heat δQ needed to change temperature by dT:
$C = \delta Q / dT$

Claim: If $S_{\mathrm{BH}} = S_{\mathrm{TD}}$ and $T = (1/2\pi)\kappa$, then a black hole has negative C.

Proof: For a Schwarzschild black hole, $R = 2M$, $A = 4\pi R^2$, $\kappa = 1/4M$

- *So*: $S_{\mathrm{BH}} = A/4 = 4\pi M^2$ or $dS_{\mathrm{BH}} = 8\pi M dM$
- *And*: $T = (1/2\pi)\kappa = 1/8\pi M$ or $dT = -(1/8\pi M^2)dM$
- *Suppose*: $S_{\mathrm{BH}} = S_{\mathrm{TD}}$
- *Then*: $\delta Q = TdS_{\mathrm{TD}} = TdS_{\mathrm{BH}} = dM$
- *Thus*: $C = \delta Q/dT = -8\pi M^2$

2. Problem: Negative Heat Capacity

- *Recall*: A reversible process is a process that takes an initial equilibrium state to a final equilibrium state.
- *So*: If a black hole can undergo a reversible process (and thus be ascribed S_{TD}), it must be able to be in an equilibrium state with its environment.

Problem: A black hole has a negative heat capacity!

Heat capacity $C =$	amount of absorbed heat δQ needed to change temperature by dT: $C = \delta Q/dT$

- If C is negative, then when heat is absorbed, temperature *decreases!*
- *This is a problem*: How can a black hole be in a stable equilibrium state with another physical system with positive heat capacity?

As that system cools, it emits heat, which is absorbed by the black hole, which also cools as a result.

The black hole and the other system can never reach an equilibrium state!

Hawking's response: (Hawking 1976)

- S_{BH} must be statistical mechanical microcanonical entropy.

"The fact that the temperature of a black hole decreases as the mass increases means that black holes cannot be in stable thermal equilibrium in the situations in which there is an indefinitely large amount of energy available... [T]his implies that the normal statistical mechanical canonical ensemble cannot be applied to gravitating systems. Instead one has to use microcanonical ensembles in which one considers all the possible configurations of a system with a given energy."

"A black hole of given mass, angular momentum, and charge can have a large number of different unobservable internal configurations which reflect the possible internal configurations of the matter which collapsed to produce the hole. The logarithm of this number can be regarded as the entropy of the black hole and is a measure of the amount of information about the initial state which was lost in the formation of the black hole."

$$S_{\text{BH}} = S_{\text{Gibbs}}(\rho_{\text{mc}}) = k \ln \Omega(E)$$

- *The number of microstates of the black hole?*
- *The information "about the initial state which was lost in the formation of the black hole"?*

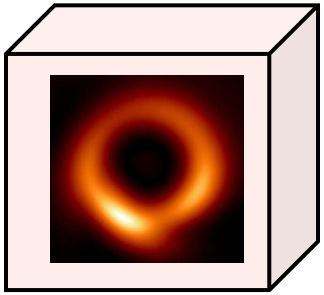
Prunkl & Timpson's response:

- "Even though black holes cannot be in stable equilibrium with an infinite heat bath, they can be in stable equilibrium with a photon gas and enclosed in a box, for a certain range of parameters."

Specific claim: For a black hole of mass M in a photon gas of volume V, there are values of M and V that make the equilibrium condition $T_{\mathrm{BH}} = T_{\mathrm{gas}}$ stable against fluctuations in temperature, in the sense:

$$\frac{dT_{\mathrm{BH}}}{dM} < \frac{dT_{\mathrm{gas}}}{dM}$$

"The gas can react quickly enough to any fluctuations in the black hole."

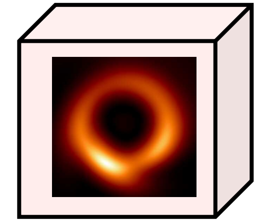


Black hole of mass M in photon gas of volume V

Task: Find values of M and V that satisfy:

$$(1) \quad T_{\mathrm{BH}} = T_{\mathrm{gas}}$$

$$(2) \quad \frac{dT_{\mathrm{BH}}}{dM} < \frac{dT_{\mathrm{gas}}}{dM}$$



Black hole of mass M in photon gas of volume V

- *First*: Rewrite (1) in terms of M: $E_{\mathrm{total}} = E_{\mathrm{BH}} + E_{\mathrm{gas}} = M + \frac{\pi^2}{15} V T_{\mathrm{gas}}^4$

Energy of photon gas from Planck's Law.

- Now solve for T_{gas}: $$T_{\mathrm{gas}} = \left[\frac{E_{\mathrm{total}} - M}{\frac{\pi^2}{15}V}\right]^{1/4}$$

- Now substitute into (1) with $T_{\mathrm{BH}} = 1/8\pi M$: $$\left[\frac{E_{\mathrm{total}} - M}{\frac{\pi^2}{15}V}\right]^{1/4} = \frac{1}{8\pi M}$$

Assumes a black hole has a temperature given by Hawking radiation.

- *Or*: $$M^4(M - E_{\mathrm{total}}) + \beta V = 0, \qquad \beta = \frac{1}{15(8)^4\pi^2}$$

To solve this for M, construct the graph of the function

$$f(M) = M^4(M - E_{\mathrm{total}}) + \beta V$$

and see where it intersects the M axis...

- Let $f(M) = M^4(M - E_{\text{total}}) + \beta V$ $\quad f(M) = 0 \Leftrightarrow T_{\text{BH}} = T_{\text{gas}}$
  - Max/min occur at values of M for which $df/dM = 0$:

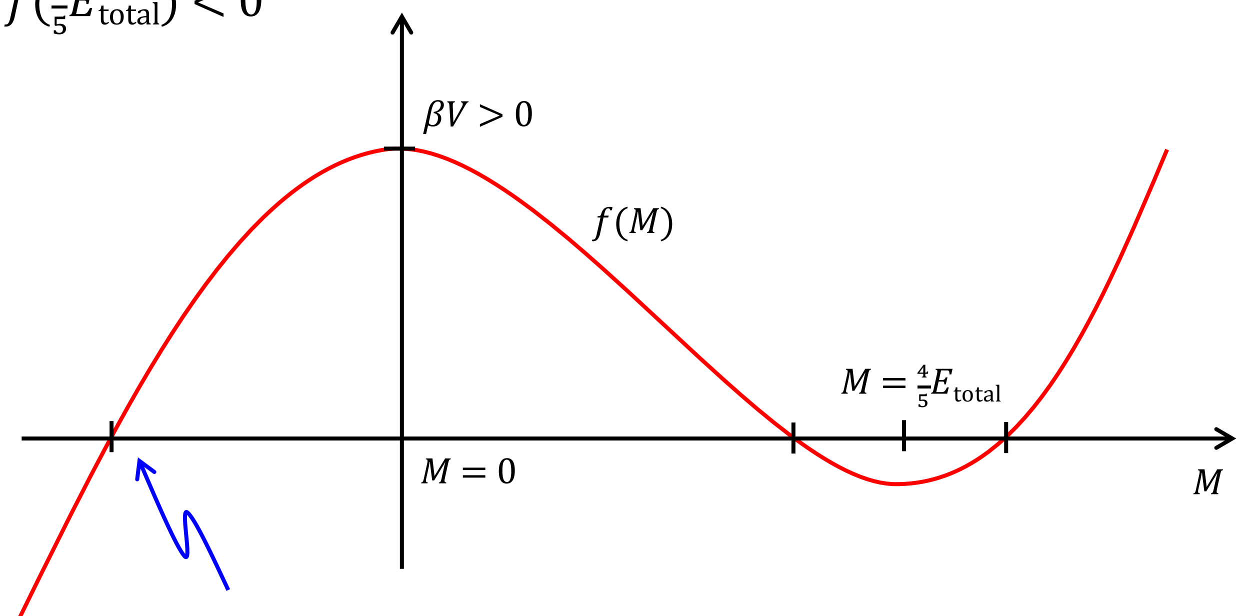
$$df/dM = 5M^4 - 4M^3 E_{\text{total}} = 0 \quad \text{for} \quad M = 0 \text{ and } M = \tfrac{4}{5}E_{\text{total}}$$

  - *Note*: $f(M) \to -\infty$ as $M \to -\infty$ and $f(M) \to +\infty$ as $M \to +\infty$

- So $M = 0$ must be a maximum!
- There are no max/min's to the left of $M = 0$, and $f(M)$ is decreasing to $-\infty$.

- So $M = \frac{4}{5}E_{\text{tot}}$ must be a minimum!
- There are no max/mins to the right of $M = \frac{4}{5}E_{\text{total}}$, and $f(M)$ is increasing to $+\infty$.

  - *Case 1*: $f(\frac{4}{5}E_{\text{total}}) < 0$



Value of M for which $f(M) = 0$!
- *But*: Unphysical since M is negative.

Two physically reasonable values of M for which $f(M) = 0$!

- Let $f(M) = M^4(M - E_{\text{total}}) + \beta V$ $f(M) = 0 \Leftrightarrow T_{\text{BH}} = T_{\text{gas}}$
  - Max/min occur at values of M for which $df/dM = 0$:

$$df/dM = 5M^4 - 4M^3 E_{\text{total}} = 0 \quad \text{for} \quad M = 0 \text{ and } M = \tfrac{4}{5}E_{\text{total}}$$

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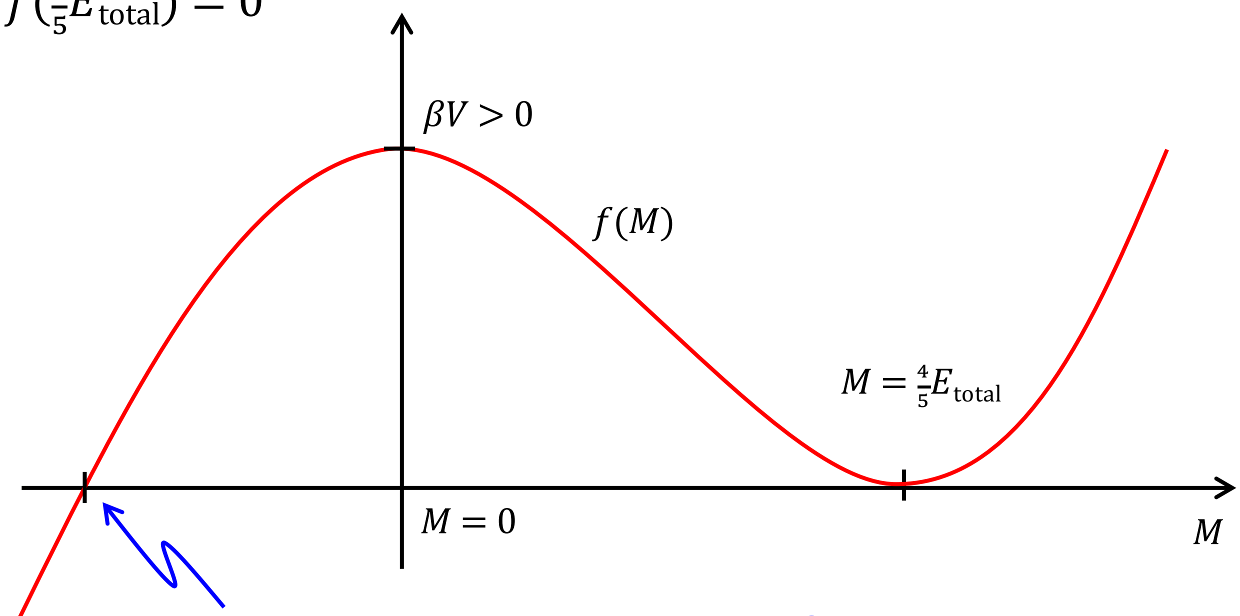
- So $M = \frac{4}{5}E_{\text{tot}}$ must be a minimum!
- There are no max/mins to the right of $M = \frac{4}{5}E_{\text{total}}$, and $f(M)$ is increasing to $+\infty$.

  - *Case 2*: $f(\frac{4}{5}E_{\text{total}}) = 0$



Value of M for which $f(M) = 0$!
- *But*: Unphysical since M is negative.

One physically reasonable value of M for which $f(M) = 0$!

- Let $f(M) = M^4(M - E_{\text{total}}) + \beta V$ $f(M) = 0 \Leftrightarrow T_{\text{BH}} = T_{\text{gas}}$
  - Max/min occur at values of M for which $df/dM = 0$:

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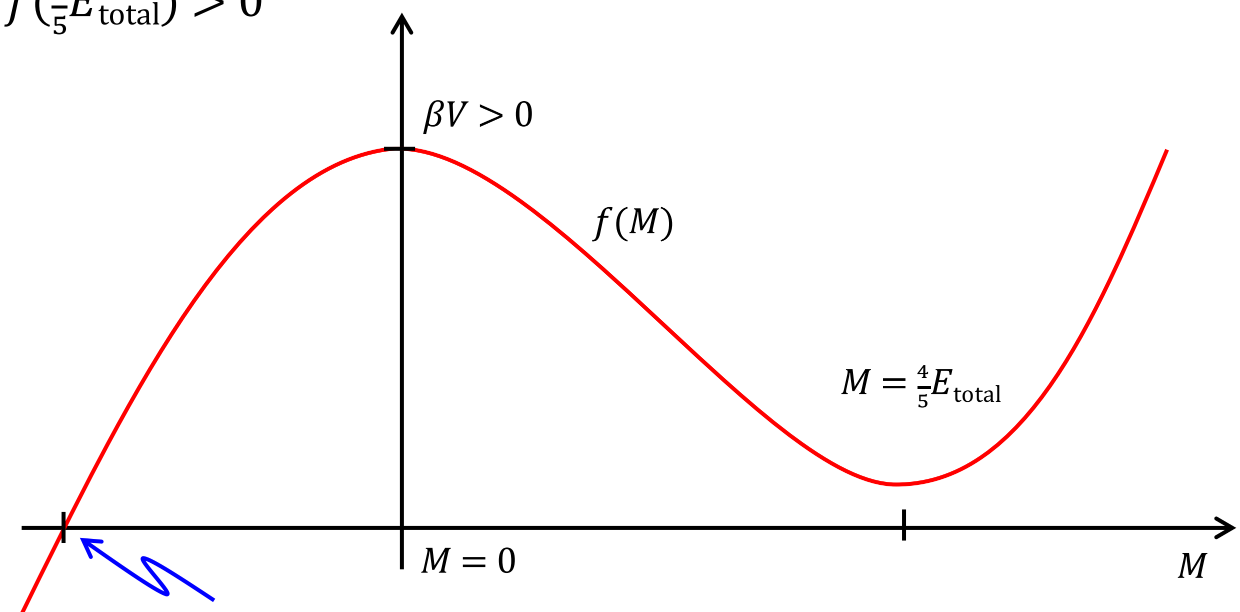
- So $M = \tfrac{4}{5}E_{\text{tot}}$ must be a minimum!
- There are no max/mins to the right of $M = \tfrac{4}{5}E_{\text{total}}$, and $f(M)$ is increasing to $+\infty$.

  - *Case 3*: $f(\tfrac{4}{5}E_{\text{total}}) > 0$



Value of M for which $f(M) = 0$!
- But: Unphysical since M is negative.

- Let $f(M) = M^4(M - E_{\text{total}}) + \beta V$ $f(M) = 0 \Leftrightarrow T_{\text{BH}} = T_{\text{gas}}$

  - *So*: Condition for there to be physically reasonable values of M for which $T_{\text{BH}} = T_{\text{gas}}$:

$$f(\tfrac{4}{5}E_{\text{total}}) \leq 0 \quad \text{or} \quad \beta V \leq 0.082 E_{\text{tot}}{}^5$$

Are these values of M stable equilibrium points? Do they satisfy Condition (2) $dT_{\text{BH}}/dM < dT_g/dM$?

$$\frac{dT_{\text{BH}}}{dM} = \frac{d}{dM}\left(\frac{1}{8\pi M}\right) = -\frac{1}{8\pi M^2}$$

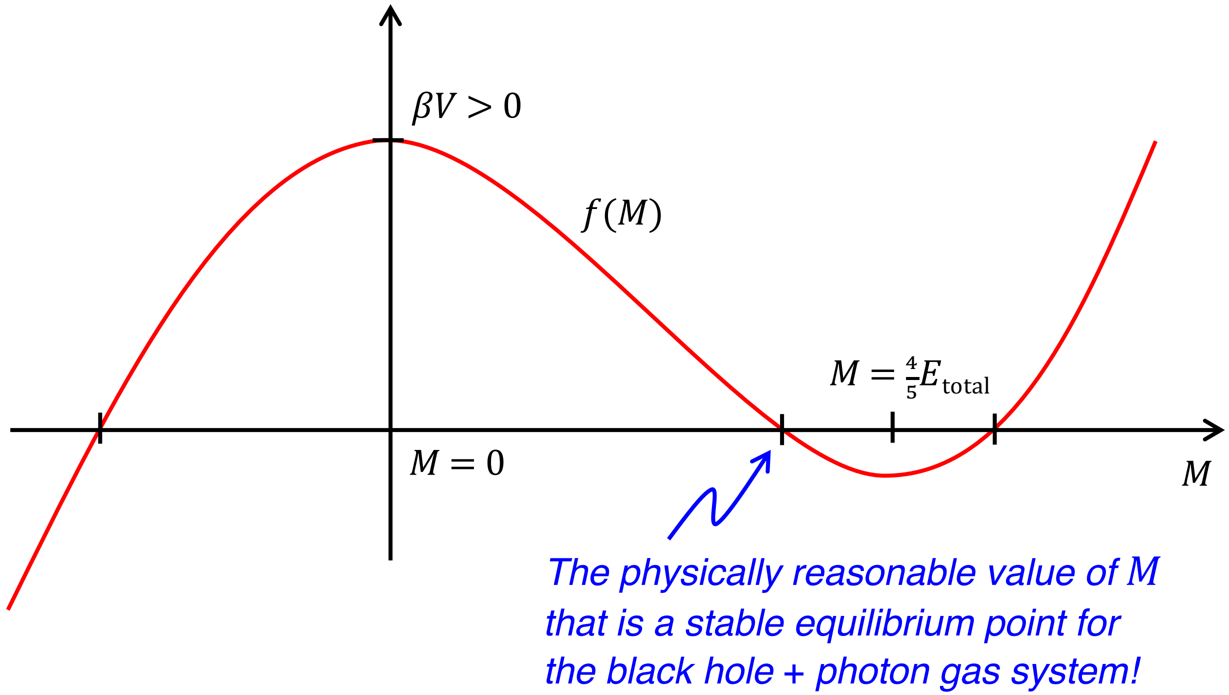
$$\frac{dT_{gas}}{dM} = \frac{d}{dM}\left(\left[\frac{E_{\text{total}} - M}{\frac{\pi^2}{15}V}\right]^{1/4}\right) = -\frac{1}{4}\left(\frac{\pi^2}{15}V\right)^{-1/4}(E_{\text{total}} - M)^{-3/4}$$

So Condition (2) is: $$-\frac{1}{8\pi M^2} < -\frac{1}{4}\left(\frac{\pi^2}{15}V\right)^{-1/4}(E_{\text{total}} - M)^{-3/4}$$

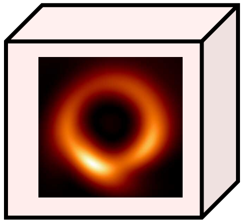
Or: $$\frac{1}{4^4}\frac{M^8}{(E_{\text{total}} - M)^3} < \beta V$$

  - For physically reasonable values of M, this becomes: $$\frac{1}{4^4}\frac{M^8}{(E_{\text{total}} - M)^3} \leq 0.082 E_{\text{total}}^5 \quad \text{or} \quad M < \tfrac{4}{5}E_{\text{total}}$$

- *Case 1*: $f(\frac{4}{5}E_{\text{total}}) < 0,\ 0 < \beta V < 0.082 E_{\text{tot}}{}^5$



3. A Black Hole Carnot Cycle



- *So*: It looks like a black hole *can* be in equilibrium with a photon gas, and hence potentially undergo a reversible process.

- *Next task*: Describe such a process.
  - *Construct a Carnot cycle using a black hole and photon gas as the working fluid.*

- *Slight catch*: Black hole + photon gas working fluid has negative heat capacity.

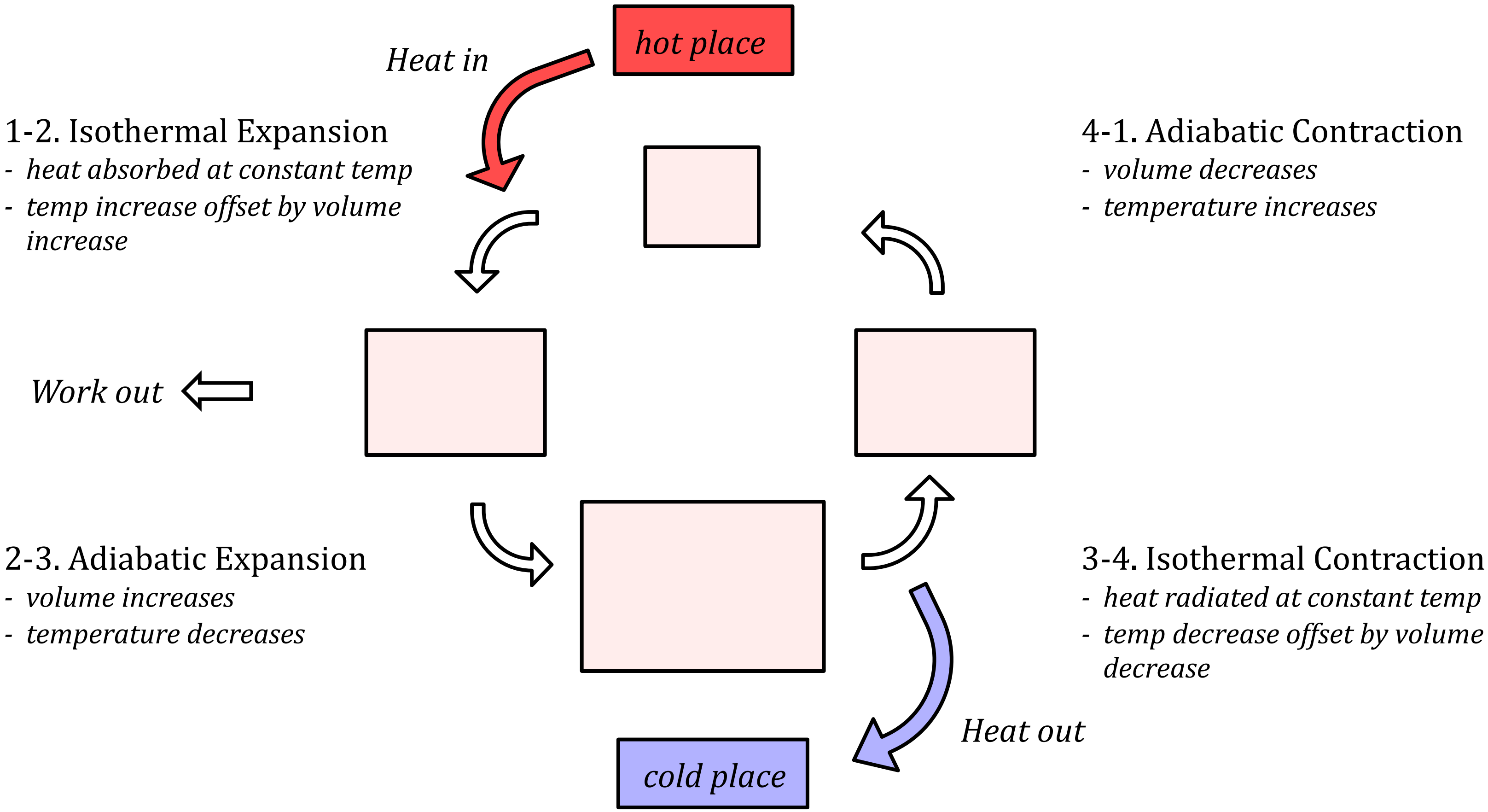
To use it to power a heat engine, need to reverse the standard (positive heat capacity working fluid) Carnot cycle.

Standard Carnot cycle heat engine

Positive heat capacity working fluid:

- *heat absorbed → temp increase*
- *heat radiated → temp decrease*

Heat engine: takes heat from a hot place to a cold place and produces work

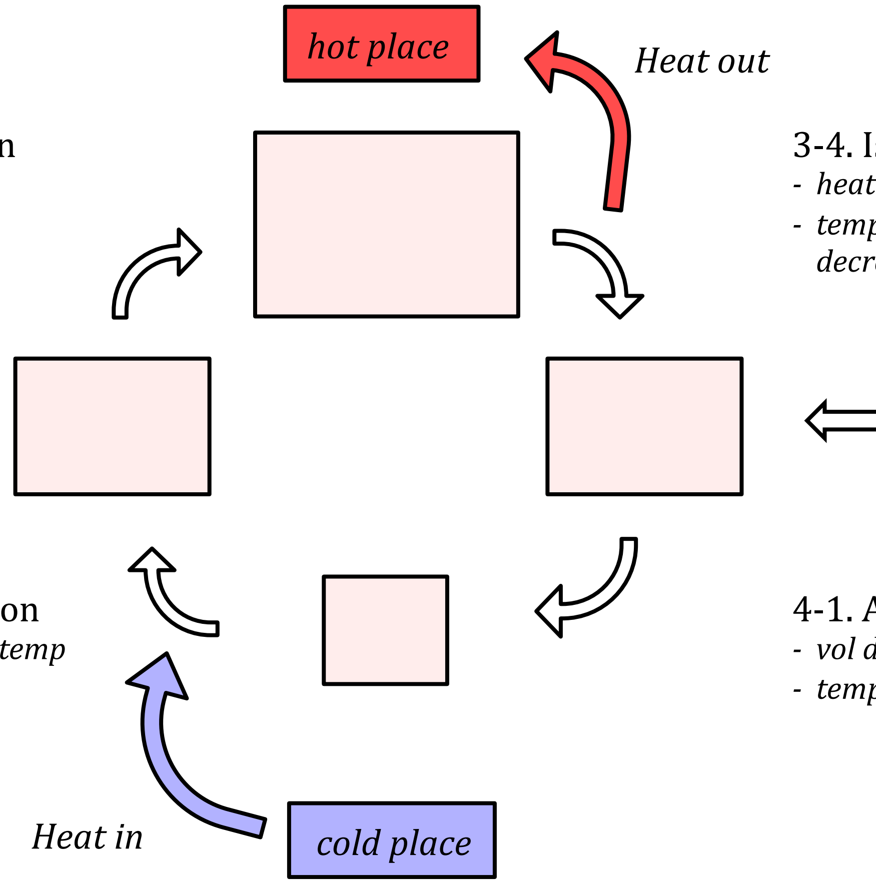


Standard Carnot cycle refrigerator

Refrigerator: takes heat from a hot place to a cold place with work input

Positive heat capacity working fluid:

- *heat absorbed → temp increase*
- *heat radiated → temp decrease*



2-3. Adiabatic Expansion

- *vol increases*
- *temp decreases*

3-4. Isothermal Contraction

- *heat radiated at constant temp*
- *temp decrease offset by vol decrease*

Work in

- *to decrease volume (squeezing heat out)*

1-2. Isothermal Expansion

- *heat absorbed at constant temp*
- *temp increase offset by vol increase*

4-1. Adiabatic Contraction

- *vol decreases*
- *temp increases*

A reverse positive heat capacity heat engine!

Negative heat capacity Carnot cycle heat engine

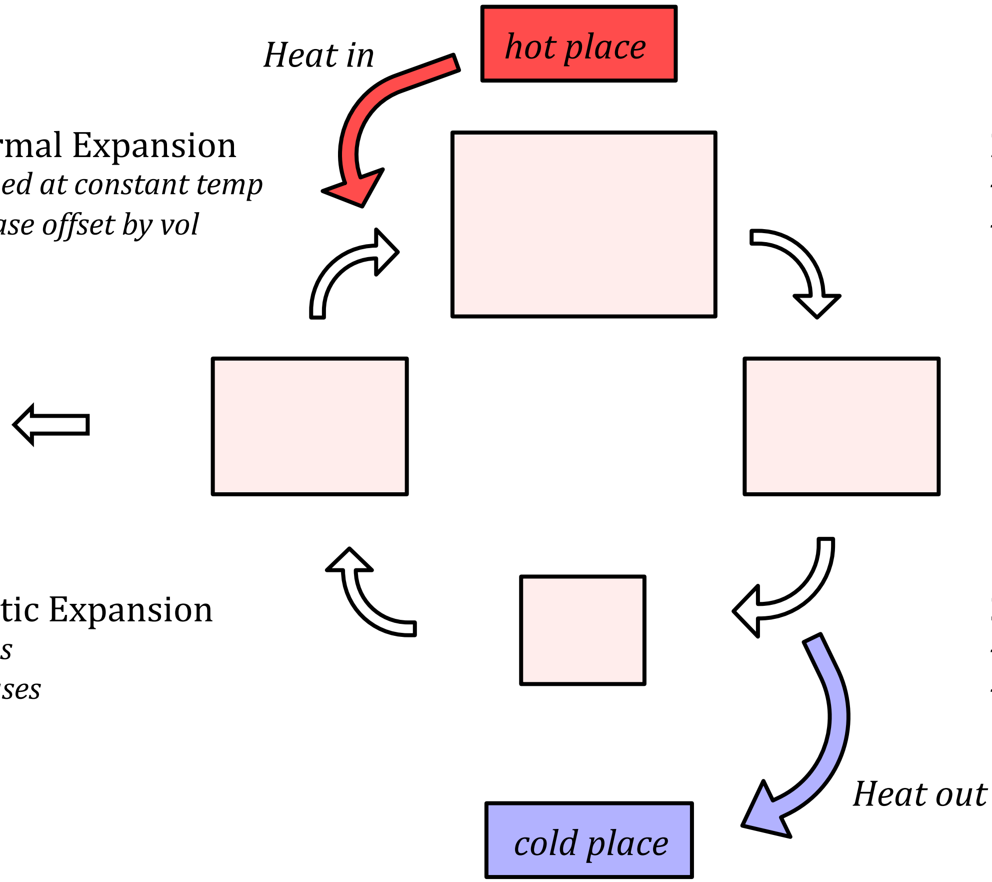
- *heat absorbed → temp decrease*
- *heat radiated → temp increase*

Heat engine: takes heat from a hot place to a cold place and produces work

1-2. Isothermal Expansion
- *heat absorbed at constant temp*
- *temp decrease offset by vol increase*

2-3. Adiabatic Contraction
- *vol decreases*
- *temp decreases*

Work out



4-1. Adiabatic Expansion
- *vol increases*
- *temp increases*

3-4. Isothermal Contraction
- *heat radiated at constant temp*
- *temp increase offset by vol decrease*

A "reverse" positive heat capacity refrigerator!

Black hole + photon gas Carnot cycle heat engine

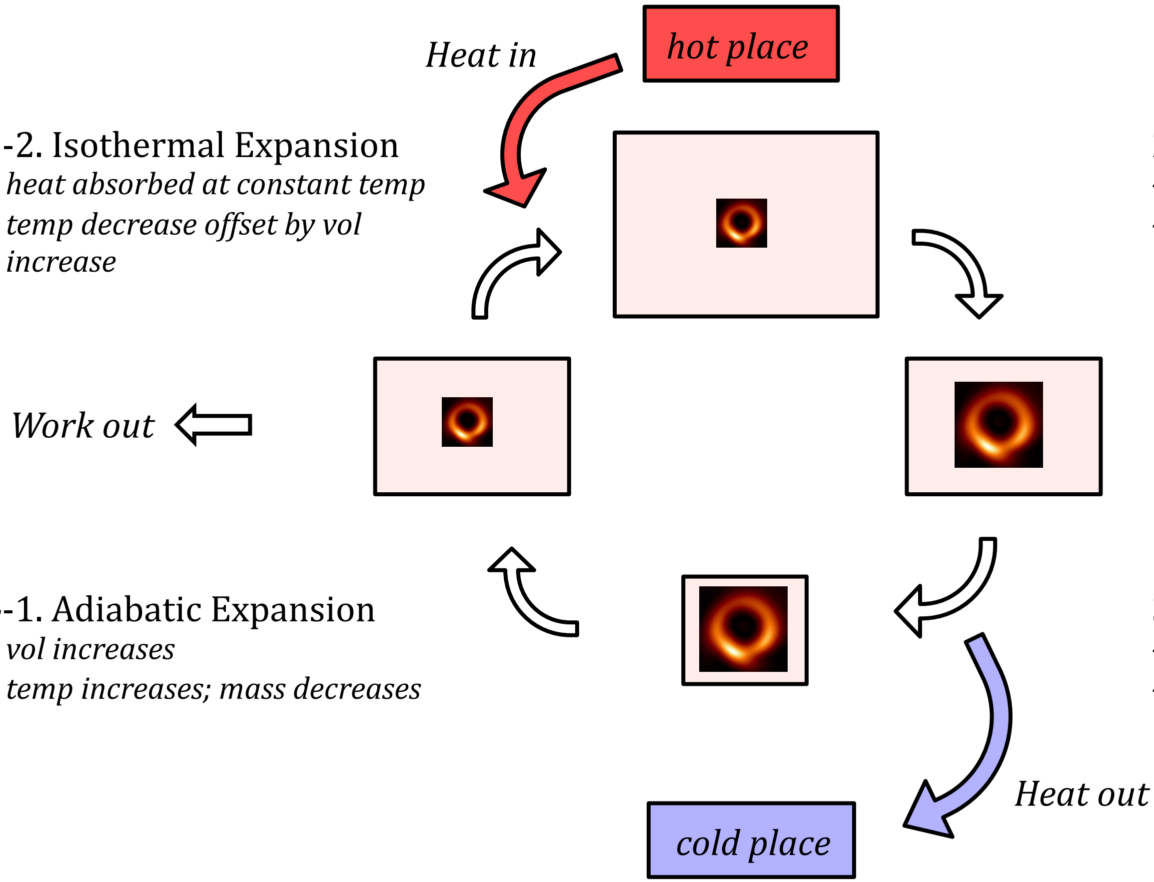
- *heat absorbed → temp decrease; mass increase*
- *heat radiated → temp increase; mass decrease*

$T_{\mathrm{BH}} = 1/8\pi M$

Heat engine: takes heat from a hot place to a cold place and produces work

1-2. Isothermal Expansion
- *heat absorbed at constant temp*
- *temp decrease offset by vol increase*

2-3. Adiabatic Contraction
- *vol decreases*
- *temp decreases; mass increases*



4-1. Adiabatic Expansion
- *vol increases*
- *temp increases; mass decreases*

3-4. Isothermal Contraction
- *heat radiated at constant temp*
- *temp increase offset by vol decrease*

Tasks:

(a) *Show that efficiency* $\mu = (Q_{\mathrm{in}} - Q_{\mathrm{out}})/Q_{\mathrm{in}} = 1 - T_2/T_1$

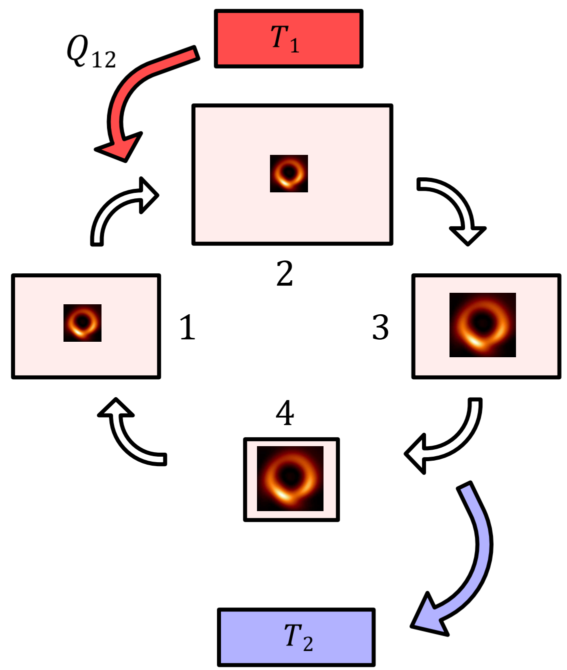
(b) *Show that* S_{TD} *of BH subsystem is* $1/16\pi T^2 = 4\pi M^2 = A/4 = S_{\mathrm{BH}}$

Black hole + photon gas Carnot cycle

$$E_{\text{BH+gas}} = U_{\text{BH+gas}} = M + \alpha V T^4, \quad \alpha = \frac{\pi^2}{15}$$
$$S_{\text{gas}} = \int dU_{\text{gas}}/T = \int (4\alpha V T^3 dT)/T = \int 4\alpha V T^2 dT = \tfrac{4}{3}\alpha V T^3$$
$$P_{\text{gas}} = \tfrac{1}{3}(U_{\text{gas}}/V) = \tfrac{1}{3}\alpha T^4$$

(1→2) *Isothermal Expansion*

$V_1 \rightarrow V_2$ at constant $T = T_1$



- ***Note***: $$\delta Q_{12} = dU_{\text{BH+gas}} + P_{\text{gas}} dV$$ $$= (\partial U_{\text{BH+gas}}/\partial V)_T dV + \tfrac{1}{3}\alpha T_1^4 dV$$ $$= \alpha T_1^4 dV + \tfrac{1}{3}\alpha T_1^4 dV$$

- ***So***: $$\boxed{Q_{12} = \int_{V_1}^{V_2} \left(\alpha T_1^{\,4} + \tfrac{1}{3}\alpha T_1^{\,4}\right) dV = \tfrac{4}{3}\alpha T_1^{\,4}(V_2 - V_1)}$$

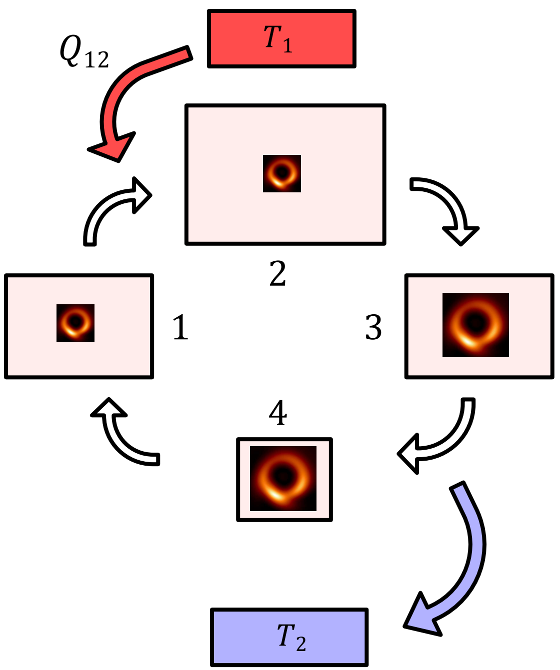
Heat absorbed

- ***And***: $$\boxed{\Delta S_{12} = \int_1^2 (\delta Q_{12}/T_1) = \tfrac{4}{3}\alpha T_1^{\,3}(V_2 - V_1)}$$

Resulting change in entropy

Black hole + photon gas Carnot cycle

$$E_{\text{BH+gas}} = U_{\text{BH+gas}} = M + \alpha V T^4, \quad \alpha = \frac{\pi^2}{15}$$
$$S_{\text{gas}} = \int dU_{\text{gas}}/T = \int (4\alpha V T^3 dT)/T = \int 4\alpha V T^2 dT = \tfrac{4}{3}\alpha V T^3$$
$$P_{\text{gas}} = \tfrac{1}{3}(U_{\text{gas}}/V) = \tfrac{1}{3}\alpha T^4$$



(2→3) Adiabatic Compression

$Q_{23} = 0, V_2 \to V_3, T_1 \to T_2$

- *So*: $\boxed{\Delta S_{23} = \int_2^3 (\delta Q_{23}/T_1) = 0}$

- *Adiabatic relation*: $dU_{\text{BH+gas}} = dW_{\text{gas}}$

$dM + 4\alpha V T^3 dT + \alpha T^4 dV$

$-\frac{1}{3}\alpha T^4 dV =$ work done on gas

Or... $$V T^3 + \frac{3\gamma}{8\alpha T^2} = \text{const.}$$

- $\gamma = \frac{1}{8\pi}$
- For photon gas by itself, adiabatic relation is just $VT^3 = \text{const.}$

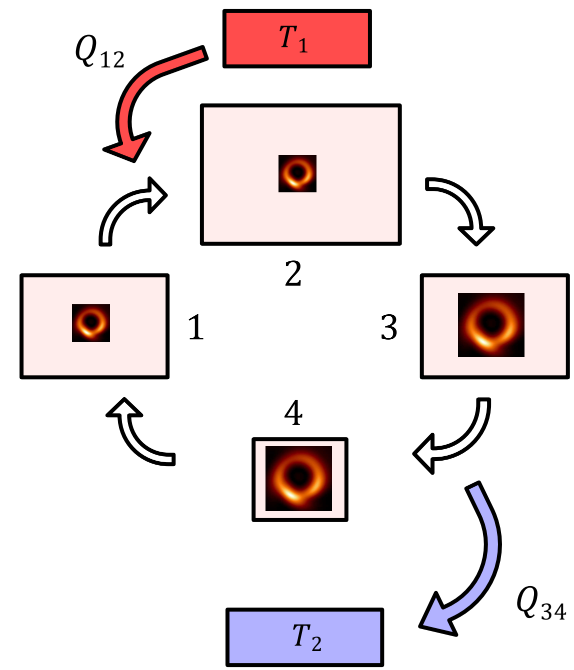
Or... $$\boxed{V_3 {T_2}^3 + \frac{3\gamma}{8\alpha {T_2}^2} = V_2 {T_1}^3 + \frac{3\gamma}{8\alpha {T_1}^2}}$$

Black hole + photon gas Carnot cycle

$$E_{\text{BH+gas}} = U_{\text{BH+gas}} = M + \alpha V T^4, \quad \alpha = \frac{\pi^2}{15}$$
$$S_{\text{gas}} = \int dU_{\text{gas}}/T = \int (4\alpha V T^3 dT)/T = \int 4\alpha V T^2 dT = \tfrac{4}{3}\alpha V T^3$$
$$P_{\text{gas}} = \tfrac{1}{3}(U_{\text{gas}}/V) = \tfrac{1}{3}\alpha T^4$$

(3→4) Isothermal Compression

$V_3 \to V_4$ at constant $T = T_2$



- *In derviation similar to (1→2)*:

$$Q_{34} = \int_{V_3}^{V_4} \left(\alpha {T_2}^4 + \tfrac{1}{3}\alpha {T_2}^4\right) dV = \tfrac{4}{3}\alpha {T_2}^4 (V_4 - V_3)$$

Heat radiated

$$\Delta S_{34} = \int_3^4 (\delta Q_{34}/T_2) = \tfrac{4}{3}\alpha {T_2}^3 (V_4 - V_3)$$

Resulting change in entropy

Black hole + photon gas Carnot cycle

$$E_{\text{BH+gas}} = U_{\text{BH+gas}} = M + \alpha V T^4, \quad \alpha = \frac{\pi^2}{15}$$
$$S_{\text{gas}} = \int dU_{\text{gas}}/T = \int (4\alpha V T^3 dT)/T = \int 4\alpha V T^2 dT = \tfrac{4}{3}\alpha V T^3$$
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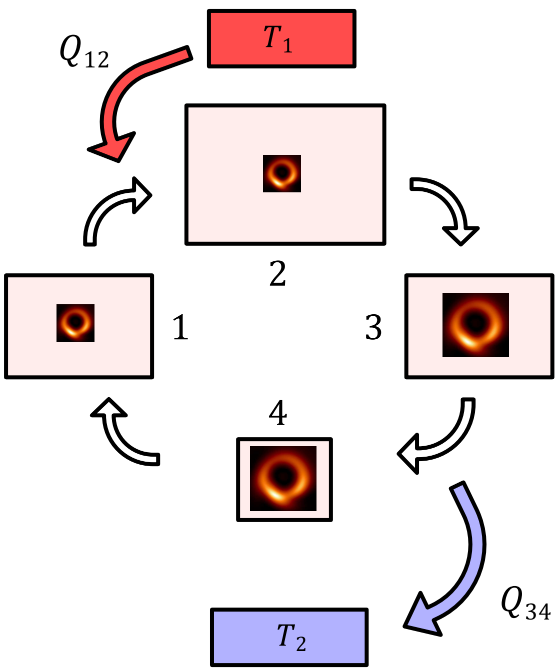


(4→1) Adiabatic Expansion

$Q_{41} = 0, V_4 \to V_1, T_2 \to T_1$

- *So*: $$\boxed{\Delta S_{41} = \int_4^1 (\delta Q_{41}/T_2) = 0}$$

- *Recall adiabatic relation for (2→3)*:

$$V_3 {T_2}^3 + \frac{3\gamma}{8\alpha {T_2}^2} = V_2 {T_1}^3 + \frac{3\gamma}{8\alpha {T_1}^2}$$

- *For (4→1) this becomes*:

$$V_4 {T_2}^3 + \frac{3\gamma}{8\alpha {T_2}^2} = V_1 {T_1}^3 + \frac{3\gamma}{8\alpha {T_1}^2}$$

Combine to get:

$$\boxed{V_4 {T_2}^3 - V_3 {T_2}^3 = V_1 {T_1}^3 - V_2 {T_1}^3}$$

Task (a): Show that efficiency $\mu = (Q_{\mathrm{in}} - Q_{\mathrm{out}})/Q_{\mathrm{in}} = 1 - T_2/T_1$

$$\mu = \frac{Q_{12} - Q_{34}}{Q_{12}}$$

$$= \frac{\frac{4}{3}\alpha {T_1}^4 (V_2 - V_1) - \frac{4}{3}\alpha {T_2}^4 (V_4 - V_3)}{Q_{12}}$$

$$= \frac{\frac{4}{3}\alpha\left[T_1\left(V_2 {T_1}^3 - V_1 {T_1}^3\right)\right] - \frac{4}{3}\alpha\left[T_2\left(V_4 {T_2}^3 - V_3 {T_2}^3\right)\right]}{Q_{12}}$$

$$= \frac{\frac{4}{3}\alpha\left[T_1\left(V_2 {T_1}^3 - V_1 {T_1}^3\right)\right] - \frac{4}{3}\alpha\left[T_2\left(V_1 {T_1}^3 - V_2 {T_1}^3\right)\right]}{Q_{12}}$$

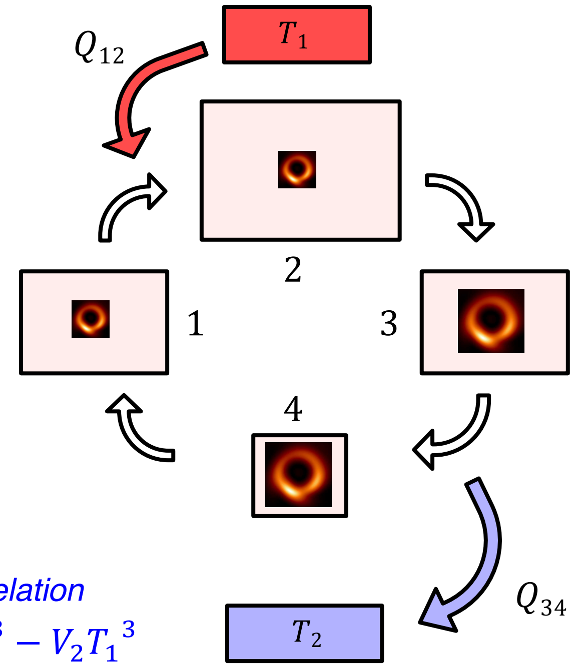
adiabatic relation

$$V_4 {T_2}^3 - V_3 {T_2}^3 = V_1 {T_1}^3 - V_2 {T_1}^3$$

$$= \frac{\frac{4}{3}\alpha {T_1}^4 (V_2 - V_1)\left(1 - \frac{T_2}{T_1}\right)}{Q_{12}}$$

$$= 1 - \frac{T_2}{T_1}$$

Task (a) accomplished!



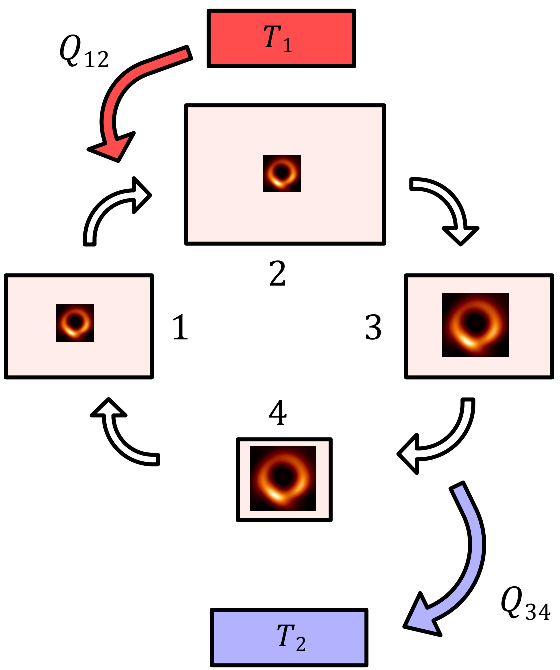
Task (b): *Show that* S_{TD} *of BH subsystem is* $1/16\pi T^2$

- Isothermal processes (1→2), (3→4):

$$\Delta S_{12} = \tfrac{4}{3}\alpha {T_1}^3 (V_2 - V_1)$$

$$\Delta S_{34} = \tfrac{4}{3}\alpha {T_2}^3 (V_4 - V_3)$$

Entropy change in gas only (no change of state of BH)



- Adiabatic processes (2→3), (4→1): $\Delta S_{\mathrm{BH+gas}} = 0$
  - *But: Adiabatic relation of gas by itself is different from adiabatic relation of BH+gas.*
  - *Claim: There must be heat flux from gas to BH during adiabatic compression* (2→3).

- *So*:

$$\Delta S_{\mathrm{BH+gas,23}} = \Delta S_{\mathrm{BH,23}} + \Delta S_{\mathrm{gas,23}}$$
$$= \Delta S_{\mathrm{BH,23}} + \frac{4}{3}\alpha (V_3 T_2^3 - V_2 T_1^3)$$
$$= \Delta S_{\mathrm{BH,23}} + \frac{4}{3}\alpha \left(\frac{3\gamma}{8\alpha {T_2}^2} - \frac{3\gamma}{8\alpha {T_1}^2} \right)$$
$$= \Delta S_{\mathrm{BH,23}} + \frac{1}{2}\gamma \left(\frac{1}{{T_2}^2} - \frac{1}{{T_1}^2} \right) = 0$$

adiabatic relation

$$V_3 {T_2}^3 + \frac{3\gamma}{8\alpha {T_2}^2} = V_2 {T_1}^3 + \frac{3\gamma}{8\alpha {T_1}^2}$$

- *Or*: $\Delta S_{\mathrm{BH,23}} = \frac{1}{2}\gamma \left(\frac{1}{{T_1}^2} - \frac{1}{{T_2}^2} \right)$

- *In general*: $S_{\mathrm{BH}} = \frac{1}{2}\gamma (1/T^2) = 1/16\pi T^2$

Task (b) accomplished!