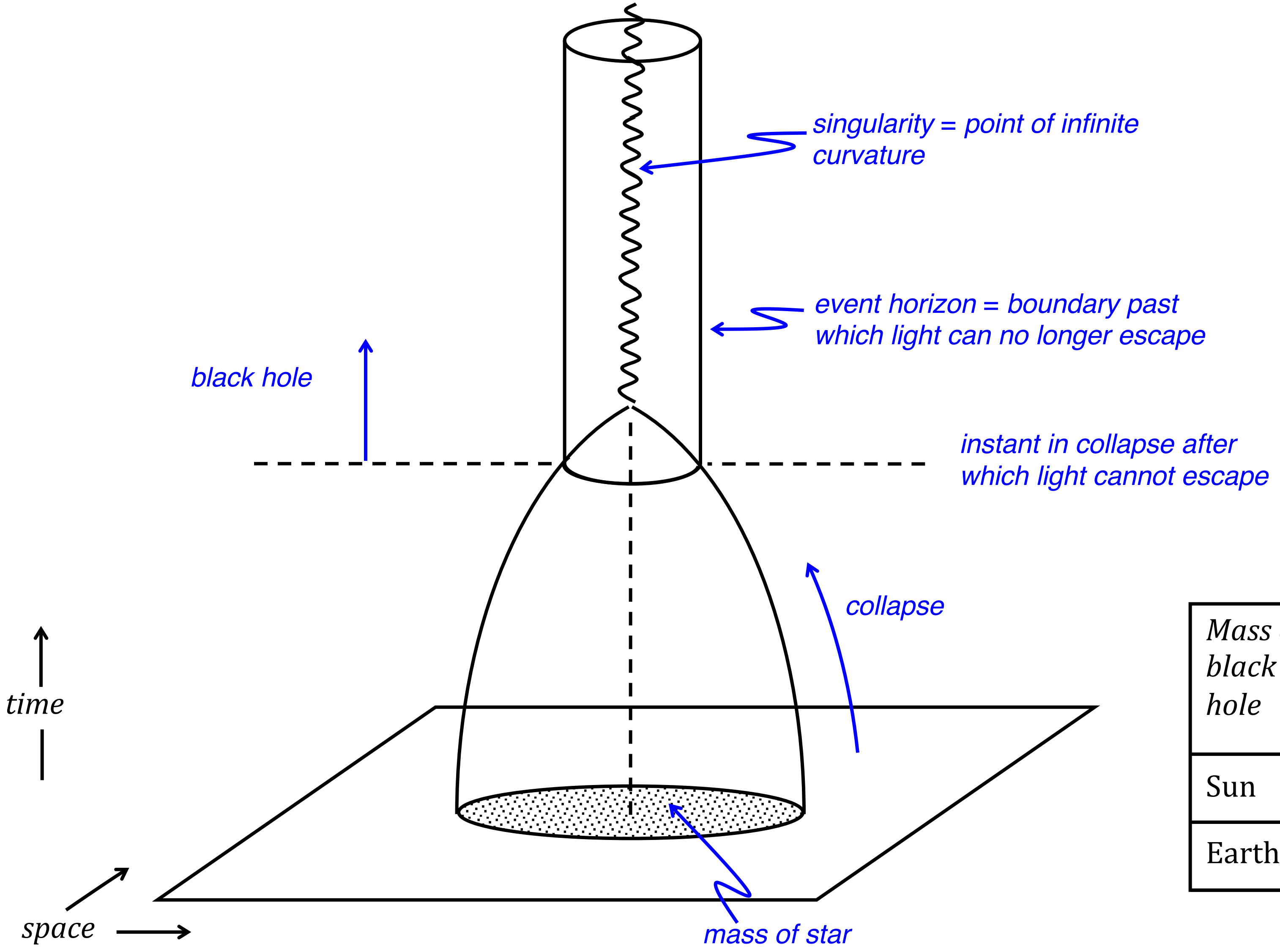
13. Black Hole Thermodynamics

1. General Properties
2. Laws of B.H. Mechanics
3. Area & Entropy
4. Surface Gravity & Temp

1. General Properties of Relativistic Black Holes

<u>*Simplest case*</u>: Schwarzschild black hole (no charge or rotation).



Mass of black hole	*Radius of event horizon*
Sun	2.95 *km*
Earth	8.86 *mm*

- *No Hair Conjecture*: A black hole is completely characterized by its mass M, charge Q, and angular momentum J.

Four types of black hole:		
	non-rotating ($J = 0$)	*rotating* ($J \neq 0$)
uncharged ($Q = 0$)	Schwarzschild	Kerr
charged ($Q \neq 0$)	Reissner-Nordström	Kerr-Newman

Properties of event horizon

- *radius*: $R = M + \sqrt{M^2 - Q^2 - (J/M)^2}$
- *area*: $A = 4\pi[R^2 + (J/M)^2]$
- *surface gravity*: $\kappa = \sqrt{M^2 - Q^2 - (J/M)^2}/2MR$

acceleration needed to keep an object at event horizon

Schwarzschild black hole

$R = 2M$

$A = 4\pi R^2$

$\kappa = 1/4M$

Area Theorem. (Hawking 1971)

$\Delta A \geq 0$ in any process.

Looks like 2nd Law of Thermodynamics!

Stephen Hawking (1942-2018)

- *Ex*. Suppose two black holes with areas A_1, A_2 collide to form black hole with area A_3. Then $A_3 \geq A_1 + A_2$.

2. The Laws of Black Hole Mechanics (Bardeen, Carter, Hawking 1973)

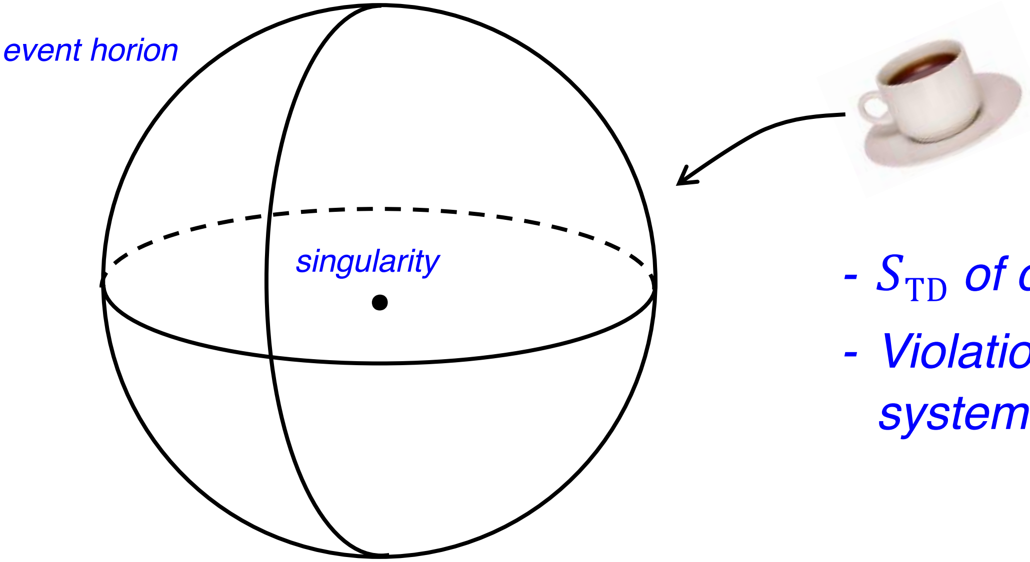
	Black Hole Mechanics	*Thermodynamics*
0th Law	Surface gravity κ is constant over the event horizon of a stationary black hole.	Temperature T is constant throughout a body in thermal equilibrium.
1st Law	$\Delta M = (1/8\pi)\kappa\Delta A + \cdots$ *M = mass = energy*	$\Delta U = T\Delta S_{\mathrm{TD}} + \cdots$ *U = energy*
2nd Law	$\Delta A \geq 0$ in any process.	$\Delta S_{\mathrm{TD}} \geq 0$ in any process. *S_{TD} = thermodynamic entropy*
3rd Law	$\kappa = 0$ is not achievable by any process.	$T = 0$ is not achievable by any process.

- Formally identical if $A/4 = S_{\mathrm{TD}}$ and $(1/2\pi)\kappa = T$.

Is this merely a formal equivalence, or does it have a physical basis?

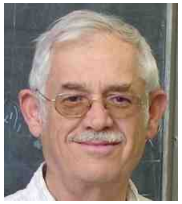
3. Area and Thermodhynamic Entropy

- *Question*: What happens when a physical system with a large amount of thermodynamic entropy falls into a black hole?



- *S_{TD} of coffee cup disappears!*
- *Violation of 2nd Law for closed system of black hole + coffee cup?*

- *Bekenstein (1973)*: Suppose black holes have an "entropy" S_{BH} proportional to their area: $S_{\mathrm{BH}} = A/4$.



Jacob Bekenstein
2012 Wolf Prize
(NYU-Tandon grad!)
(1947-2015)

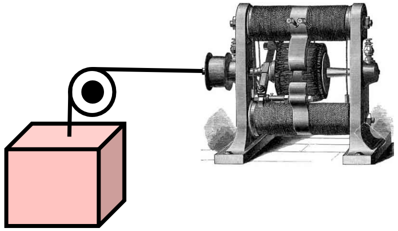
Generalized Second Law of Thermodynamics (GSL)

$$\underset{\text{Always positive!}}{\Delta S_{\mathrm{BH}}} + \underset{\text{Negative for coffee cup!}}{\Delta S_{\mathrm{TD}}} \geq 0$$

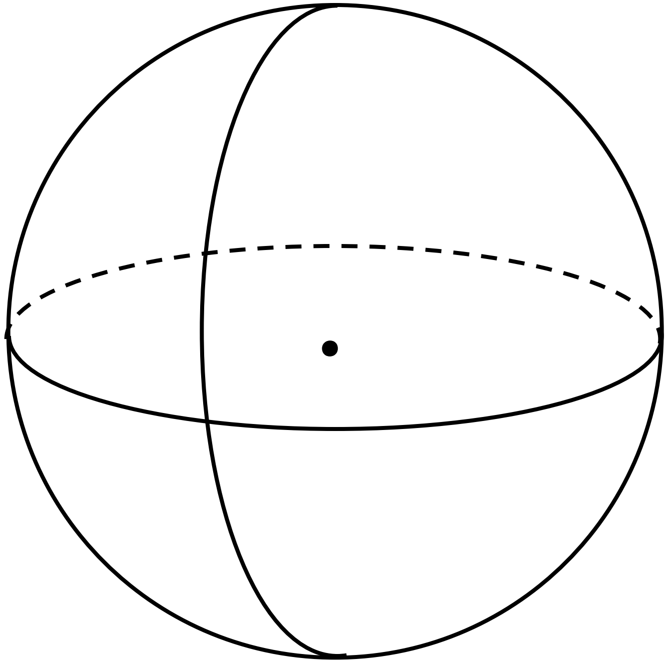
- *Suggests S_{BH} is S_{TD}...*
- *But then why the need for a "generalized" 2^{nd} Law?*

Problem (Geroch 1971)

- Can coffee cup example be made a bit more precise?



1. Lower box of radiation with high entropy toward event horizon.
2. Use weight to generate work.



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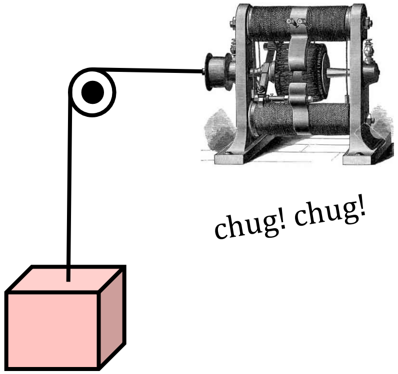

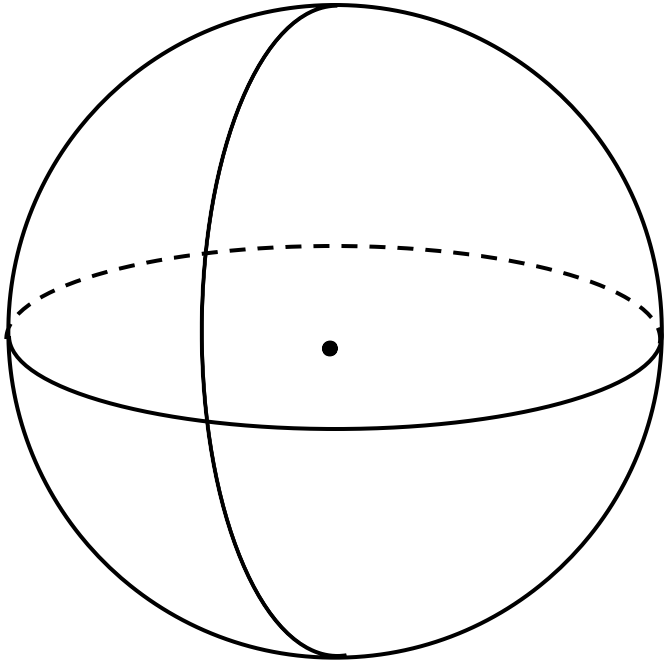

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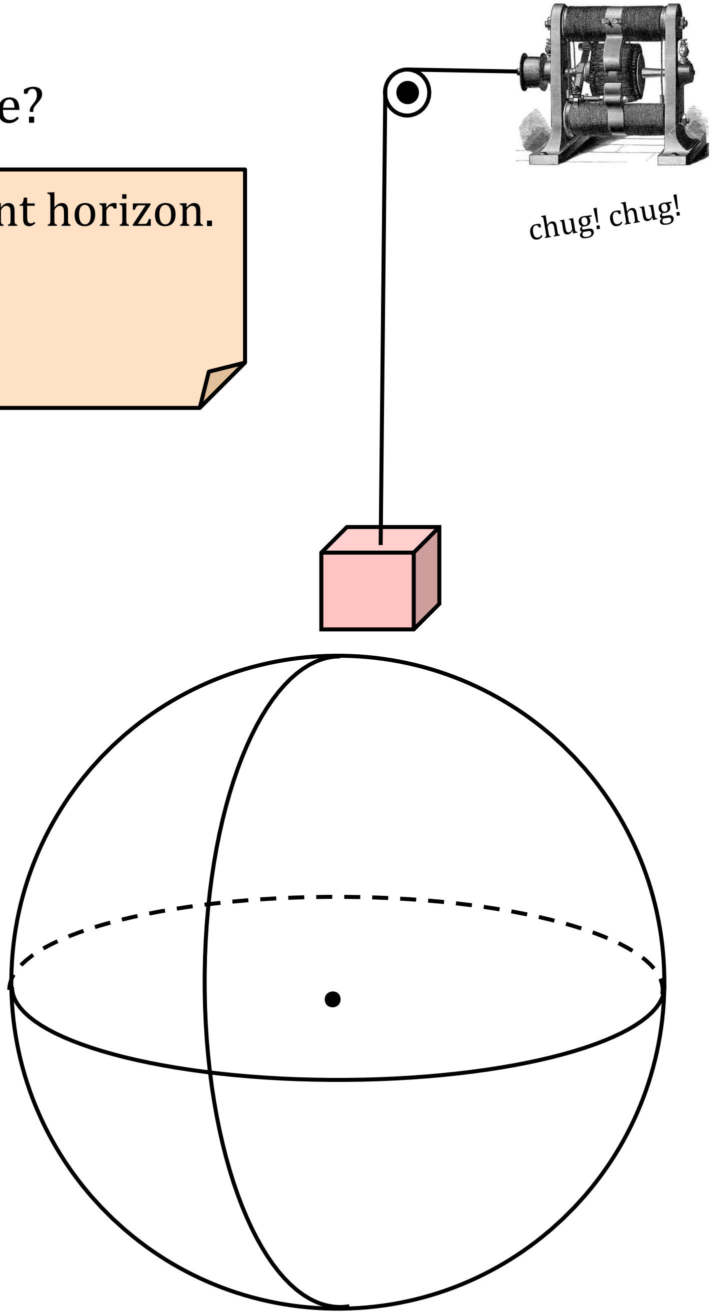

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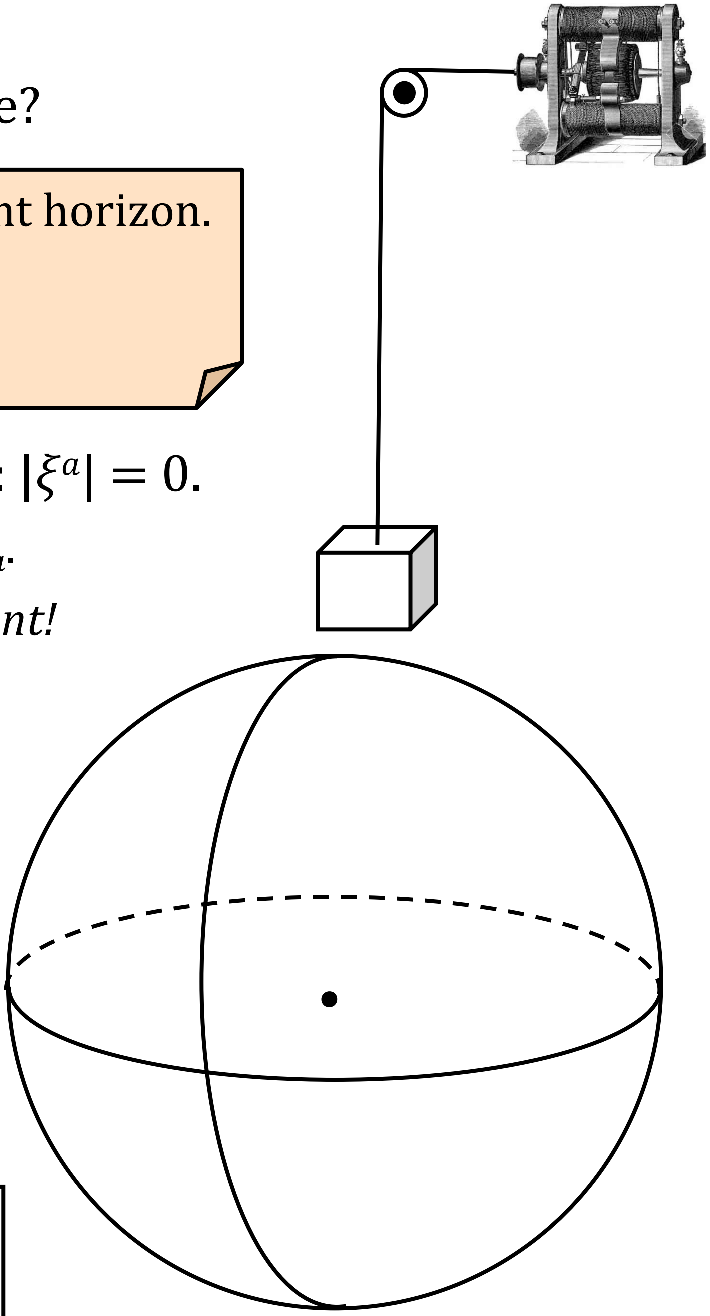


- At event horizon, time-translation vector ξ^a is *null*: $|\xi^a| = 0$.
  - *So*: *At event horizon, the box has zero energy* $E = -\xi^a p_a$.
  - *All of the box's energy has gone into work:* 100% *efficient!*

- *So*: If box can reach horizon, then no increase in area at Step (3).

$\Delta M \propto \kappa \Delta A$

  - *Thus*: $\Delta S_{\text{BH}} = 0$.
  - *But*: $\Delta S_{\text{TD}} < 0$.

Violation of GSL!

- *Thus*: $\Delta S_{\text{BH}} + \Delta S_{\text{TD}} < 0$

Bekenstein's response (1973):

Box has finite size, so it's never entirely at the horizon; hence it's energy is never fully zero.

4. Surface Gravity κ and Temperature T

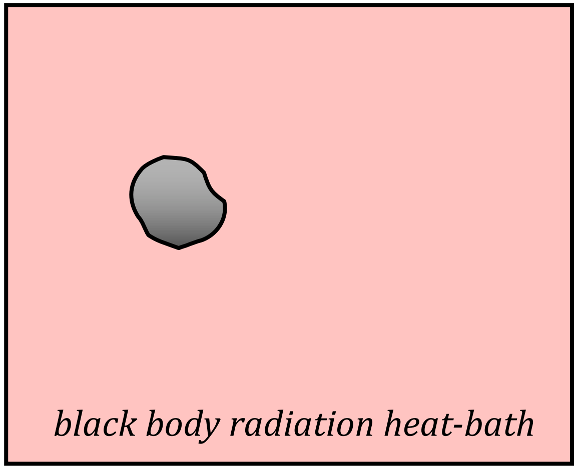
- *Recall*: Laws of Black Hole Mechanics look like Laws of Thermodynamics if we equate surface gravity κ with temperature: $(1/2\pi)\kappa = T$.

How seriously should we take this?

Claim 1. A black hole should be assigned *zero* "effective" temperature.

Proof:

To measure the "effective" temperature of an object, put the object in thermal equilibrium with black body radiation and measure temperature of latter.



- *Object in equilibirum with heat bath.*
- $T_{object} = T_{heat\text{-}bath}$

- *Black body* = object that absorbs all incident radiation.
- *Black body radiation* = radiation emitted by a black body in thermal equilibrium.
- *Effective temp* of an object = temp of a black body that emits same total amount of radiation as the object.

Why use a black body to measure effective temp?

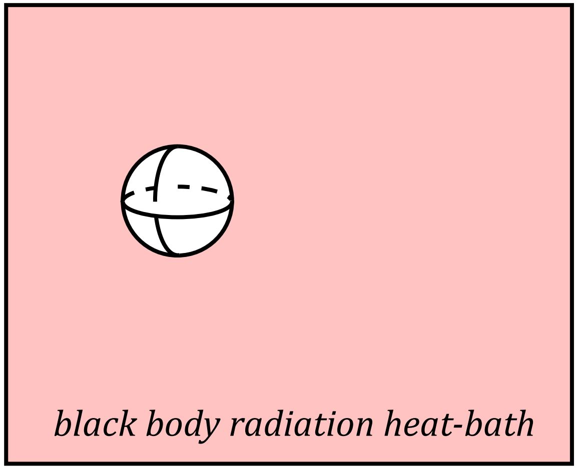
- *A black body in thermal equilibrium is an ideal emitter: No "impurities" in black body radiation that might affect its use as a measure of effective temp.*

Proof, cont.

> "...a black hole cannot be in equilibrium with black body radiation at any non-zero temperature, because *no radiation could be emitted from the hole* whereas some radiation would always cross the horizon into the black hole." (Bardeen, Carter, Hawking 1973, pg. 168)

- *Conclusion*: "In classical black hole physics, κ has nothing to do with the physical temperature of a black hole..." (Wald 1994, pg. 149)

But: This argument depends on quantum mechanics (black body radiation can only be characterized quantum-mechanically).



- *Black hole in heat bath.*
- *Equilibrium cannot be established.*

Planck's (1900) quantum-mechanical formula for energy distribution of black body radiation: $E(\nu) = h\nu/(e^{h\nu/kT} - 1)$.

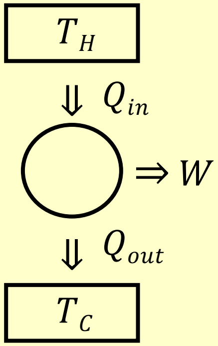
Is there a "classical" proof that a black hole must have zero temperature?

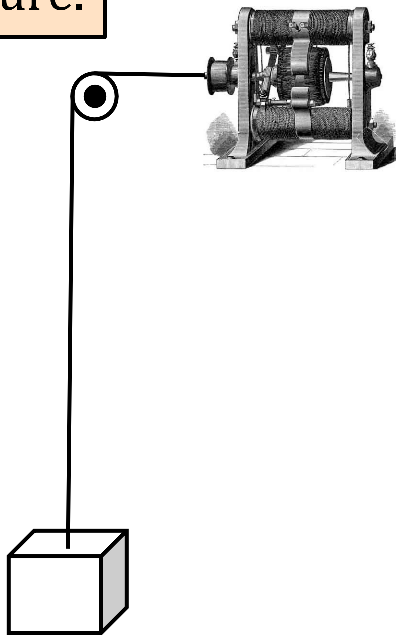
Claim 2. A black hole should be assigned zero *absolute* temperature.

Proof: Consider "Geroch heat engine" again:

- $T_H =$ temperature of box at initial position.
- $T_C =$ temperature of black hole.
- $Efficiency = W/Q_{in} = (Q_{in} - Q_{out})/Q_{in}$

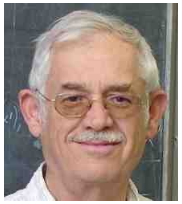
$$\begin{aligned} &= 1 - Q_{out}/Q_{in} \\ &= 1 - T_C/T_H \\ &= 1 \quad \textit{if} \text{ all energy of box goes into work} \end{aligned}$$



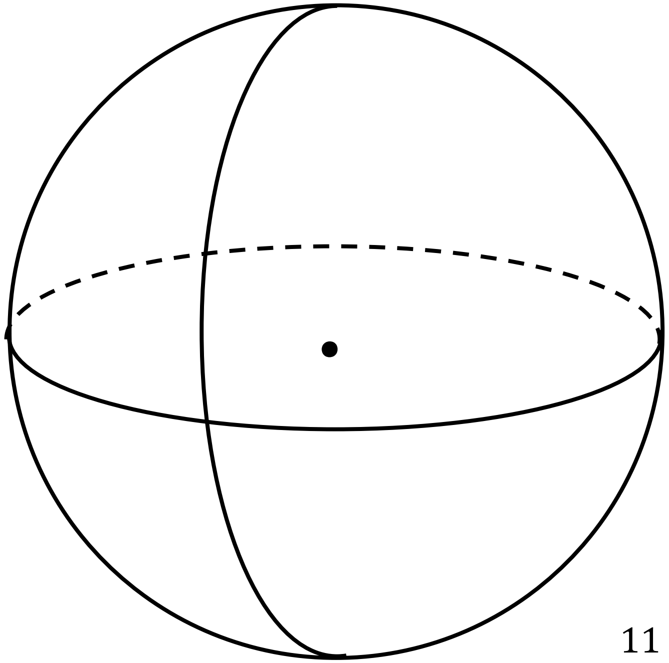


- *So*: $T_C = 0$, *if* all energy of box goes into work.
- *In other words*: $T_C = 0$, *if* box can reach horizon.

But (again):



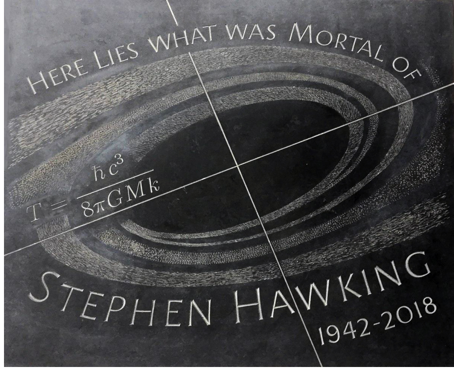
Finite box can't reach horizon! (Bekenstein 1973)



Lingering concern: Can a black hole be in equilibrium with its environment?

4. Hawking Radiation.

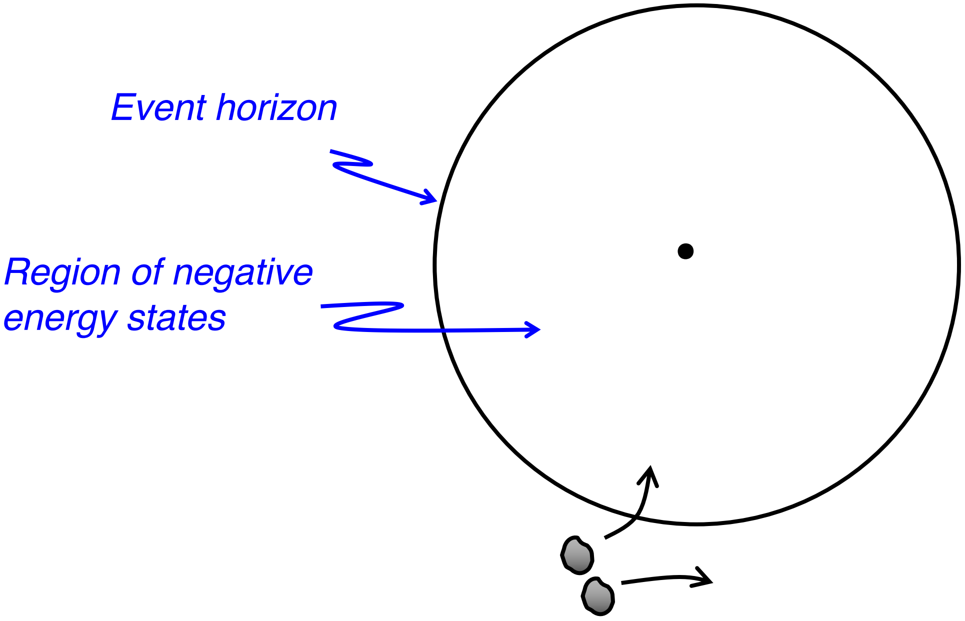
- *Hawking (1975)*: Black holes emit radiation at the same rate that a black body would at temperature $T = (1/2\pi)\kappa$*!*



Memorial stone, Westminster Abbey

$T = \hbar c^3/8\pi GMk$
$= (2\pi/h)c^3/8\pi GMk$
$= (1/2\pi)\kappa$
in units in which $h = c = G = k = 1$, and $\kappa = 1/4M$ for Schwarzschild black hole.

> "One might picture this...in the following way. Just outside the event horizon there will be virtual pairs of particles, one with negative energy and one with positive energy. The negative particle is in a region which is classically forbidden but it can tunnel through the event horizon to the region inside the black hole where the Killing vector which represents time translations is spacelike. In this region the particle can exist as a real particle with a timelike momentum vector even though its energy relative to infinity as measured by the time translation Killing vector is negative. The other particle of the pair, having a positive energy, can escape to infinity where it constitutes a part of the thermal emission described above. The probability of the negative energy particle tunnelling through the horizon is governed by the surface gravity κ since this quantity measures the gradient of the magnitude of the Killing vector or, in other words, how fast the Killing vector is becoming spacelike."



- *Particle/antiparticle pair production in quantum vacuum near event horizon.*

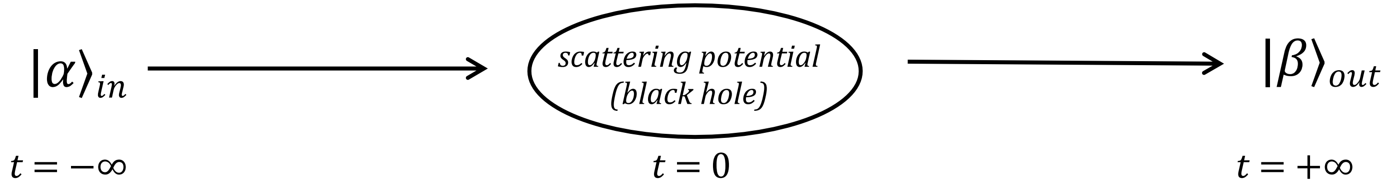
- *Negative energy antiparticle falls through event horizon and falls into singularity, decreasing black hole's area.*

- *Positive energy particle escapes in form of thermal radiation.*

"It should be emphasized that these pictures of the mechanism responsible for the thermal emission and area decrease are heuristic only and should not be taken too literally... The real justification of the thermal emission is the mathematical derivation..."

Technical Aside. *"...the mathematical derivation..."*

Black hole acts as scattering potential for particle states of a quantum field φ.



$|\alpha\rangle_{in}$ → *scattering potential (black hole)* → $|\beta\rangle_{out}$

$t = -\infty$ $t = 0$ $t = +\infty$

Particle states in distant past:

- Expand φ in basis $\{f_\omega\}$ of positive frequency solutions with respect to past:

$$\varphi = \int d\omega (a_\omega f_\omega + a_\omega^\dagger f_\omega^*)$$

- $a_\omega^\dagger$, a_ω are raising/lowering operators for "in" particle states.
- "In" vacuum $|0\rangle_{in}$ = state with no "in" particles.

Particle states in distant future:

- Expand φ in basis $\{p_\omega, q_\omega\}$, where p_ω are +freq solutions w.r.t. future, and q_ω are solutions w.r.t. event horizon:

$$\varphi = \int d\omega (b_\omega p_\omega + b_\omega^\dagger p_\omega^* + c_\omega q_\omega + c_\omega^\dagger q_\omega^*)$$

- $b_\omega^\dagger$, b_ω are raising/lowering operators for "out" particle states.
- "Out" vacuum $|0\rangle_{out}$ = state with no "out" particles.

The Main Result:

$${}_{in}\langle 0|b_\omega^\dagger\, b_\omega|0\rangle_{in} = \frac{\Gamma_\omega}{e^{2\pi\omega/\kappa} - 1}$$

number of "out" particles in "in" vacuum

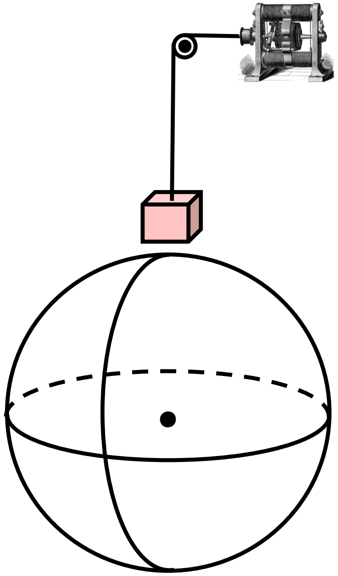
energy distribution of black body radiation with temperature $\kappa/2\pi$

- *So*: "In" vacuum of a quantum field in region of black hole is full of black body radiation at temperature $\kappa/2\pi$!

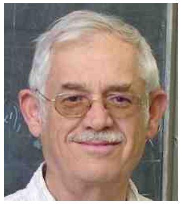
- *But*: $|0\rangle_{in}$ and $|0\rangle_{out}$ belong to *unitarily inequivalent representations* of the quantum field.
- *Which means*: It's mathematically incoherent to write ${}_{in}\langle 0|b_\omega^\dagger\, b_\omega|0\rangle_{in}$.

Ooops!

Claim. Hawking radiation prevents Geroch heat engine from violating Generalized Second Law. (Unruh and Wald 1982)



- *Recall*: *If* box can reach horizon, then $\Delta S_{\text{BH}} = 0$, $\Delta S_{\text{TD}} < 0$, and thus $\Delta S_{\text{BH}} + \Delta S_{\text{TD}} < 0$. *Violation of GSL!*
- *But*: Hawking radiation generates *buoyancy* that prevents box from reaching horizon!
  - *Hawking radiation justifies Bekenstein's conjecture!*



Finite box can't reach horizon!
(Bekenstein 1973)

So: Should S_{BH} be equated with S_{TD}?

Concerns

- Hawking's derivation of black hole radiation and temperature involves a semi-classical mash-up of quantum field theory and general relativity.

Eventually leads to the "Black Hole Information Loss Paradox"!!

- The formal similarity between the laws of black hole mechanics and the laws of thermodynamics just supports an analogy.

We can model a pendulum or an LC circuit as a harmonic oscillator, but does that mean a pendulum and an LC circuit literally are harmonic oscillators?

- S_{TD} involves reversible quasi-static processes involving transitions between equilibrium states. Can a black hole undergo such processes?

If $T_{\mathrm{BH}} = (1/2\pi)\kappa$, then black holes have negative heat capacity:

- T_{BH} is inversely proportional to mass.

- As mass/energy increases, temp decreases!

- So how can a black hole be in equilibrium with a heat reservoir?