12. Von Neumann Entropy vs. Thermodynamic Entropy

$$S_{\text{vN}}(\rho) = -\text{Tr}(\rho \ln \rho) = -\sum_i p_i \ln p_i$$

A measure of the degree to which the density operator state ρ is mixed.

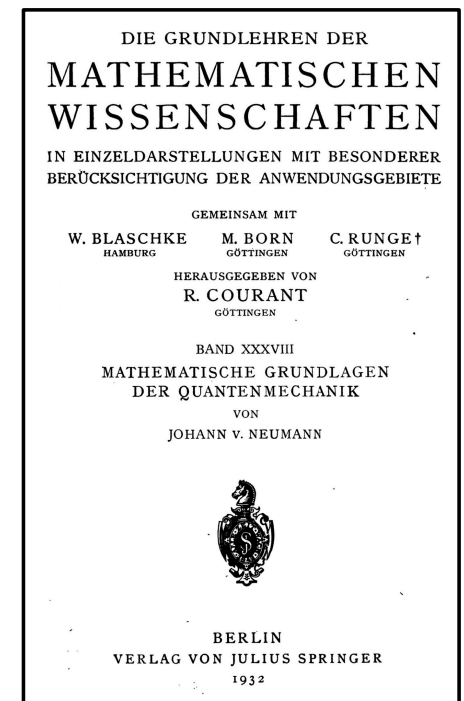$$S_{\text{TD}}(\sigma_2) \equiv \int_{\sigma_1}^{\sigma_2} \frac{\delta Q_R}{T} + S_0$$

The ratio of the change in heat to temperature of a reversible process that connects an initial state σ_1 to a final state σ_2.

- *Claim*:

  S_{vN} is the quantum mechanical generalization of S_{TD}!

John von Neumann
(1903-1957)

DIE GRUNDLEHREN DER

MATHEMATISCHEN
WISSENSCHAFTEN

IN EINZELDARSTELLUNGEN MIT BESONDERER
BERÜCKSICHTIGUNG DER ANWENDUNGSGEBIETE

GEMEINSAM MIT

W. BLASCHKE   M. BORN   C. RUNGE†
HAMBURG   GÖTTINGEN   GÖTTINGEN

HERAUSGEGEBEN VON
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MATHEMATISCHE GRUNDLAGEN
DER QUANTENMECHANIK
VON
JOHANN v. NEUMANN

BERLIN
VERLAG VON JULIUS SPRINGER
1932

(a) A quantum gas in an initial pure state that transitions to a mixed state upon measurement, and hence undergoes an increase in S_{vN}.

(b) Gas then goes through a reversible expansion and compression process that brings it back to initial state, and produces a decrease in S_{TD}.

Claim: There must be a compensating increase in S_{TD} to offset (b), and this occurs in (a) if we equate S_{vN} with S_{TD}.

Issues to consider:

#1. Can a quantum measurement be characterized by a transition between a pure state and a mixed state?

#2. Can a quantum measurement be characterized by an increase in S_{TD}?

Set-Up In Detail: 7 stage cyclical process
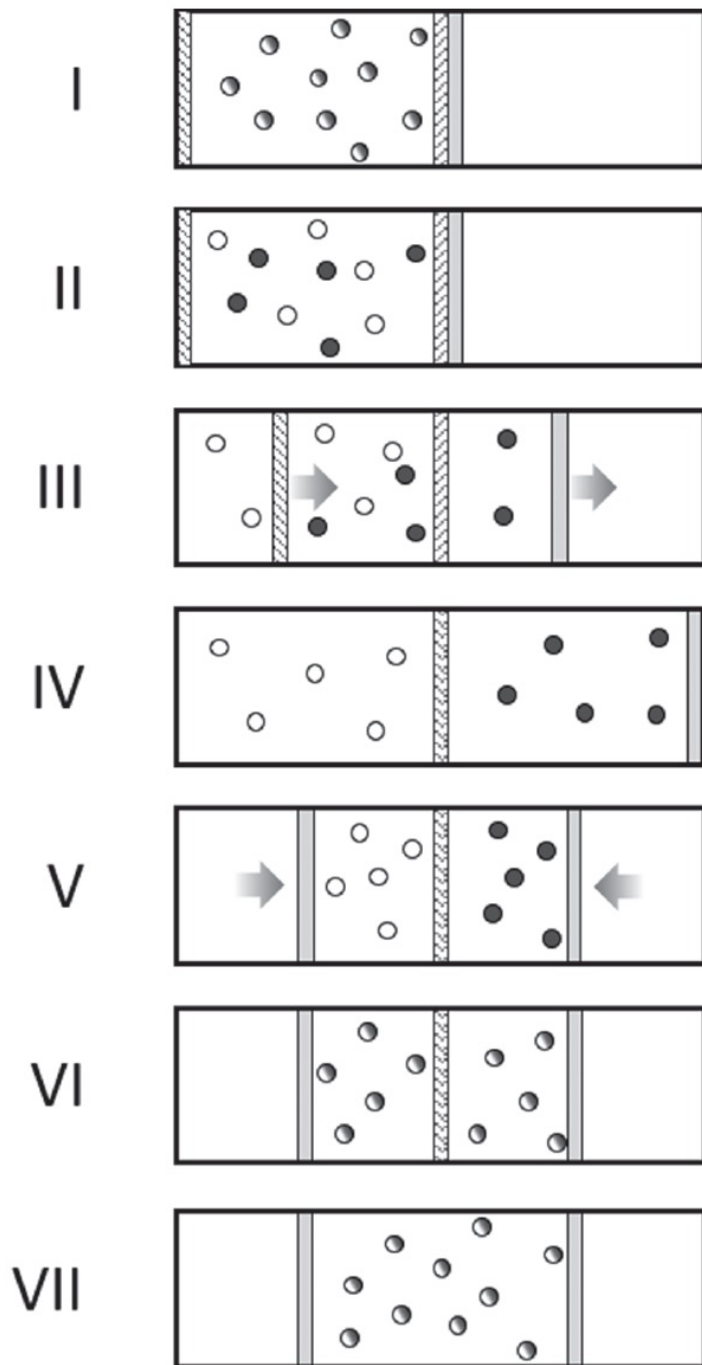
Stage I:

- N-particle gas in left side of chamber with vol $2V_0$ in equilibrium with reservoir at temp T.

- Each particle in vector state:

$$|0\rangle = \sqrt{\tfrac{1}{2}}(|+\rangle + |-\rangle)$$

- *Pure* pre-measurement gas state:

$$\rho_{\text{pre}} = |0_1 0_2 \cdots 0_N\rangle\langle 0_1 0_2 \cdots 0_N|$$

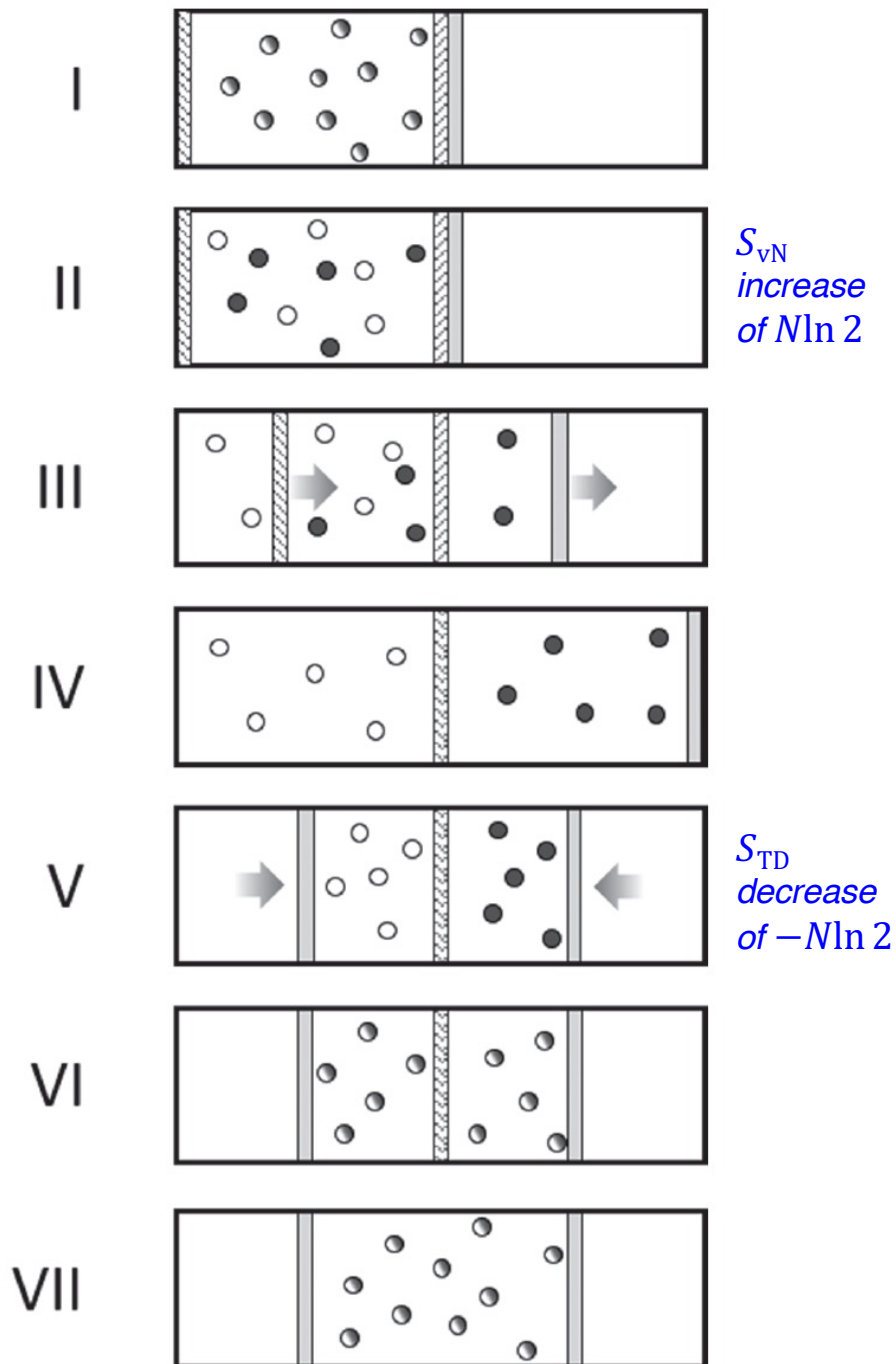- *So:* $\boxed{S_{\text{vN}}(\rho_{\text{pre}}) = 0}$

Stage II:

- Each particle is measured.

- *Maximally mixed* post measurement gas state:

$$\rho_{\text{post}} = \frac{1}{2^N}\{|+_1\cdots+_N\rangle\langle+_1\cdots+_N| + |+_1-_2+_3\cdots+_N\rangle\langle+_1-_2+_3\cdots+_N|$$

$$+\cdots+ |-_1\cdots-_N\rangle\langle-_1\cdots-_N|\}$$

- *So:* $\boxed{S_{\text{vN}}(\rho_{\text{post}}) = \ln 2^N = N\ln 2}$

N-partite vector space $\mathcal{H}$ has dim $= 2^N$

S_{vN} increase of $N\ln 2$

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Set-Up In Detail: *7 stage cyclical process*

I

II
S_{vN}
increase
of $N\ln 2$

III

IV

V
S_{TD}
decrease
of $-N\ln 2$

VI

VII

Stages III-IV:

- Particles are reversibly separated into left and
  right chambers by semi-permeable membrane.
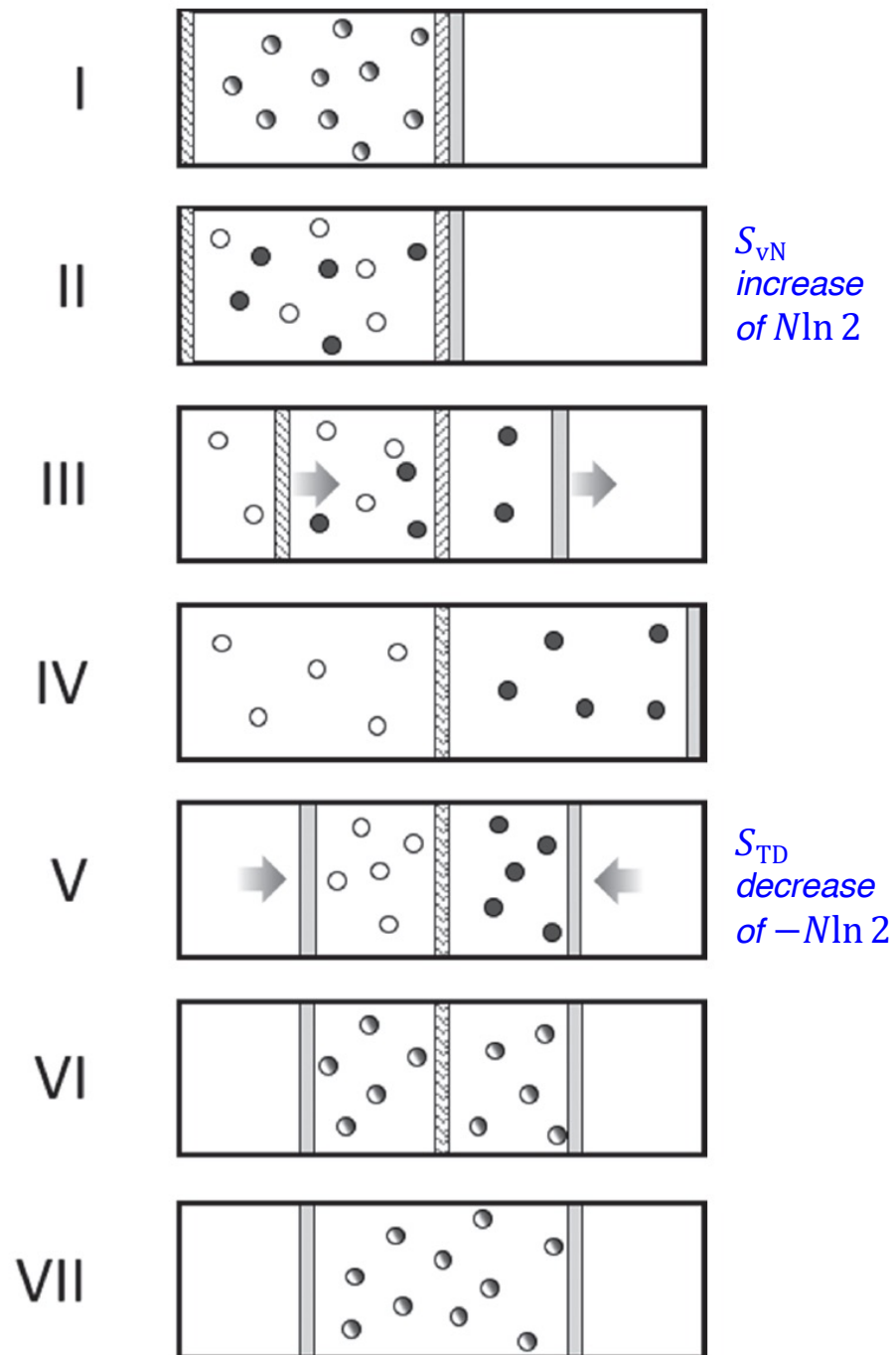
- No work done, no heat exchanged.

Stage V:

- Isothermal compression of left and right
  chambers from $2V_0$ to V_0.

- Work done on gas is:

$$W = -\int_{2V_0}^{V_0} \left(\frac{NT}{V}\right) dV = NT\ln 2$$

- Heat emitted by gas is $Q = -W = -NT\ln 2$

- TD entropy change in gas is:

$$\Delta S_{TD} = \int_{\sigma_1}^{\sigma_2} \frac{\delta Q_R}{T} = \frac{-NT\ln 2}{T} - 0 = -N\ln 2$$
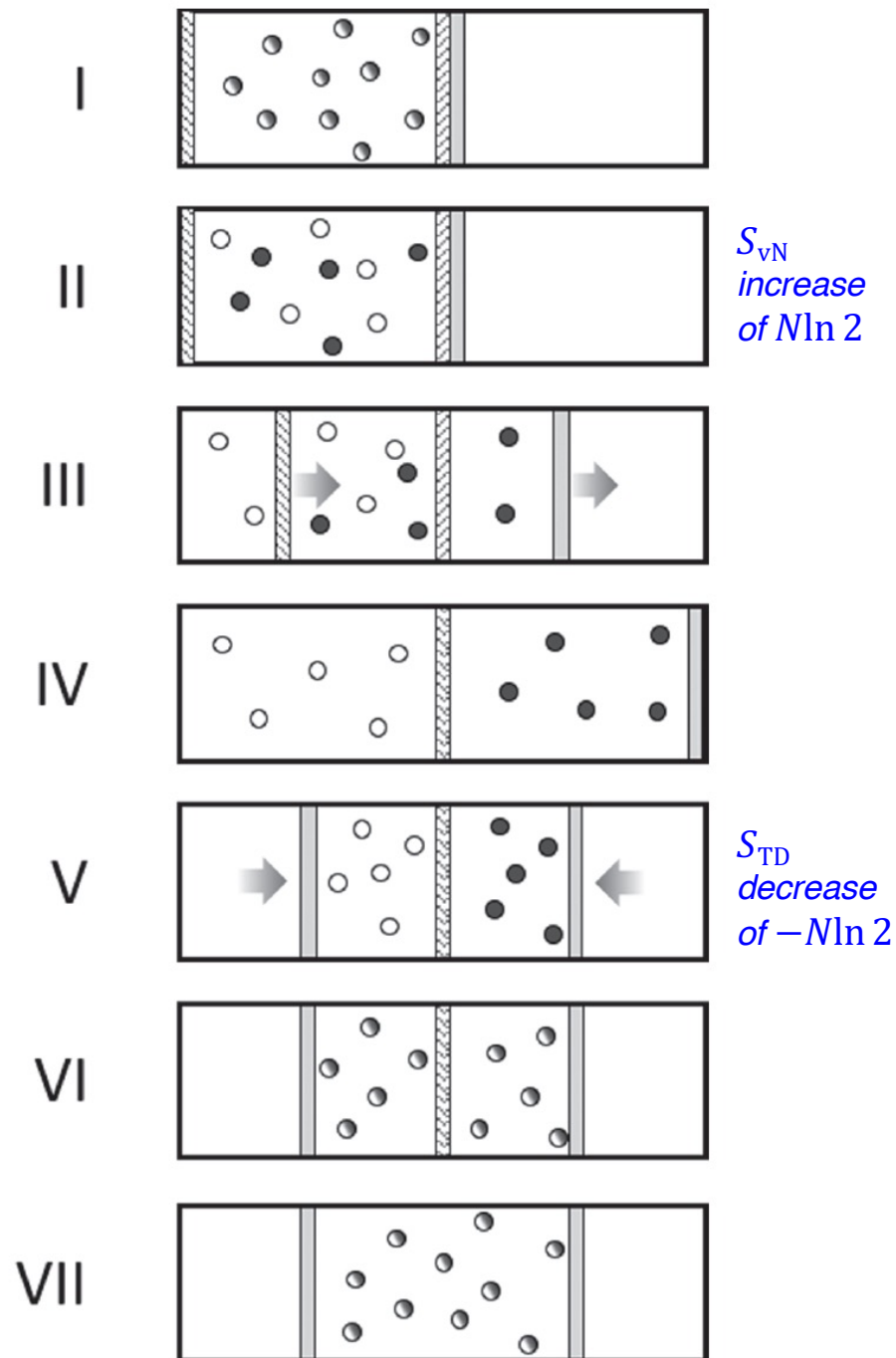
4

Set-Up In Detail: *7 stage cyclical process*

I

II — S_{vN} *increase of* $N \ln 2$

III

IV

V — S_{TD} *decrease of* $-N \ln 2$

VI

VII

Stage VI:
- The $|+\rangle$ and $|-\rangle$ gases are reversibly transformed back into a $|0\rangle$ gas via unitary operations.

Stages VII:
- Partition is removed; gas returns to initial state.

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Set-Up In Detail: _7 stage cyclical process_

I

II

S_{vN}
increase
_of_ $N\ln 2$

III

IV

V

S_{TD}
decrease
_of_ $-N\ln 2$

VI

VII

von Neumann's Argument:

- All transitions between II and VII are reversible, so 2nd Law requires $\Delta S_{TD} = 0$.

- There is a decrease in S_{TD} of $-N\ln 2$ during Stage V.

- _So_: There must be an increase in S_{TD} of $N\ln 2$ prior to Stage V.

- _And_: There is an increase in S_{vN} of $N\ln 2$ due to measurement at Stage II.

- _So_: If $S_{vN} = S_{TD}$, then no violation of 2nd law.

Issue #1: Can a quantum measurement be characterized by a transition between a pure state and a mixed state?

What is a quantum measurement?

- *According to von Neumann*:

$$|0\rangle = \sqrt{\tfrac{1}{2}}(|+\rangle + |-\rangle) \xrightarrow{\ collapse\ } either \ |+\rangle \ or \ |-\rangle$$

pre-measurement vector state　　　　　*each with probability ½*

　Or:
$$\rho_{\text{pre}} = |0\rangle\langle 0| \longrightarrow \rho_{\text{post}} = \tfrac{1}{2}|+\rangle\langle +| + \tfrac{1}{2}|-\rangle\langle -|$$

pure density operator state　　　　　*mixed density operator state*

- *But*: To the experimentalist, there is a definite result! ⟵ *Her experience is that the post-measurement state is a definite (i.e., pure) state.*

- *Moreover*: The Schrödinger dynamics is unitary: it cannot transform a pure state to a mixed state.

  - *So if post-measurement states are mixed states, then we need an explanation of how the Projection Postulate "takes over"* ⟵ *The "Measurement Problem"!*
  from the Schrödinger dynamics during a measurement.

Issue #2: Can a quantum measurement be characterized by an increase in S_{TD}?

- *This is Szilard's Principle*: *Gaining* information that allows us to discern between $n \, (= 2^N)$ equally likely states is associated with a minimum increase in S_{TD} of $k \ln n$.

- What about Landauer's Principle?
  - *In general*: *What about the critique of information theoretic attempts to secure the 2nd Law?*

Issue #3: Did von Neumann really intend to equate S_{vN} with S_{TD}, or did he intend to equate S_{vN} with S_{Gibbs}?

- Sheridan (2020)*: In von Neumann's gas, each "molecule" is supposed to represent the state of an entire gas, as opposed to a single gas molecule.
  - *The "gas" is really an ensemble of quantum vector states.*

- *Thus*: The decrease in entropy in Stage V is a decrease in the Gibbs entropy S_{Gibbs} of a (classical) distribution that corresponds to the density operator state of the quantum system.

*Sheridan, E. (2020) "A Man Misunderstood: Von Neumann did not claim that his entropy corresponds to the phenomenological thermodynamic entropy", arXiv:2007.06673.