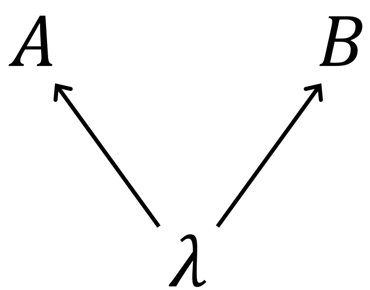
10. Entanglement Correlations

Let's prove the claim that correlations between some observables in an entangled vector state cannot be due to a common cause.

1. What is a Common Cause?



λ as common cause of A and B

Example:
A = storm
B = drop in mercury in barometer
λ = drop in atmospheric pressure

- *B* is *relevant* to *A* in the absence of *λ*, but *irrelevant* in its presence.

If we didn't know there was a drop in atmospheric pressure, then a drop in our barometer would be relevant to whether a storm will develop.

If we did know that there was a drop in atmospheric pressure, then this alone would be relevant to whether a storm will develop.

- *Which means*: *λ* screens *A* off from *B*.

- *Which means*: *A* and *B* are "conditionally statistically independent" with respect to *λ*...

Def. 1 (*Conditional statistical independence*). Observables A and B are **conditionally statistically independent** in vector state $|\psi\rangle$ with respect to a random variable λ just when

$$\Pr_\psi(a, b|A, B, \lambda) = \Pr_\psi(a|A, \lambda)\Pr_\psi(b|B, \lambda)$$

(*The joint probability of getting the value a of A and the value b of B in vector state $|\psi\rangle$, given λ.*) = (*The probability of getting the value a of A in vector state $|\psi\rangle$, given λ.*) × (*The probability of getting the value b of B in vector state $|\psi\rangle$, given λ.*)

Claim. Conditional statistical independence of A and B with respect to λ is a *necessary* condition for λ to be a common cause of A and B.

Hans Reichenbach (1891-1953)

- *Which means*: *If λ is a common cause of A and B, then A and B are conditionally statistically independent with respect to λ.*

So: If A and B are correlated, and there is no λ such that A and B are conditionally statistically independent with respect to λ, then their correlation cannot be due to a common cause.

Def. 2 (*Common cause-violating correlation*). The observables represented by *A* and *B* exhibit a **common cause-violating correlation** just when they are correlated and there is no random variable λ such that they are *conditionally statistically independent* with respect to λ.

- *Thus*: To show that there can be correlations between observables in an entangled vector state that cannot be due to a common cause, we have to show that there is no random variable λ with respect to which these observables are conditionally statistically independent.

Claim. There are pair-wise correlations in the entangled vector state $|\Psi^-\rangle = \sqrt{½}\{|0_A 1_B\rangle - |1_A 0_B\rangle\}$ between four spin-½ observables such that a particular sum of their expectation values violates a "Bell" inequality that it must satisfy if the correlated observables are conditionally statistically independent.

In other words:

(a) If these correlated observables are conditionally statistically independent, then a particular sum of their expectation values must satisfy a Bell inequality.

(b) This sum does not satisfy the Bell inequality.

So: If we can prove (a) *and* (b)*, then these correlations are common cause-violating.*

2. The CHSH Inequality

One type of "Bell" inequality

(Clauser, Horne, Shimony, Holt 1969)

Claim (a) (*CHSH inequality*). Let $A_x, B_y, x, y \in \{0, 1\}$ be four spin-½ operators that act on 2-dim vector spaces $\mathcal{H}_A, \mathcal{H}_B$, respectively, with values $a, b \in \{-1, +1\}$, and let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. If A_x, B_y are conditionally statistically independent, then

$$S \equiv \langle A_0 \otimes B_0 \rangle_\psi + \langle A_0 \otimes B_1 \rangle_\psi + \langle A_1 \otimes B_0 \rangle_\psi - \langle A_1 \otimes B_1 \rangle_\psi \leq 2$$

Proof: Note first that conditional statistical independence of A_x, B_y requires

$$\Pr_\psi(a, b|A_x, B_y, \lambda) = \Pr_\psi(a|A_x, \lambda)\Pr_\psi(b|B_y, \lambda)$$

or

$$\int d\lambda q(\lambda)\Pr_\psi(a, b|A_x, B_y, \lambda) = \int d\lambda q(\lambda)\Pr_\psi(a|A_x, \lambda)\Pr_\psi(b|B_y, \lambda)$$

$q(\lambda)$ is a probability distribution for the general case of a continuous range of values of the random varible λ

Left-hand-side is the joint probability $\Pr_\psi(a, b|A_x, B_y)$

So: $\langle A_x \otimes B_y \rangle_\psi = \sum_{a,b} ab \Pr_\psi(a,b|A_x, B_y)$

$$= \sum_{a,b} ab \int d\lambda q(\lambda)\Pr_\psi(a|A_x, \lambda)\Pr_\psi(b|B_y, \lambda)$$

conditional statistical independence assumption!

$$= \int d\lambda q(\lambda) \sum_a a \Pr_\psi(a|A_x, \lambda) \sum_b b \Pr_\psi(b|B_y, \lambda)$$

$$= \int d\lambda q(\lambda) \langle A_x \rangle_{\psi,\lambda} \langle B_y \rangle_{\psi,\lambda}$$

where, e.g., $\langle A_x \rangle_{\psi,\lambda} \equiv \sum_a a \Pr_\psi(a|A_x, \lambda)$

2. The CHSH Inequality

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Claim (a) (*CHSH inequality*). Let A_x, B_y, $x, y \in \{0, 1\}$ be four spin-½ operators that act on 2-dim vector spaces $\mathcal{H}_A$, $\mathcal{H}_B$, respectively, with values $a, b \in \{-1, +1\}$, and let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. If A_x, B_y are conditionally statistically independent, then

$$S \equiv \langle A_0 \otimes B_0\rangle_\psi + \langle A_0 \otimes B_1\rangle_\psi + \langle A_1 \otimes B_0\rangle_\psi - \langle A_1 \otimes B_1\rangle_\psi \leq 2$$

Proof:

So: If A_x, B_y are conditionally statistically independent, then

$$\langle A_x \otimes B_y\rangle_\psi = \int d\lambda q(\lambda)\langle A_x\rangle_{\psi,\lambda}\langle B_y\rangle_{\psi,\lambda}$$

Where, e.g., $\langle A_x\rangle_{\psi,\lambda} \equiv \sum_a a \Pr_\psi(a|A_x,\lambda) = -\Pr_\psi(-1|A_x,\lambda) + \Pr_\psi(+1|A_x,\lambda)$

Thus: $S = \int d\lambda q(\lambda)\{\langle A_0\rangle_{\psi,\lambda}\langle B_0\rangle_{\psi,\lambda} + \langle A_0\rangle_{\psi,\lambda}\langle B_1\rangle_{\psi,\lambda} + \langle A_1\rangle_{\psi,\lambda}\langle B_0\rangle_{\psi,\lambda} - \langle A_1\rangle_{\psi,\lambda}\langle B_1\rangle_{\psi,\lambda}\}$

$$= \int d\lambda q(\lambda)\{\langle A_0\rangle_{\psi,\lambda}[\langle B_0\rangle_{\psi,\lambda} + \langle B_1\rangle_{\psi,\lambda}] + \langle A_1\rangle_{\psi,\lambda}[\langle B_0\rangle_{\psi,\lambda} - \langle B_1\rangle_{\psi,\lambda}]\}$$

$$\leq \int d\lambda q(\lambda)\{|\langle B_0\rangle_{\psi,\lambda} + \langle B_1\rangle_{\psi,\lambda}| + |\langle B_0\rangle_{\psi,\lambda} - \langle B_1\rangle_{\psi,\lambda}|\}$$

since the max value of $\langle A_x\rangle_{\psi,\lambda}$ is $+1$

- *Now note*: The maximum value of $\langle B_y\rangle_{\psi,\lambda}$ is $+1$, so the maximum value of $|\langle B_0\rangle_{\psi,\lambda} + \langle B_1\rangle_{\psi,\lambda}| + |\langle B_0\rangle_{\psi,\lambda} - \langle B_1\rangle_{\psi,\lambda}|$ is 2.
- *So*: $S \leq 2$!

3. A Violation of the CHSH Inequality

Now we'll show that a particular choice of A_0, A_1, B_0, B_1 violates the CHSH Inequality with respect to the entangled vector state $|\Psi^-\rangle = \sqrt{½}\{|01\rangle - |10\rangle\}$.

- Consider the following spin-½ operators:

$$A_0 = \hat{x} \cdot \vec{\sigma} \qquad A_1 = \hat{y} \cdot \vec{\sigma}$$

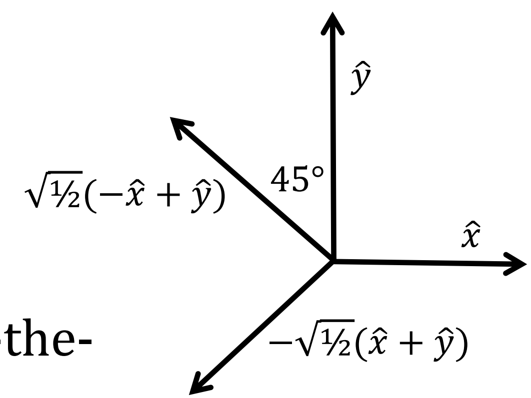
$$B_0 = -\sqrt{½}(\hat{x} + \hat{y}) \cdot \vec{\sigma} \qquad B_1 = \sqrt{½}(-\hat{x} + \hat{y}) \cdot \vec{\sigma}$$

- <u>*Note* 1</u>: The "vector" $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ encodes the Pauli operators that act on 2-dim vectors in the following way:

$$\sigma_x|0\rangle = |1\rangle \qquad \sigma_x|1\rangle = |0\rangle$$

$$\sigma_y|0\rangle = i|1\rangle \qquad \sigma_y|1\rangle = -i|0\rangle$$

$$\sigma_z|0\rangle = |0\rangle \qquad \sigma_z|1\rangle = -|1\rangle$$



- <u>*Note* 2</u>: A_0 represents the spin-½ observable "spin-along-the-$\hat{x}$-axis", and A_1 represents "spin-along-the-$\hat{y}$-axis". The axes of B_0 and B_1 are at 45° from the $\hat{x}$ and $\hat{y}$ axes.

Let's explicitly calculate

$$S = \langle A_0 \otimes B_0 \rangle_{\Psi^-} + \langle A_0 \otimes B_1 \rangle_{\Psi^-} + \langle A_1 \otimes B_0 \rangle_{\Psi^-} - \langle A_1 \otimes B_1 \rangle_{\Psi^-}$$

Calculating $S = \langle A_0 \otimes B_0 \rangle_{\Psi^-} + \langle A_0 \otimes B_1 \rangle_{\Psi^-} + \langle A_1 \otimes B_0 \rangle_{\Psi^-} - \langle A_1 \otimes B_1 \rangle_{\Psi^-}$

- $\langle A_0 \otimes B_0 \rangle_{\Psi^-} = ½\{\langle 01| - \langle 10|\}(A_0 \otimes B_0)\{|01\rangle - |10\rangle\}$

$$= ½\{\langle 01| - \langle 10|\}[(\hat{x} \cdot \vec{\sigma}) \otimes [-\sqrt{½}(\hat{x} + \hat{y}) \cdot \vec{\sigma}]]\{|01\rangle - |10\rangle\}$$

$$= ½(-\sqrt{½})\{\langle 01| - \langle 10|\}[\sigma_x \otimes (\sigma_x + \sigma_y)]\{|01\rangle - |10\rangle\}$$

$$= ½(-\sqrt{½})\{\langle 01| - \langle 10|\}[(\sigma_x \otimes \sigma_x) + (\sigma_x \otimes \sigma_y)]\{|01\rangle - |10\rangle\}$$

$$= ½(-\sqrt{½})\{\langle 01| - \langle 10|\}\{|10\rangle - |01\rangle + [-i|10\rangle - i|01\rangle]\}$$

$$= ½(-\sqrt{½})\{(-1 - i) + (-1 + i)\} = \sqrt{½}$$

$\sigma_x|0\rangle = |1\rangle \quad \sigma_x|1\rangle = |0\rangle$

$\sigma_y|0\rangle = i|1\rangle \quad \sigma_y|1\rangle = -i|0\rangle$

$\sigma_z|0\rangle = |0\rangle \quad \sigma_z|1\rangle = -|1\rangle$

- $\langle A_0 \otimes B_1 \rangle_{\Psi^-} = ½\{\langle 01| - \langle 10|\}(A_0 \otimes B_1)\{|01\rangle - |10\rangle\}$

$$= ½\{\langle 01| - \langle 10|\}[(\hat{x} \cdot \vec{\sigma}) \otimes [\sqrt{½}(-\hat{x} + \hat{y}) \cdot \vec{\sigma}]]\{|01\rangle - |10\rangle\}$$

$$= ½(\sqrt{½})\{\langle 01| - \langle 10|\}[\sigma_x \otimes (-\sigma_x + \sigma_y)]\{|01\rangle - |10\rangle\}$$

$$= ½(\sqrt{½})\{\langle 01| - \langle 10|\}[(\sigma_x \otimes -\sigma_x) + (\sigma_x \otimes \sigma_y)]\{|01\rangle - |10\rangle\}$$

$$= ½(\sqrt{½})\{\langle 01| - \langle 10|\}\{-|10\rangle + |01\rangle + [-i|10\rangle - i|01\rangle]\}$$

$$= ½(\sqrt{½})\{(1 - i) - (-1 - i)\} = \sqrt{½}$$

Calculating $S = \langle A_0 \otimes B_0 \rangle_{\Psi^-} + \langle A_0 \otimes B_1 \rangle_{\Psi^-} + \langle A_1 \otimes B_0 \rangle_{\Psi^-} - \langle A_1 \otimes B_1 \rangle_{\Psi^-}$

- $\langle A_1 \otimes B_0 \rangle_{\Psi^-} = ½\{\langle 01| - \langle 10|\}(A_1 \otimes B_0)\{|01\rangle - |10\rangle\}$

$$= ½\{\langle 01| - \langle 10|\}[(\hat{y} \cdot \vec{\sigma}) \otimes [-\sqrt{½}(\hat{x} + \hat{y}) \cdot \vec{\sigma}]]\{|01\rangle - |10\rangle\}$$

$$= ½(-\sqrt{½})\{\langle 01| - \langle 10|\}[\sigma_y \otimes (\sigma_x + \sigma_y)]\{|01\rangle - |10\rangle\}$$

$$= ½(-\sqrt{½})\{\langle 01| - \langle 10|\}[(\sigma_y \otimes \sigma_x) + (\sigma_y \otimes \sigma_y)]\{|01\rangle - |10\rangle\}$$

$$= ½(-\sqrt{½})\{\langle 01| - \langle 10|\}\{i|10\rangle + i|01\rangle + [|10\rangle - |01\rangle]$$

$$= ½(-\sqrt{½})\{(i - 1) - (i + 1)\} = \sqrt{½}$$

$\sigma_x|0\rangle = |1\rangle \quad \sigma_x|1\rangle = |0\rangle$

$\sigma_y|0\rangle = i|1\rangle \quad \sigma_y|1\rangle = -i|0\rangle$

$\sigma_z|0\rangle = |0\rangle \quad \sigma_z|1\rangle = -|1\rangle$

- $\langle A_1 \otimes B_1 \rangle_{\Psi^-} = ½\{\langle 01| - \langle 10|\}(A_0 \otimes B_1)\{|01\rangle - |10\rangle\}$

$$= ½\{\langle 01| - \langle 10|\}[(\hat{y} \cdot \vec{\sigma}) \otimes [\sqrt{½}(-\hat{x} + \hat{y}) \cdot \vec{\sigma}]]\{|01\rangle - |10\rangle\}$$

$$= ½(\sqrt{½})\{\langle 01| - \langle 10|\}[\sigma_y \otimes (-\sigma_x + \sigma_y)]\{|01\rangle - |10\rangle\}$$

$$= ½(\sqrt{½})\{\langle 01| - \langle 10|\}[(\sigma_y \otimes -\sigma_x) + (\sigma_y \otimes \sigma_y)]\{|01\rangle - |10\rangle\}$$

$$= ½(\sqrt{½})\{\langle 01| - \langle 10|\}\{-i|10\rangle - i|01\rangle + [|10\rangle - |01\rangle]\}$$

$$= ½(\sqrt{½})\{(-i - 1) - (-i + 1)\} = -\sqrt{½}$$

Calculating $S = \langle A_0 \otimes B_0 \rangle_{\Psi^-} + \langle A_0 \otimes B_1 \rangle_{\Psi^-} + \langle A_1 \otimes B_0 \rangle_{\Psi^-} - \langle A_1 \otimes B_1 \rangle_{\Psi^-}$

- *So*: $S = \langle A_0 \otimes B_0 \rangle_{\Psi^-} + \langle A_0 \otimes B_1 \rangle_{\Psi^-} + \langle A_1 \otimes B_0 \rangle_{\Psi^-} - \langle A_1 \otimes B_1 \rangle_{\Psi^-}$

$$= \sqrt{½} + \sqrt{½} + \sqrt{½} - \sqrt{½}$$

$$= 2\sqrt{2} > 2$$ ← *A violation of the CSHS inequality!*

- *What this means*: In the entangled vector state $|\Psi^-\rangle = \sqrt{½}\{|01\rangle - |10\rangle\}$, and for our choice of spin-½ observables:

$$A_0 = \hat{x} \cdot \vec{\sigma} \qquad A_1 = \hat{y} \cdot \vec{\sigma}$$

$$B_0 = -\sqrt{½}(\hat{x} + \hat{y}) \cdot \vec{\sigma} \qquad B_1 = \sqrt{½}(-\hat{x} + \hat{y}) \cdot \vec{\sigma}$$

A_0 and B_0 are correlated, as are A_0 and B_1, and A_1 and B_0, and A_1 and B_1.

- *And*: These correlations are *not* conditionally statistically independent (because their expectation values with respect to $|\Psi^-\rangle$ violate the CHSH inequality).
- *So*: These correlations are common-cause violating!

To Sum Up:

- When a bipartite system is in a state represented by the entangled vector $|\Psi^-\rangle = \sqrt{½}\{|01\rangle - |10\rangle\}$, there are correlations between spin-½ properties of the two subsystems that cannot be due to a common cause.
- *And*: If the two subsystems are separated by a distance large enough so that a direct cause cannot propagate between them, these correlations cannot be due to a direct cause, either.

- Is there a way to quantify these non-classical entanglement correlations?

- Is there a way to quantify entanglement?

- Yes!

... And it has to do with entropy!