09. Quantum Mechanics: Entanglement

1. Multipartite Vector States

Entanglement Involves Multipartite States!

- Let $\mathcal{H}_A, \mathcal{H}_B$ be vector spaces for *two* quantum 2-state systems.
  - The state space for the combined "bipartite" system is represented by the *product vector space* $\mathcal{H}_A \otimes \mathcal{H}_B$.

- Suppose $\{|0_A\rangle, |1_A\rangle\}$ is a basis for $\mathcal{H}_A$ and $\{|0_B\rangle, |1_B\rangle\}$ is a basis for $\mathcal{H}_B$.
  - *Then*: $\{|0_A\rangle|0_A\rangle, |0_A\rangle|1_B\rangle, |1_A\rangle|0_B\rangle, |1_A\rangle|1_B\rangle\}$ is a basis for $\mathcal{H}_A \otimes \mathcal{H}_B$.
  - *And*: Any bipartite state $|Q\rangle$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ can be expanded in this basis:

$$|Q\rangle = a|0_A\rangle|0_B\rangle + b|0_A\rangle|1_B\rangle + c|1_A\rangle|0_B\rangle + d|1_A\rangle|1_B\rangle$$

Def. 1 (*Product/non-product vector state*). A **product vector state** in a product vector space $\mathcal{H}_A \otimes \mathcal{H}_B$ is a vector $|\psi\rangle$ that can be written as a product of two vectors $|\psi\rangle = |v_A\rangle \otimes |v_B\rangle$, where $|v_A\rangle \in \mathcal{H}_A$ and $|v_B\rangle \in \mathcal{H}_B$. A **non-product vector** in $\mathcal{H}_A \otimes \mathcal{H}_B$ is a vector that is not a product vector.

Examples:

- *Non-product*: $|\Psi^+\rangle = \sqrt{1/2}\,\{|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle\}$

- *Product*:

$$|Q\rangle = \sqrt{1/4}\{|0_A\rangle|0_B\rangle + |0_A\rangle|1_B\rangle + |1_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle\}$$
$$= \sqrt{1/4}\{|0_A\rangle + |1_A\rangle\}\{|0\rangle_2 + |1\rangle_2\}$$
$$|Q'\rangle = \sqrt{1/2}\,\{|0_A\rangle|0_B\rangle + |1_A\rangle|0_B\rangle\} = \sqrt{1/2}\,\{|0_A\rangle + |1_A\rangle\}|0_B\rangle$$
$$|Q''\rangle = |0_A\rangle|0_B\rangle$$

- Suppose $|0\rangle$ and $|1\rangle$ are eigenstates of Hardness ($|0\rangle = |hard\rangle$ and $|1\rangle = |soft\rangle$).

According to the Eigenvalue-Eigenvector Rule:

- In vector states $|\Psi^+\rangle$ and $|Q\rangle$, both electrons have no determinate Hardness value. ← *Both are in superposed states*

- In vector state $|Q'\rangle$, electron A has no determinate Hardness value, but electron B *does* (*i.e.*, hard). ← *Electron A is in a superposed state*

- In state $|Q''\rangle$, both electrons have determinate Hardness values.

Examples:

- *Non-product*: $|\Psi^+\rangle = \sqrt{½}\{|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle\}$

- *Product*:

$$|Q\rangle = \sqrt{¼}\{|0_A\rangle|0_B\rangle + |0_A\rangle|1_B\rangle + |1_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle\}$$
$$= \sqrt{¼}\{|0_A\rangle + |1_A\rangle\}\{|0\rangle_2 + |1\rangle_2\}$$
$$|Q'\rangle = \sqrt{½}\{|0_A\rangle|0_B\rangle + |1_A\rangle|0_B\rangle\} = \sqrt{½}\{|0_A\rangle + |1_A\rangle\}|0_B\rangle$$
$$|Q''\rangle = |0_A\rangle|0_B\rangle$$

- Suppose $|0\rangle$ and $|1\rangle$ are eigenstates of Hardness ($|0\rangle = |hard\rangle$ and $|1\rangle = |soft\rangle$).

According to the Projection Postulate:

- In the non-product state $|\Psi^+\rangle$, when a measurement is performed on electron A, its state collapses (to either $|0_A\rangle$ or $|1_A\rangle$), *and this instantaneously affects the state of* electron B!

Spooky action at a distance?

Can this be detected?

- In any of the product states, a measurement performed on electron A will not affect the state of electron B.

2. Correlations in Vector States

Entanglement Involves Correlations!

- *Idea*: When a bipartite system is in an engangled vector state, its subsystems can have properties that are correlated in a *non-classical* way.

Can't be explained by a direct cause!

Can't be explained by a common cause!

- *Quantum Information Theory*: How to exploit entanglement correlations to solve computational problems.

Task #1: How can we represent correlated observables with respect to vector states?

Task #2: How can we represent correlations that cannot be due to either a direct cause or a common cause?

Def. 2 (*Expectation value for vector state*). The **expectation value** $\langle O\rangle_\psi$ of an observable O with respect to a vector state $|\psi\rangle$ is given by $\langle O\rangle_\psi \equiv \langle\psi|O|\psi\rangle$.

- *Idea*: $\langle O\rangle_\psi$ is the average value of O in the vector state $|\psi\rangle$.

  - *Suppose*: The observable is represented by operator B with eigenvectors $|b_i\rangle$ and eigenvalues b_i, and let $|\psi\rangle = \sum_i \alpha_i|b_i\rangle$, $\langle\psi| = \sum_j \alpha_j^*\langle b_j|$.

  - *Then*:

$$\langle B\rangle_\psi = \langle\psi|B|\psi\rangle = \sum_j \alpha_j^*\langle b_j|B\sum_i \alpha_i|b_i\rangle$$

$$= \sum_{i,j} b_i\alpha_j^*\alpha_i\langle b_j|b_i\rangle$$

$$= \sum_i b_i \mathrm{Pr}_{|\psi\rangle}(b_i|B)$$

$\langle b_j|b_i\rangle = 1$ for $i = j$, and 0 otherwise.
$\alpha_i^*\alpha_i = |\alpha_i|^2 = \mathrm{Pr}_{|\psi\rangle}(b_i|B)$

The average value of the set of values $\{b_1, ..., b_n\}$ with the Born probabilities $\{\mathrm{Pr}_{|\psi\rangle}(b_1|B), ..., \mathrm{Pr}_{|\psi\rangle}(b_n|B)\}$ assigned to its members.

Def. 3 (*Expectation value for density operator state*). The **expectation value** $\langle O\rangle_\rho$ of an observable O with respect to a density operator state ρ is given by $\langle O\rangle_\rho \equiv \mathrm{Tr}(\rho O)$.

- *Idea*: $\langle O\rangle_\rho$ is the *average value* of O in density operator state ρ.

  - *So*: $\langle O\rangle_\rho = \sum_j p_j \langle O\rangle_{\psi_j}$ ← *The weighted sum of the average value of O in each vector state $|\psi_j\rangle$ of an ensemble $\{|\psi_j\rangle, p_j\}$.*

  - *Suppose*: The observable is represented by operator B with eigenvectors $|b_i\rangle$ and eigenvalues b_i, and let $|\psi_j\rangle = \sum_k \alpha_{jk}|b_k\rangle$, and $\rho = \sum_j p_j|\psi_j\rangle\langle\psi_j| = \sum_{j,k,l} p_j\alpha_{jk}\alpha^*_{jl}|b_k\rangle\langle b_l|$.

  - *Then*:

$$\mathrm{Tr}(\rho B) = \sum_i \langle b_i|\rho B|b_i\rangle$$

$$= \sum_{i,j,k,l} p_j b_i \alpha_{jk}\alpha^*_{jl}\langle b_i|b_k\rangle\langle b_l|b_i\rangle$$

← *$\langle b_i|b_k\rangle = 1$ for $i = k$, and 0 otherwise. $\langle b_l|b_i\rangle = 1$ for $l = i$, and 0 otherwise.*

$$= \sum_j p_j \sum_i b_i \alpha_{ji}\alpha^*_{ji}$$

← $\alpha_{ji}\alpha^*_{ji} = |\alpha_j|^2 = \mathrm{Pr}_{|\psi_j\rangle}(b_i|B)$

$$= \sum_j p_j \sum_i b_i \mathrm{Pr}_{|\psi_j\rangle}(b_i|B)$$

$$= \sum_j p_j \langle O\rangle_{\psi_j}$$

Def. 4 (*Product operator*). Let O_A and O_B be operators on vector spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, and let $|\psi_A\rangle \in \mathcal{H}_A$, $|\psi_B\rangle \in \mathcal{H}_B$, and $|\psi_A \otimes \psi_B\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. The **product operator** $O_A \otimes O_B$ is defined by

$$(O_A \otimes O_B)|\psi_A \otimes \psi_B\rangle \equiv O_A|\psi_A\rangle \otimes O_B|\psi_B\rangle$$

- *Idea*: The "A" part of $O_A \otimes O_B$ only acts on the "A" part of $|\psi_A \otimes \psi_B\rangle$, and the "$B$" part of $O_A \otimes O_B$ only acts on the "B" part of $|\psi_A \otimes \psi_B\rangle$.

- O_A and O_B represent observables (properties) of the two subsystems of a composite bipartite system.

What does it mean to say these observables are correlated with each other?

Def. 5 (*Correlated observables for vector states*). Let O_A and O_B be operators on vector spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with identity operators I_A and I_B, and let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Then the observables represented by O_A and O_B are **correlated in vector state $|\psi\rangle$** just when

$$\langle O_A \otimes O_B \rangle_\psi \neq \langle O_A \otimes I_B \rangle_\psi \langle I_A \otimes O_B \rangle_\psi$$

The expectation value of $O_A \otimes O_B$ cannot be factored into a product of the expectation values of its "parts"

- <u>*Motivation*</u>: The observables O_A and O_B are correlated in the sense of Def. 4 *if and only if* they are *statistically dependent*; i.e.,

$$\Pr_{|\psi\rangle}(a_i, b_j | O_A, O_B) \neq \Pr_{|\psi\rangle}(a_i | O_A) \Pr_{|\psi\rangle}(b_j | O_B), \quad \text{for all } i, j$$

(The joint probability of getting the value a_i of O_A and the value b_j of O_B in the state $|\psi\rangle$) ≠ (The probability of getting the value a_i of O_A in the state $|\psi\rangle$) × (The probability of getting the value b_j of O_B in the state $|\psi\rangle$)

Task #1 accomplished!

Claim. Observables represented by the operators O_A and O_B that appear in a product operator $O_A \otimes O_B$ are uncorrelated in a product vector state and correlated in a non-product vector state.

- *Example*: Let $|\psi_{\text{prod}}\rangle = |\psi_A \psi_B\rangle$ and $|\psi_{\text{non}}\rangle = \sqrt{½}\{|\psi_A \phi_B\rangle + |\phi_A \psi_B\rangle\}$ be a product vector and a non-product vector in $\mathcal{H}_A \otimes \mathcal{H}_B$.

$$\langle O_A \otimes O_B \rangle_{\psi_{\text{prod}}} = \langle \psi_A \psi_B | (O_A \otimes O_B) | \psi_A \psi_B \rangle$$

$$= \langle \psi_A \psi_B | (O_A |\psi_A\rangle \otimes O_B |\psi_B\rangle)$$

def. of product operator

$$= \langle \psi_A | O_A | \psi_A \rangle \langle \psi_B | O_B | \psi_B \rangle$$

def. of product space inner-product

$$= \langle \psi_A | O_A | \psi_A \rangle (\langle \psi_B | I_B | \psi_B \rangle)(\langle \psi_A | I_A | \psi_A \rangle) \langle \psi_B | O_B | \psi_B \rangle$$

insertion of identity

$$= \langle \psi_A \psi_B | (O_A \otimes I_B) | \psi_A \psi_B \rangle \, \langle \psi_A \psi_B | (I_A \otimes O_B) | \psi_A \psi_B \rangle$$

rearranging…

$$= \langle O_A \otimes I_B \rangle_{\psi_{\text{prod}}} \langle I_A \otimes O_B \rangle_{\psi_{\text{prod}}}$$

No correlation between O_A and O_B in $|\psi_{\text{prod}}\rangle$!

Claim. Observables represented by the operators O_A and O_B that appear in a product operator $O_A \otimes O_B$ are uncorrelated in a product vector state and correlated in a non-product vector state.

- *Example*: Let $|\psi_{\text{prod}}\rangle = |\psi_A \psi_B\rangle$ and $|\psi_{\text{non}}\rangle = \sqrt{1/2}\{|\psi_A \phi_B\rangle + |\phi_A \psi_B\rangle\}$ be a product vector and a non-product vector in $\mathcal{H}_A \otimes \mathcal{H}_B$.

$$\langle O_A \otimes O_B \rangle_{\psi_{\text{non}}} = 1/2\{\langle \psi_A \phi_B| + \langle \phi_A \psi_B|\}(O_A \otimes O_B)\{|\psi_A \phi_B\rangle + |\phi_A \psi_B\rangle\}$$

$$= 1/2\{\langle O_A\rangle_{\psi_A}\langle O_B\rangle_{\phi_B} + \langle \phi_A|O_A|\psi_A\rangle\langle \psi_B|O_B|\phi_B\rangle + \langle \psi_A|O_A|\phi_A\rangle\langle \phi_B|O_B|\psi_B\rangle + \langle O_A\rangle_{\phi_A}\langle O_B\rangle_{\psi_B}\}$$

$$\langle O_A \otimes I_B\rangle_{\psi_{\text{non}}}\langle I_A \otimes O_B\rangle_{\psi_{\text{non}}}$$

$$= 1/4\{\langle O_A\rangle_{\psi_A}\langle O_B\rangle_{\phi_B} + \langle O_A\rangle_{\psi_A}\langle O_B\rangle_{\psi_B} + \langle O_A\rangle_{\phi_A}\langle O_B\rangle_{\phi_B} + \langle O_A\rangle_{\phi_A}\langle O_B\rangle_{\psi_B}\}$$

So: $\langle O_A \otimes O_B\rangle_{\psi_{\text{non}}} \neq \langle O_A \otimes I_B\rangle_{\psi_{\text{non}}}\langle I_A \otimes O_B\rangle_{\psi_{\text{non}}}$

Correlation between O_A and O_B in $|\psi_{\text{non}}\rangle$!

3. Entangled Vector States

So: Non-product vector states exhibit correlations between observables.

Can these correlations be explained by direct causes and/or common causes?

- In $|\psi_{\text{non}}\rangle = \sqrt{½}\{|0_A 1_B\rangle - |1_A 0_B\rangle\}$, there is a *correlation* between the Hardness properties of electron A & electron B: When one is *hard*, the other is *soft*.

- *Direct cause explanation?*
  - *No!* Projection Postulate entails that when a measurement is performed on electron A, its state collapses (to either $|0_A\rangle$ or $|1_A\rangle$), *and this instantaneously affects the state of electron B*!

- *Common cause explanation?*
  - *No!* Can show that in $|\psi_{\text{non}}\rangle$ there are pair-wise correlations between four spin-½ observables such that a particular sum of their expectation values violates a "Bell" inequality that it must satisfy if the correlations are due to a common cause.

A bit messy to demonstrate! Ultimately due to the way we represent expectation values of bipartite properties in a non-product vector state.

<u>So</u>: Non-product vector states exhibit correlations between observables that cannot be explained either by a direct cause or a common cause.

- *Classical* correlations can always be explained by a direct cause and/or a common cause.

- This suggests that the correlations exhibited by non-product vector states are unique to quantum mechanics.

    - *Call them "entanglement correlations".*
    - *Call the vector states in quantum mechanics that support them, "entangled vector states".*

Def. 6 (*Entangled vector state*). A state represented by a multipartite vector $|\psi\rangle$ is **entangled** just when $|\psi\rangle$ is a non-product vector state.