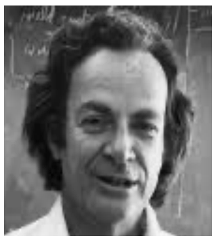


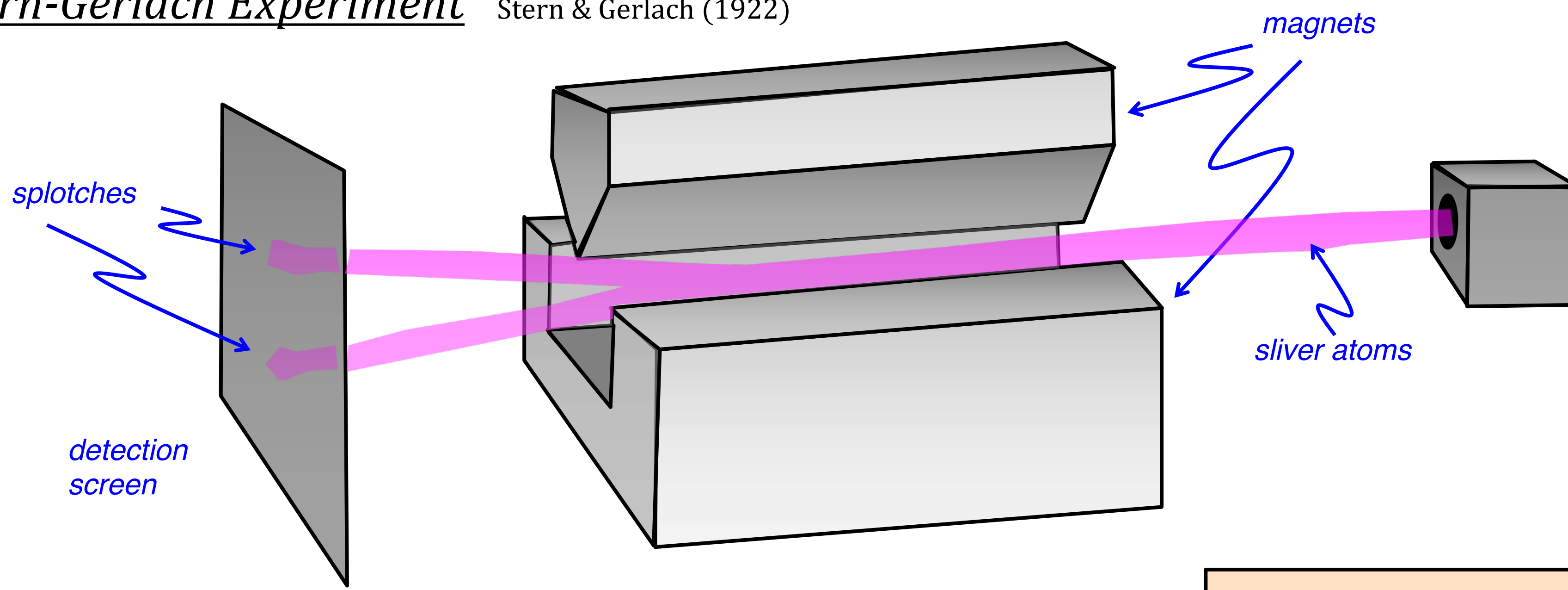
Richard Feynman
(1918-1988)

"I am going to tell you what nature behaves like... Do not keep saying to yourself, if you can possibly avoid it, 'But how can it be like that?' because you will get 'down the drain,' into a blind alley from which nobody has yet escaped. Nobody knows how it can be like that." (*The Character of Physical Laws* 1965, pg. 129.)

07. Quantum Mechanics: Basics

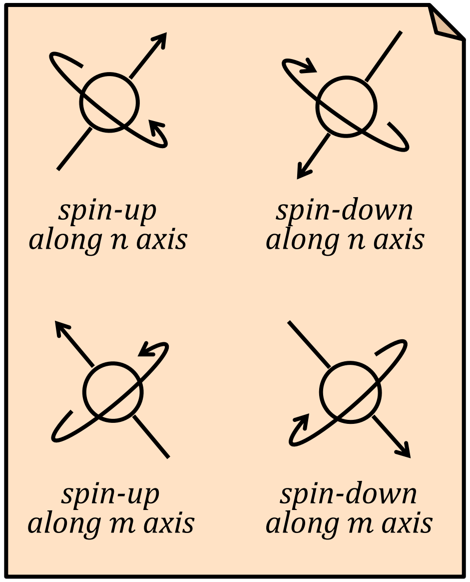
1. Motivation

Stern-Gerlach Experiment Stern & Gerlach (1922)

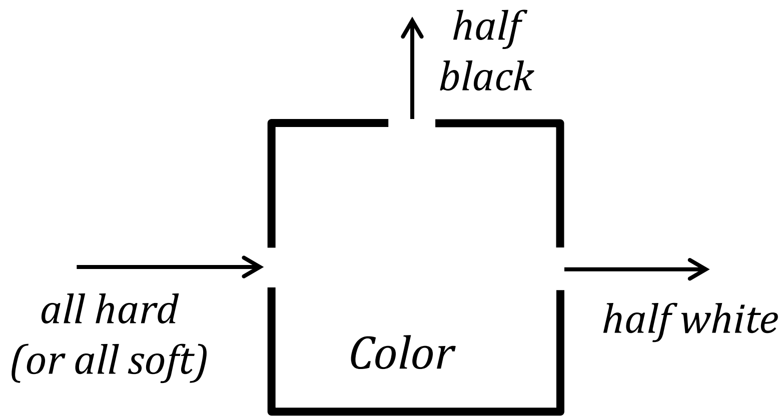


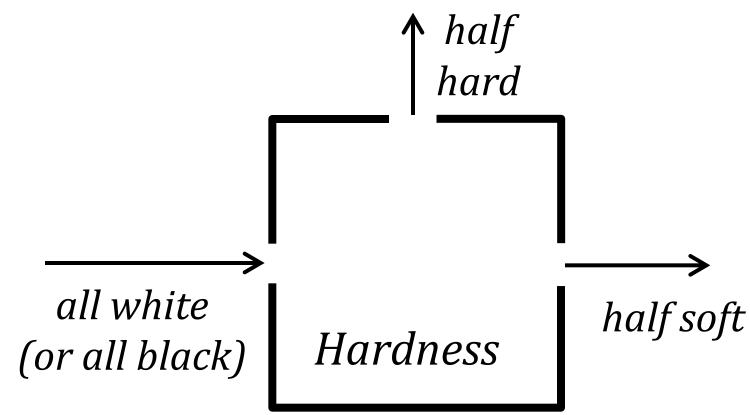
Suggests: Electrons possess 2-valued "spin" properties.
(Goudsmit & Uhlenbeck 1925)

- With respect to a given axis (direction), an electron can possess either the value "spin-up" or the value "spin-down".
- There are as many spin properties as there are possible axes!
- Call two such spin properties with perpendicular axes "Color" (with values *white* and *black*) and "Hardness" (with values *hard* and *soft*).



Experimental Result #1: There is no correlation between Color and Hardness





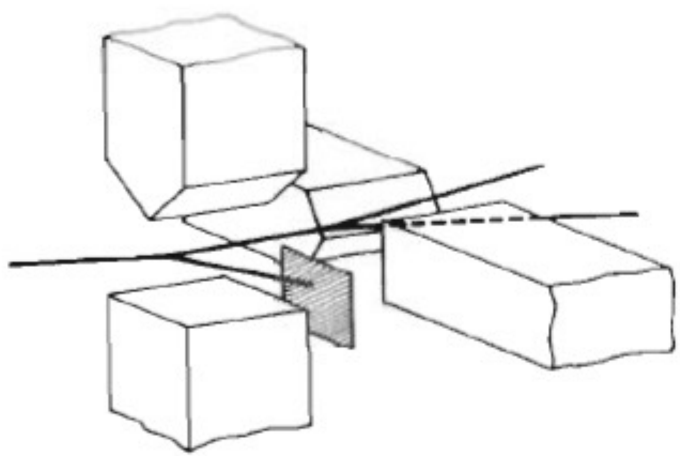
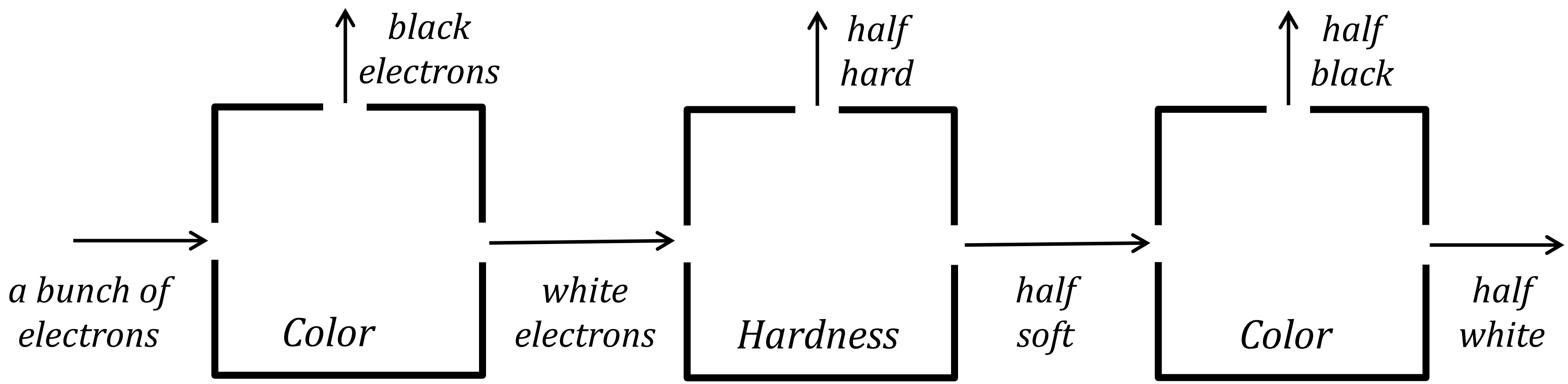


Figure 1.4 Experiment VH.

<u>*Experimental Result #2*</u>:

Hardness measurements "disrupt" Color measurements, and *vice-versa*.



- Can we build a Hardness measuring box that doesn't "disrupt" Color values?
  - *All evidence suggests "No"!*

- Can we determine which electrons get their Color values "disrupted" by a Hardness measurement?
  - *All evidence suggests "No"!*

- <u>*Thus*</u>: All evidence suggests Hardness and Color cannot be simultaneously measured.

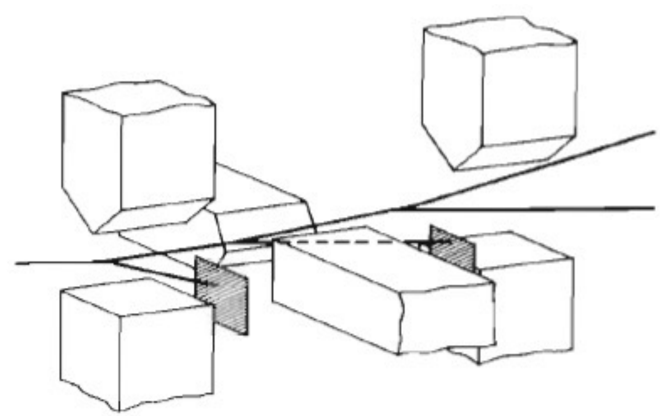
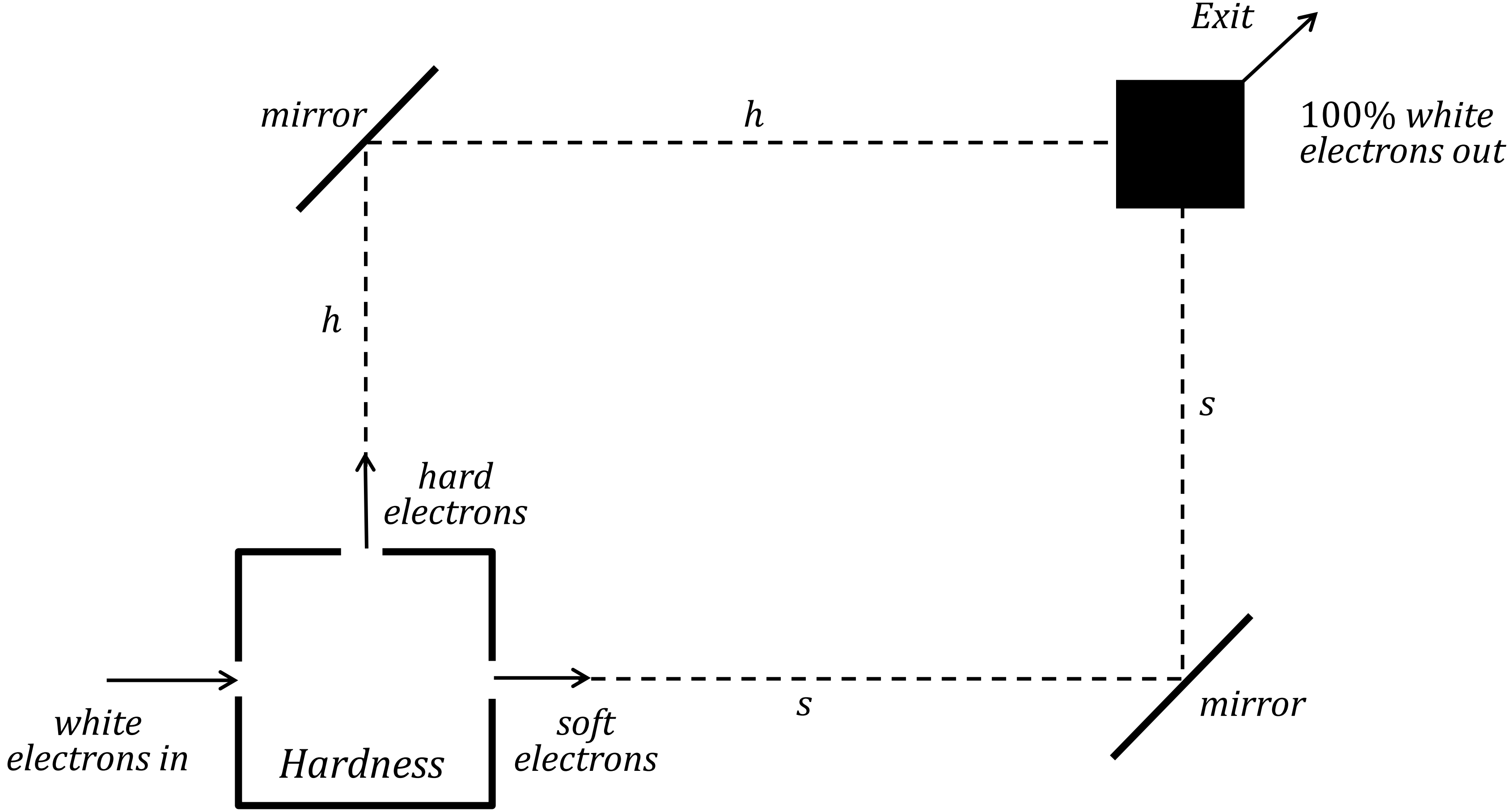


Figure 1.5 Experiment VHV.

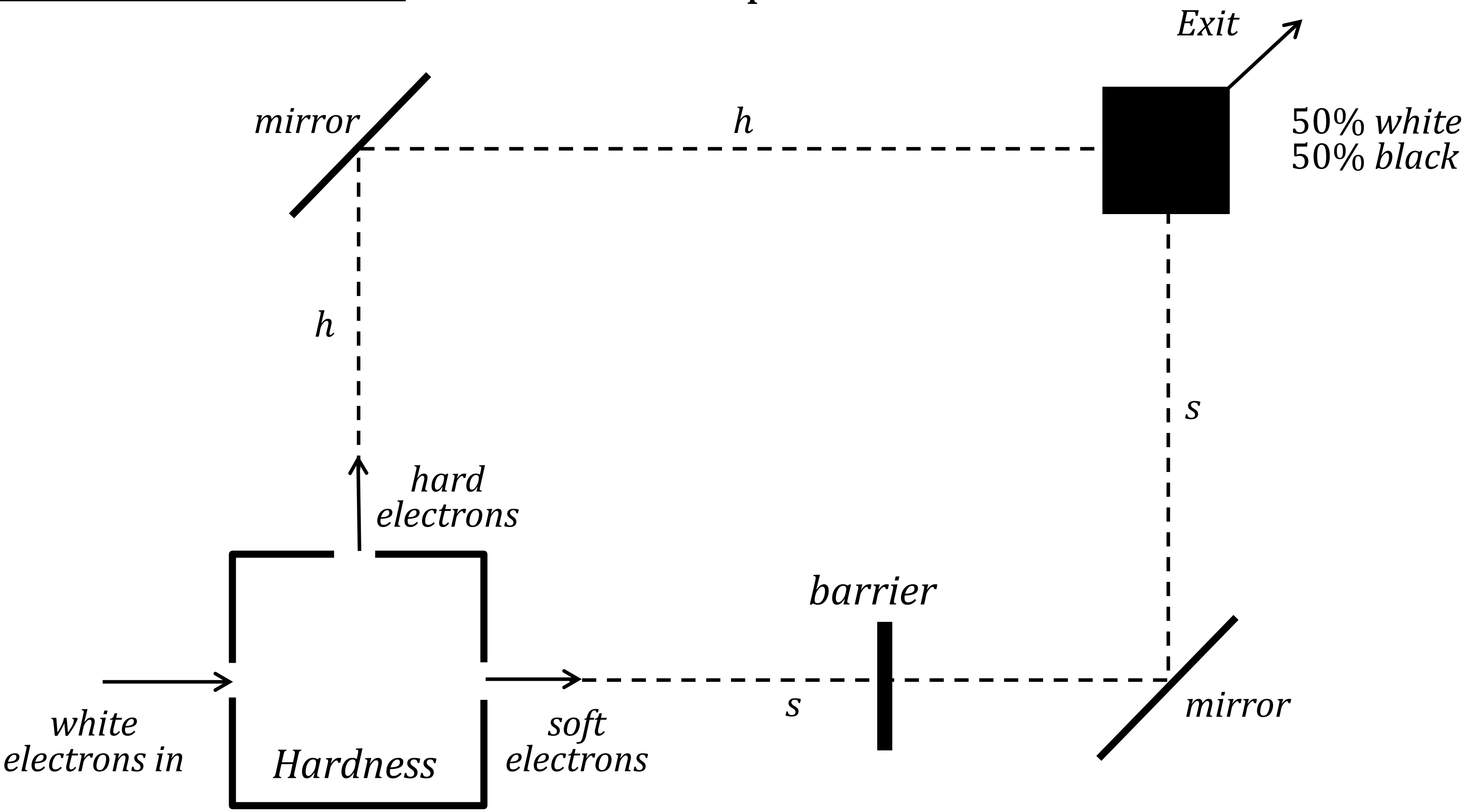
<u>*Experimental Result #3*</u>: The "2-Path" Experiment.



- Feed white electrons into the device and measure their Color as they exit.
- From previous experiments, we should expect 50% white and 50% black...

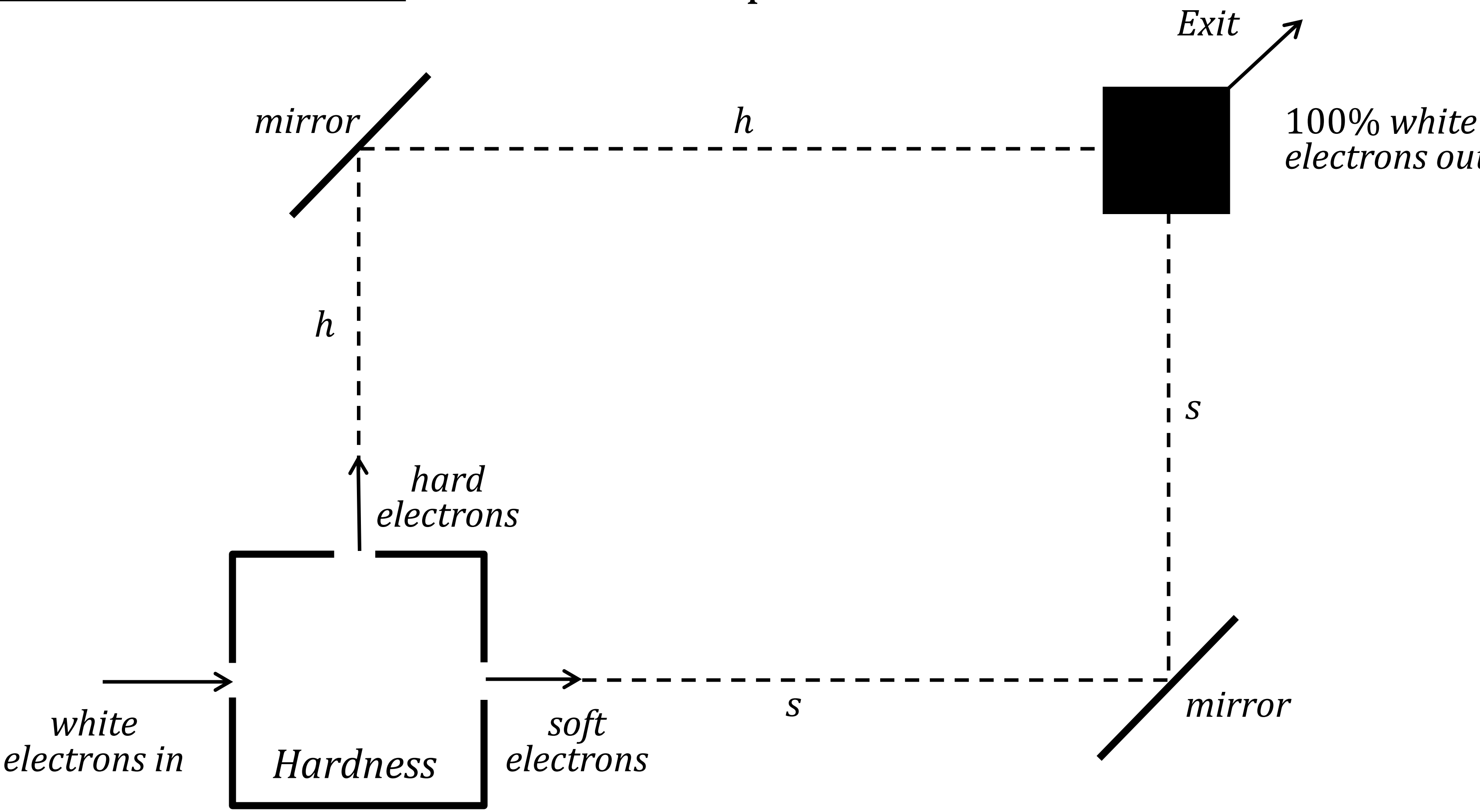
<u>But</u>: Experimentally, 100% are white!

Experimental Result #3: The "2-Path" Experiment.



- Now insert a barrier along the *s* path.
- 50% less electrons register at the Exit.
- *And*: Experimentally, of these 50% are white and 50% are black.

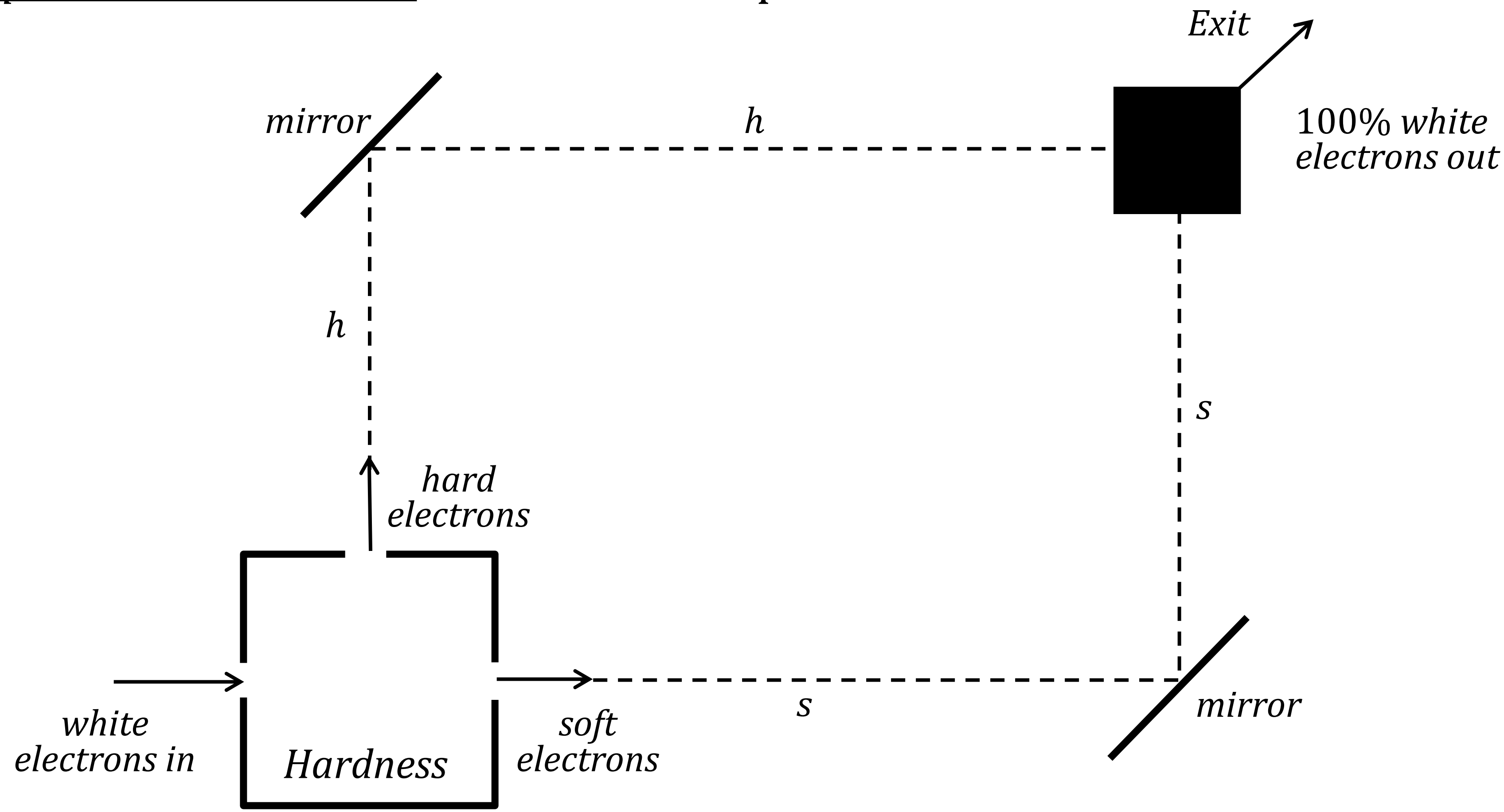
Experimental Result #3: The "2-Path" Experiment.



What path does an individual electron take without the barrier present?

- Not *h*. The Color statistics of hard electrons is 50/50.
- Not *s*. The Color statistics of soft electrons is 50/50.
- Not *both*. Place detectors along the paths and only one will register.
- Not *neither*. Block both paths and no electrons register at Exit.

Experimental Result #3: The "2-Path" Experiment.



What path does an individual electron take without the barrier present?

- Not *h*.
- Not *s*.
- Not *both*.
- Not *neither*.

Suggests that white electrons have no determinate value of Hardness.

How to Describe Physical Phenomena: 5 Basic Notions

(a) **Physical system**.

Classical example: baseball

Quantum example: electron

(b) **Properties** of a physical system.

Classical examples
- momentum
- position
- energy

Quantum examples
- Hardness (spin along a given direction)
- Color (spin along another direction)
- momentum
- position
- energy

(c) **State** of a physical system. Description of system at an instant in time in terms of its properties.

Classical example
- baseball moving at 95mph, 5 ft from batter

Quantum example
- white electron entering a Hardness box

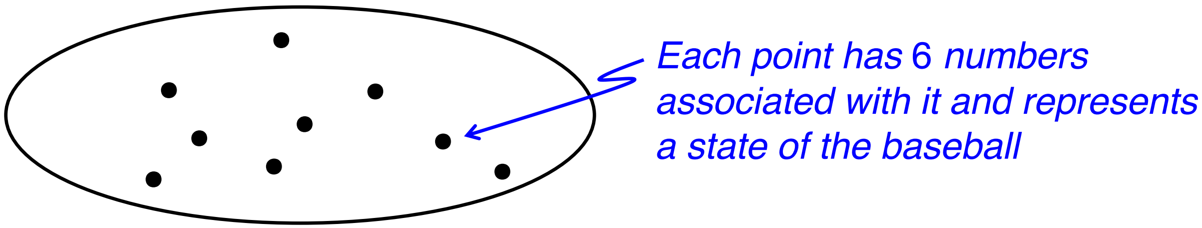
(d) **State space**. The collection of all possible states of a system.

(e) **Dynamics**. A description of how the states of a system evolve in time.

<u>Mathematical Description of Classical Physical System (baseball example)</u>

(i) A **state** of the baseball: Specified by giving *momentum* (p_1, p_2, p_3) and *position* (q_1, q_2, q_3). (Baseball has 6 "degrees of freedom".)

(ii) The **state space** of the baseball: Represented by a 6-dim *set of points* (*phase space*):



(iii) **Properties** of the baseball: Represented by *functions* on the phase space. *These are in-principle <u>always well-defined</u> for any point in phase space.*

<u>*Ex*</u>: baseball's *energy* $= E(p_i, q_i) = (p_1{}^2 + p_2{}^2 + p_3{}^2)/2m$

(iv) **Dynamics** of the baseball: Provided by Newton's equations of motion (in their Hamiltonian form).

Will this mathematical description work for electrons?

No!

- Experiments suggest the "spin" properties of Hardness and Color are not always well-defined.
- *So*: We can't represent them mathematically as functions on a set of points.

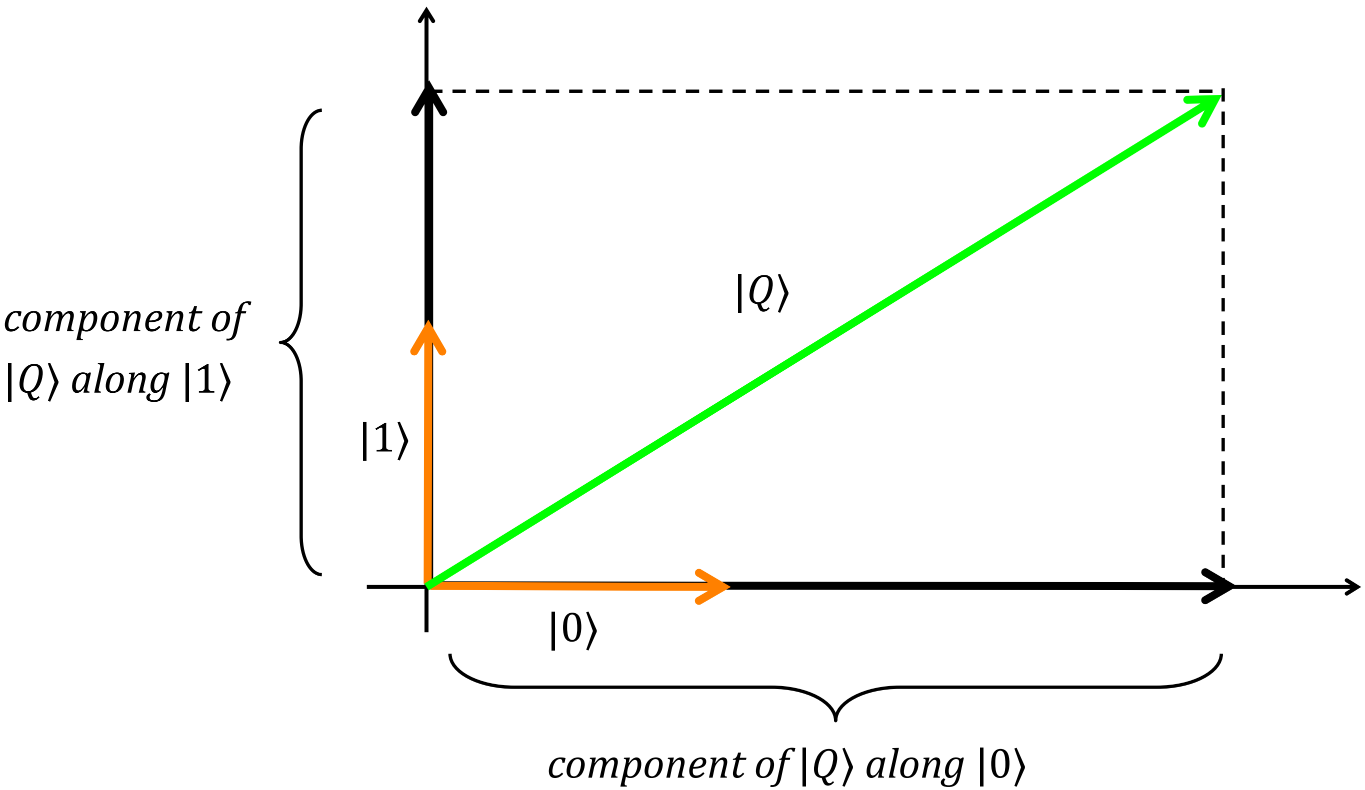
- *Early 20th century task*: Construct a new theory (quantum mechanics) for physical systems like electrons that represents states, state space, and properties in a different way than classical mechanics:

physical concept	*mathematical representation*	
	Classical mechanics	*Quantum mechanics*
states	points	vectors
state space	set of points (phase space)	vector space
properties	functions of points	operators on vectors

2. States as vectors

- Restrict attention to quantum properties with only two values (like *Hardness* and *Color*).
  - *Associated state vectors are 2-dimensional:*

quantum 2-state system



$$|Q\rangle = a|0\rangle + b|1\rangle$$

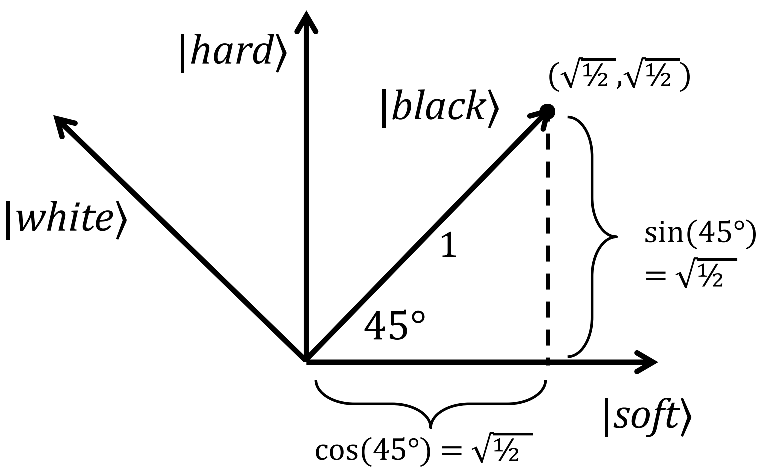
Require state vector $|Q\rangle$ to have unit length:

$$|a|^2 + |b|^2 = 1$$

- Set of all vectors decomposible in basis $\{|0\rangle, |1\rangle\}$ forms a *vector space* $\mathcal{H}$.

Why is this helpful?

- ***Recall***: Black electrons appear to have no determinate value of Hardness.
- Let's represent the *values* of Color and Hardness as basis vectors.
- Let's suppose the Hardness basis $\{|hard\rangle, |soft\rangle\}$ is rotated by 45° with respect to the Color basis $\{|white\rangle, |black\rangle\}$:



Then: $|black\rangle = \sqrt{1/2}\,|hard\rangle + \sqrt{1/2}\,|soft\rangle$

A black vector state of an electron...

... is in a "superposition" of hard and soft vector states.

- Let's assume:

> **Eigenvalue-eigenvector Rule**
> A quantum system possesses the value of a property *if and only if* it is in a vector state associated with that value.

Upshot: Since an electron in the vector state $|white\rangle$ cannot be in either of the vector states $|hard\rangle$, $|soft\rangle$, it cannot be said to possess values of Hardness.

- *Recall*: Experimental Result #1: There is no correlation between Hardness measurements and Color measurements.
  - *If the Hardness of a batch of white electrons is measured,* 50% *will be soft and* 50% *will be hard.*

- Let's assume:

Born Rule
The probability $\Pr_{|\psi\rangle}(b|B)$ that a quantum system in a vector state $|\psi\rangle$ possesses the value b of a property B is given by the square of the expansion coefficient of the basis state $|b\rangle$ in the expansion of $|\psi\rangle$ in the basis corresponding to all values of the property.


Max Born
(1882-1970)

- *So*: The probability $\Pr_{|black\rangle}(hard|\text{Hardness})$ that a *black* electron has the value *hard* when measured for Hardness is ½!

$$|black\rangle = \sqrt{½}\,|hard\rangle + \sqrt{½}|soft\rangle$$

An electron in a black vector state...

... has a probability of ½ of being in a hard vector state upon measurement for Hardness.

3. Properties as operators

- *Motivation*: A property has *values*.
  - *And*: We've associating these values with basis vectors.
  - *And*: A certain type of linear operator (a "Hermitian operator") can be associated with a set of basis vectors.

Def. 1 (*Linear operator*). A **linear operator** O is a map that assigns to any vector $|A\rangle$, another vector $O|A\rangle$, such that, for any other vector $|B\rangle$ and numbers n, m,

$$O(n|A\rangle + m|B\rangle) = n(O|A\rangle) + m(O|B\rangle)$$

To understand the notion of a Hermitian operator, let's first consider matrix representations of vectors and linear operators…

Matrix representations

$|Q\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ 2-dim vector as 2×1 "column" matrix

$\langle Q| = (a^*, b^*)$ complex-transpose of $|Q\rangle$ as 1×2 "row" matrix

$$\langle Q|Q\rangle = (a^*, b^*)\begin{pmatrix} a \\ b \end{pmatrix} = a^*a + b^*b = |a|^2 + |b|^2 = 1$$

Matrix multiplication

$$O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$$

Operator on 2-dim vectors as 2×2 matrix

$$O|Q\rangle = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} O_{11}a + O_{12}b \\ O_{21}a + O_{22}b \end{pmatrix}$$

$$\langle Q|O^\dagger = (a^*, b^*)\begin{pmatrix} O_{11}^* & O_{21}^* \\ O_{12}^* & O_{22}^* \end{pmatrix} = (O_{11}^*a^* + O_{12}^*b^*, O_{21}^*a^* + O_{22}^*b^*)$$

$O^\dagger$ is complex-transpose (or "adjoint") of O

Def. 2 (*Hermitian operator*). An operator O is Hermitian (or "self-adjoint") just when $O = O^\dagger$.

Now: In what sense can a Hermitian operator be associated with a set of basis vectors…

Def. 3 (*Eigenvector*). An **eigenvector** of an operator O is a vector $|\lambda\rangle$ that does not change its direction when O acts on it: $O|\lambda\rangle = \lambda|\lambda\rangle$, for some number λ.

Def. 4 (*Eigenvalue*). An **eigenvalue** λ of an operator O is the number that results when O acts on one of its eigenvectors.

Claim. The eigenvectors of a Hermitian operator on a vector space $\mathcal{H}$ form a basis of $\mathcal{H}$, and its eigenvalues are real numbers.

<u>*This suggests the following correspondences*</u>

- Let a Hermitian operator O represent a *property*.
- Let its eigenvectors $|\lambda\rangle$ represent the *value states* ("eigenstates") associated with the property.
- Let its eigenvalues λ represent the (real number) *values* of the property.

- The Eigenvalue-Eigenvector Rule can now be stated as:

Eigenvalue-Eigenvector Rule. A quantum system possesses the value λ of a property represented by a Hermitian operator O *if and only if* it is in a vector state $|\lambda\rangle$ that is an eigenvector of O with eigenvalue λ.

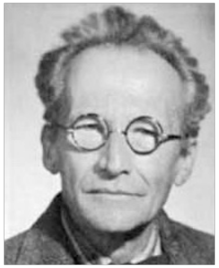
4. Dynamics and Projection Postulate

Schrödinger Dynamics

Vector states evolve in time via the Schrödinger equation.

$$|\psi(t_1)\rangle \xrightarrow{\text{Schrödinger evolution}} |\psi(t_2)\rangle$$

vector state at time t_1 — vector state at later time t_2



Erwin Schrödinger
(1887-1961)

- The Schrödinger equation can be encoded in an operator $S \equiv e^{-iH(t_2-t_1)/\hbar}$ (where H is the Hamiltonian operator that encodes the energy).

$$|A\rangle \longrightarrow |A'\rangle = S|A\rangle$$

vector state at t_1 — vector state at t_2

<u>Important property</u>: S is a *linear operator*.

$S(n|A\rangle + m|B\rangle) = n(S|A\rangle) + m(S|B\rangle)$, where n, m are numbers.

Projection Postulate (2-state systems)

When a measurement of a property represented by an operator B is made on a system in the vector state $|Q\rangle = a|\lambda_1\rangle + b|\lambda_2\rangle$ expanded in the eigenvector basis of B, and the result is the value λ_1, then $|Q\rangle$ *collapses* to the state $|\lambda_1\rangle$:

$$|Q\rangle \xrightarrow[\textit{collapse}]{} |\lambda_1\rangle$$

John von Neumann
(1903-1957)

Example: Suppose we measure a *black* electron for Hardness.

- The pre-measurement state is given by:

$$|black\rangle = \sqrt{½}\,|hard\rangle + \sqrt{½}|soft\rangle$$

- *Suppose*: The outcome of the measurement is the value *hard*.
- *Then*: The post-measurement state is given by $|hard\rangle$.

Motivations:

- Guarantees that measurements have unique outcomes.
- Guarantees that if we obtain the value λ_1 once, then we should get the same value λ_1 on a second measurement (provided the system is not interferred with).

Recap: 5 Principles of Quantum Mechanics

(1) **States** are represented by vectors of length 1.

(2) **Properties** are represented by Hermitian operators.

<u>*Eigenvalue-Eigenvector Rule*</u>: A quantum system possesses the value λ of a property represented by a Hermitian operator O *if and only if* it is in a vector state $|\lambda\rangle$ that is an eigenvector of O with eigenvalue λ.

(3) **Dynamics** is given by the linear Schrödinger equation.

$$|\psi(t_1)\rangle \xrightarrow[\text{Schrödinger evolution}]{} |\psi(t_2)\rangle$$

(4) **Born Rule**.

$\Pr_{|\psi\rangle}(b|B) = |\langle\psi|b_i\rangle|^2$ where $|b_i\rangle$ is an eigenvector of B with eigenvalue b_i

(5) **Projection Postulate**.

When a measurement of a property represented by an operator B is made on a system in the vector state $|Q\rangle = a|\lambda_1\rangle + b|\lambda_2\rangle$ expanded in the eigenvector basis of B, and the result is the value λ_1, then $|Q\rangle$ *collapses* to the state $|\lambda_1\rangle$:

$$|Q\rangle \xrightarrow[\text{collapse}]{} |\lambda_1\rangle$$