

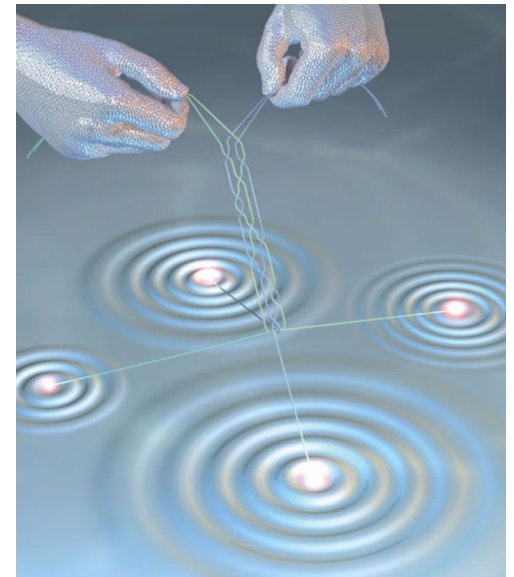
13. Topology and Quantum Error Correction

1. Motivation

- Last lecture: The *geometrical* (*metrical*) structure of spacetime affects what is and is not computable.
- This lecture: The *topological* structure of spacetime affects the effectiveness of quantum error correcting codes (QECCs).

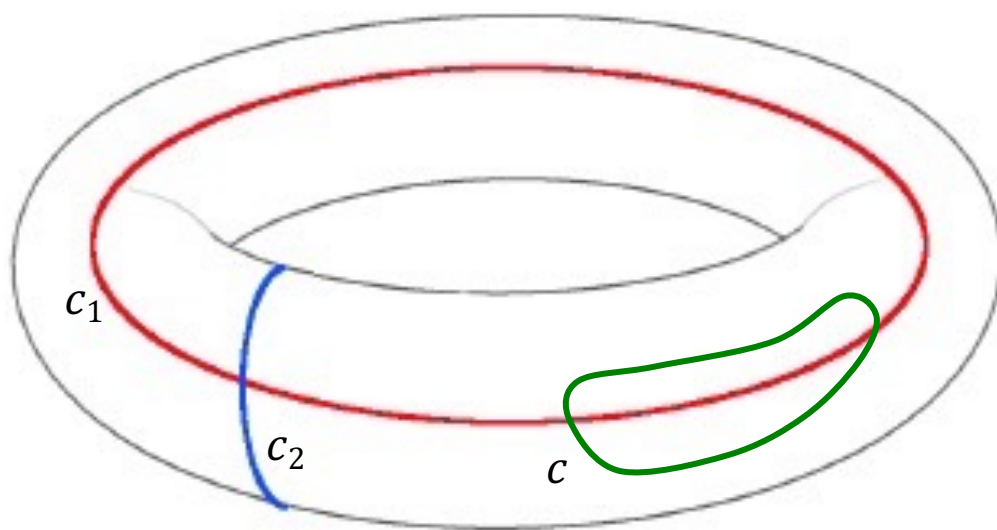
- Question: Is there a way to guarantee the KL Condition for a QECC based on the *topology* of the physical system we use to encode information in qubits?
- Immediate goal: To construct a QECC from a physical system with a non-trivial topology.
- Ultimate goal: To build a "topological" quantum computer.

Yes!



A *topological property* of a surface is a property that remains invariant under continuous deformations of the surface.

Example: Consider 2-dim surface of a torus.



Three types of closed paths:

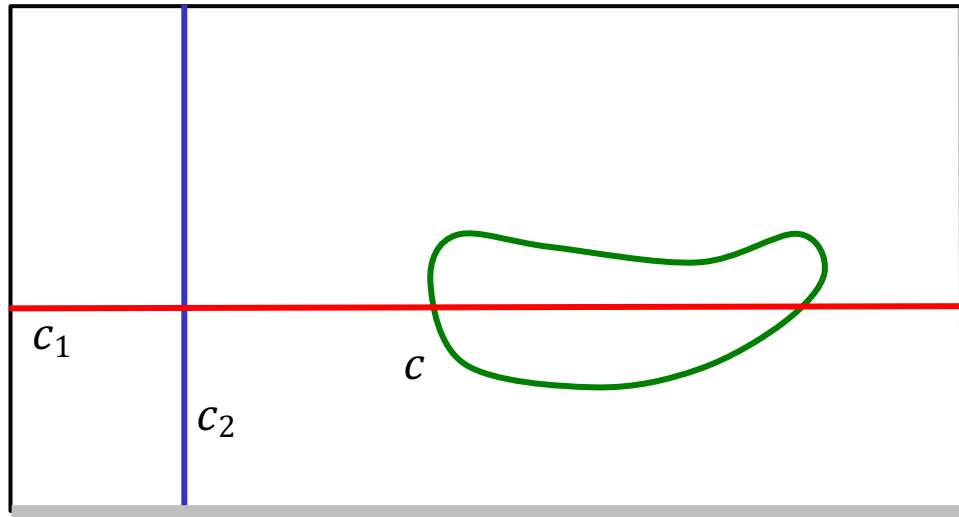
- Contractible loops c which can be continuously deformed into a point.
- Non-contractible loops c_1 which cannot be continuously deformed into a point.
- Non-contractible loops c_2 which cannot be continuously deformed into a point.

- None of these loops can be continuously deformed into the others.
- The surface of a torus is topologically characterized by these loops.



They describe features of the torus that are invariant under continuous deformations of its surface (i.e., they are topological properties).

Slightly more abstract way to represent a torus: unwind it into a flat surface with periodic boundary conditions.

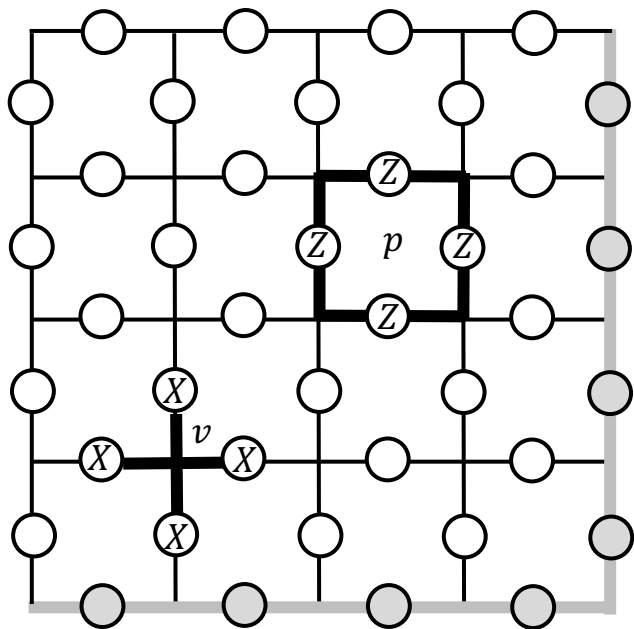


Periodic boundary conditions:

- Identify top and bottom edges.
- Identify left and right edges.

Let's add some (abstract) physics...

2. The Toric Code



- Put an $L \times L$ lattice on the torus with L^2 vertices.
- On each lattice edge, place a qubit.

$2L^2$ -qubit Hilbert space $\mathcal{H} = V_1 \otimes \cdots \otimes V_{2L^2}$,
where each V is a single-qubit Hilbert space.

- For each vertex v define a vertex operator A_v :

$$A_v = I_1 \otimes \cdots \otimes I_{(i-1)} \otimes (X_{(i)} \otimes X_{(i+1)} \otimes X_{(i+2)} \otimes X_{(i+3)}) \\ \otimes I_{(i+4)} \otimes \cdots \otimes I_{(2L^2)}$$

Acts as I on all qubits except those on edges leading to v , on which it acts as X .

- For each plaquette p define a plaquette operator B_p :

$$B_p = I_1 \otimes \cdots \otimes I_{(j-1)} \otimes (Z_{(j)} \otimes Z_{(j+1)} \otimes Z_{(j+2)} \otimes Z_{(j+3)}) \\ \otimes I_{(j+4)} \otimes \cdots \otimes I_{(2L^2)}$$

Acts as I on all qubits except those on edges around p , on which it acts as Z .

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue $+1$.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Claim: $\mathcal{C}$ is a 4-dim (i.e., two-qubit) subspace of $\mathcal{H}$ with "topologically distinct" basis states $\{|\xi\rangle_{ee}, |\xi\rangle_{eo}, |\xi\rangle_{oe}, |\xi\rangle_{oo}\}$ that are entangled with respect to the decomposition $\mathcal{H} = V_1 \otimes \cdots \otimes V_{2L^2}$.

Story to come:

- $\mathcal{C}$ will be our **code space**: use its two entangled qubits to encode information in a topologically non-local way.
- Operators that act like the identity on codewords in $\mathcal{C}$ will be "local" operators associated with contractible loops.
- Operators that act nontrivially on codewords within $\mathcal{C}$ will be "non-local" operators associated with non-contractible loops.
- **Error** operators that act on codewords to transform them out of $\mathcal{C}$ will be "local" operators associated with contractible open paths.

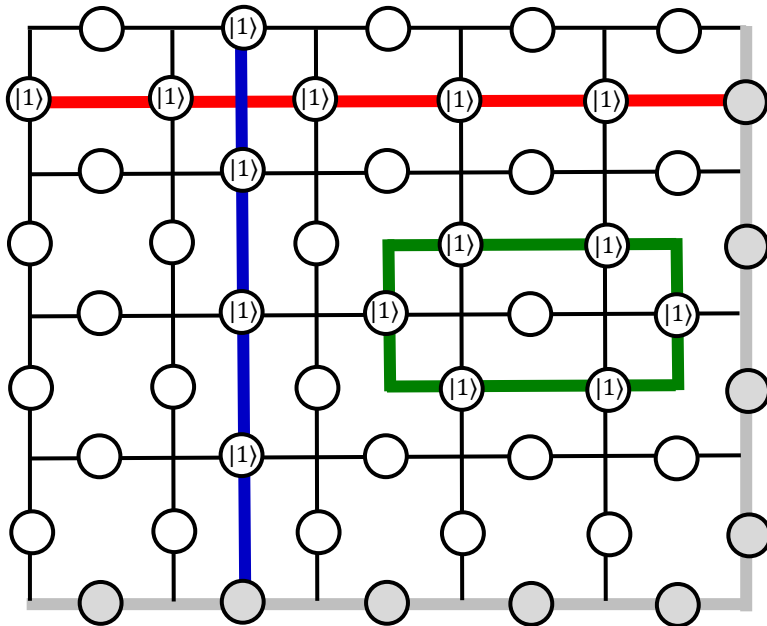
Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue +1.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Constraints:

(a) $B_p|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must either be the $|0\rangle$ $2L^2$ -qubit state, or have an *even number* of $|1\rangle$ qubits per plaquette, since $Z|1\rangle = -|1\rangle$.

Claim: Constraint (a) entails $|\xi\rangle$ is either the $|0\rangle$ $2L^2$ -qubit state or a *loop state*.



Def. A *loop state* is a $2L^2$ -qubit state that has $|1\rangle$'s along one or more closed loops that do not intersect vertices, and $|0\rangle$'s everywhere else.

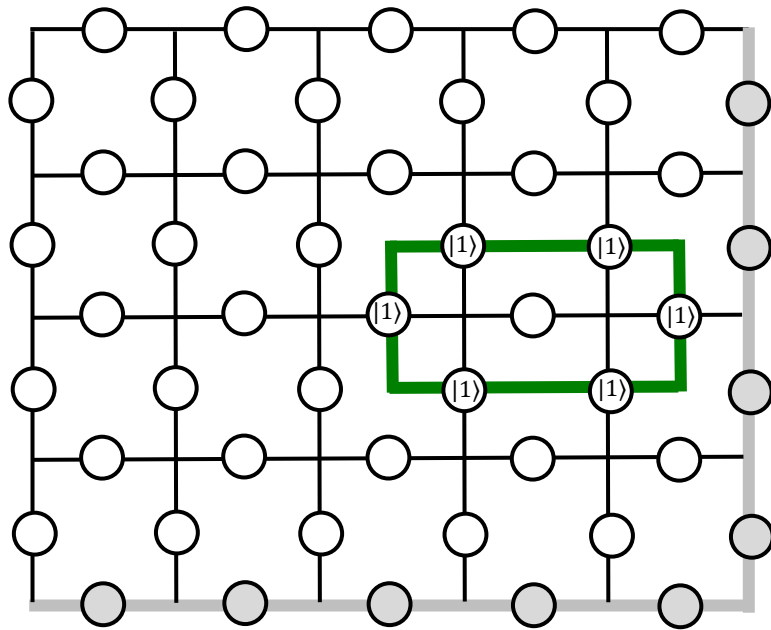
↪ A loop state consisting of three closed loops of $|1\rangle$'s. (The "empty" qubits are $|0\rangle$'s.)

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue +1.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Constraints:

(b) $A_v|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must be a superposition of a vector in $\mathcal{H}$ and its A_v -flipped counterpart, since X flips qubits.



$|\text{loop}\rangle$

Example:

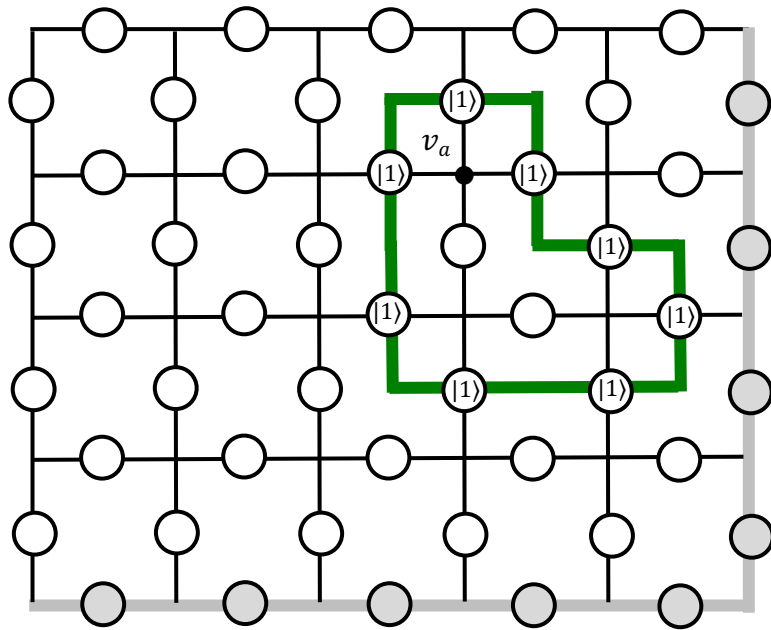
- This loop state is not a +1 eigenvector of any A_v , since they flip $|0\rangle$'s to $|1\rangle$'s and $|1\rangle$'s to $|0\rangle$'s.

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue +1.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Constraints:

(b) $A_v|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must be a superposition of a vector in $\mathcal{H}$ and its A_v -flipped counterpart, since X flips qubits.



$A_{v_a}|\text{loop}\rangle$

Example:

- But: The sum of the loop state and any of its A_v -flipped counterparts is a +1 eigenvector of that vertex operator!
- $A_{v_a}\{|\text{loop}\rangle + A_{v_a}|\text{loop}\rangle\} = \{A_{v_a}|\text{loop}\rangle + |\text{loop}\rangle\}$
- Also note: Any A_v that acts on all $|0\rangle$'s creates a loop. And any A_v that "touches" a loop deforms it into another loop.
- So: If $|\xi\rangle$ is a loop state, then so is $A_v|\xi\rangle$ for all v .

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue +1.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Constraints:

- (a) $B_p|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must either be the $|0\rangle$ $2L^2$ -qubit state or a loop state.
- (b) $A_v|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must be a superposition of a vector in $\mathcal{H}$ and its A_v -flipped counterpart.

Claim: A vector that satisfies (a) and (b) is given by:

$$|\xi\rangle_{ee} = \prod_{i=1}^{L^2} 2^{-1/2} (I + A_{v_i}) |0\rangle_1 \cdots |0\rangle_{2L^2}$$

- ← - Start with $|0\rangle$ $2L^2$ -qubit state.
 - Then add all other states that can be obtained via A_v -flips, and their unflipped counterparts.

Proof: Let $j = 1, \dots, L^2$. Then

$$\begin{aligned} B_{p_j}|\xi\rangle_{ee} &= 2^{-L^2/2} B_{p_j} (I + A_{v_1}) \cdots (I + A_{v_{L^2}}) |0 \cdots 0\rangle \\ &= 2^{-L^2/2} (I + A_{v_1}) \cdots (I + A_{v_{L^2}}) B_{p_j} |0 \cdots 0\rangle & B_{p_j} \text{ commutes with } (I + A_{v_i}) \text{ for all } i, j \\ &= 2^{-L^2/2} (I + A_{v_1}) \cdots (I + A_{v_{L^2}}) |0 \cdots 0\rangle & B_{p_j} |0 \cdots 0\rangle = |0 \cdots 0\rangle, \text{ for all } j \\ &= |\xi\rangle_{ee} \end{aligned}$$

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue +1.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Constraints:

- (a) $B_p|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must either be the $|0\rangle_{2L^2}$ -qubit state or a loop state.
- (b) $A_v|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must be a superposition of a vector in $\mathcal{H}$ and its A_v -flipped counterpart.

Claim: A vector that satisfies (a) and (b) is given by:

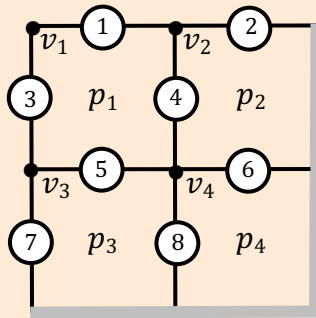
$$|\xi\rangle_{ee} = \prod_{i=1}^{L^2} 2^{-1/2} (I + A_{v_i}) |0\rangle_1 \cdots |0\rangle_{2L^2}$$

- Start with $|0\rangle_{2L^2}$ -qubit state.
- Then add all other states that can be obtained via A_v -flips, and their unflipped counterparts.

Proof: Let $j = 1, \dots, L^2$. Then

$$\begin{aligned} A_{v_j}|\xi\rangle_{ee} &= 2^{-L^2/2} A_{v_j} (I + A_{v_1}) \cdots (I + A_{v_{L^2}}) |0\dots 0\rangle \\ &= 2^{-L^2/2} (I + A_{v_1}) \cdots A_{v_j} (I + A_{v_j}) \cdots (I + A_{v_{L^2}}) |0\dots 0\rangle & A_{v_j} \text{ commutes with } (I + A_{v_i}) \\ &= 2^{-L^2/2} (I + A_{v_1}) \cdots (A_{v_j} + A_{v_j} A_{v_j}) \cdots (I + A_{v_{L^2}}) |0\dots 0\rangle \\ &= 2^{-L^2/2} (I + A_{v_1}) \cdots (I + A_{v_j}) \cdots (I + A_{v_{L^2}}) |0\dots 0\rangle & A_{v_j} A_{v_j} = I \\ &= |\xi\rangle_{ee} \end{aligned}$$

Ex: $L = 2$.



8 qubits (so $\mathcal{H}$ has $2^8 = 256$ dimensions!)

4 plaquettes: $p_1 = \{1, 3, 4, 5\}$ $p_2 = \{2, 3, 4, 6\}$

$p_3 = \{1, 5, 7, 8\}$ $p_4 = \{2, 6, 7, 8\}$

4 vertices: $v_1 = \{1, 2, 3, 7\}$ $v_2 = \{1, 2, 4, 8\}$

$v_3 = \{3, 5, 6, 7\}$ $v_4 = \{4, 5, 6, 8\}$

$$|\xi\rangle_{ee} = \prod_{i=1}^4 2^{-1/2} (I + A_{v_i}) |00000000\rangle$$

$$= \frac{1}{4} (I + A_{v_1}) (I + A_{v_2}) (I + A_{v_3}) (I + A_{v_4}) |00000000\rangle$$

$$= \frac{1}{4} (I + A_{v_1}) (I + A_{v_2}) (I + A_{v_3}) \{ |00000000\rangle + |00011101\rangle \}$$

$$= \frac{1}{4} (I + A_{v_1}) (I + A_{v_2}) \{ |00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle \}$$

$$= \frac{1}{4} (I + A_{v_1}) \{ |00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle + |11010001\rangle + |11001100\rangle + |11111111\rangle + |11100010\rangle \}$$

$$= \frac{1}{4} \{ |00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle + |11010001\rangle + |11001100\rangle + |11111111\rangle + |11100010\rangle + |11100010\rangle + |11111111\rangle + |11001100\rangle + |11010001\rangle + |00110011\rangle + |00101110\rangle + |00011101\rangle + |00000000\rangle \}$$

$$= \frac{1}{2} \{ |00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle + |11010001\rangle + |11001100\rangle + |11111111\rangle + |11100010\rangle \}$$

- Entangled state!
- Each term has even # of 1's
- Each term has an A_v -flipped counterpart

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue $+1$.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

- So: One element of $\mathcal{C}$ is

$$|\xi\rangle_{ee} = \prod_{i=1}^{L^2} 2^{-1/2} (I + A_{v_i}) |0\rangle_1 \cdots |0\rangle_{2L^2}$$

The $(I + A_{v_i})$ operators do not change the parity of the number of c_1 and c_2 loops, so $|\xi\rangle_{ee}$ also has an even number of c_1 and c_2 loops.

The $|0\rangle_{2L^2}$ -qubit state has an even number (zero!) of c_1 and c_2 loops.

- This entails: There are four *topologically distinct* types of elements of $\mathcal{C}$:

$|\xi\rangle_{ee}$: loop state with even # c_1 loops and even # c_2 loops.

$|\xi\rangle_{eo}$: loop state with even # c_1 loops and odd # c_2 loops.

$|\xi\rangle_{oe}$: loop state with odd # c_1 loops and even # c_2 loops.

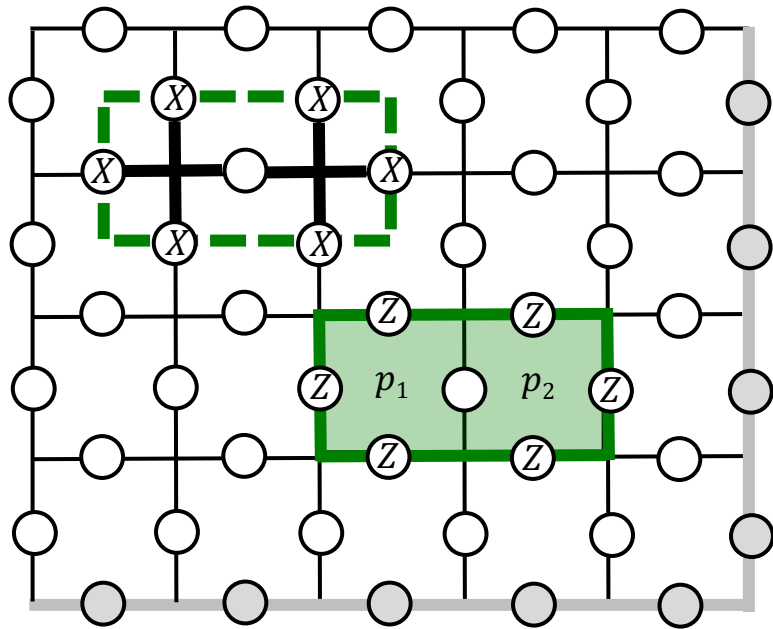
$|\xi\rangle_{oo}$: loop state with odd # c_1 loops and odd # c_2 loops.

- So: $\mathcal{C} = \text{span}\{|\xi\rangle_{ee}, |\xi\rangle_{eo}, |\xi\rangle_{oe}, |\xi\rangle_{oo}\}$
 - Topologically distinct bases vectors.*
 - Entangled with respect to $\mathcal{H} = V_1 \otimes \cdots \otimes V_{2L^2}$.*

Three types of operators that act on $\mathcal{C}$

First type: "Stabilizer" operators.

- Composing adjacent plaquette operators B_{p_1}, B_{p_2} to form $B_{p_1}B_{p_2}$ results in a *loop* of Z operators:



- B_{p_1} and B_{p_2} share an edge.
- $B_{p_1}B_{p_2}$ includes the square of the Z operator of the shared edge, and $Z^2 = I$.
- So: The Z 's that appear in $B_{p_1}B_{p_2}$ will act on the qubits that form the *boundary* of the two plaquettes!
- The same holds for any number of adjacent plaquette operators.
- The same holds for vertex operators A_v .

- Note: These loops are of type c on the torus.

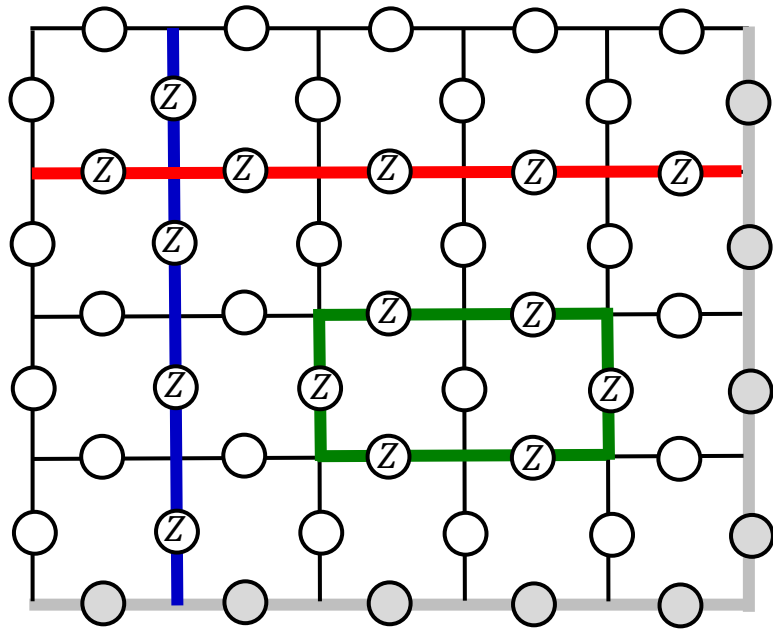
Type c loop operators are called "stabilizer" operators:

- They act like the identity on $\mathcal{C}$ (since they are compositions of A_v and B_p operators).
- They are "local" (in the sense that they are associated with contractible loops).

Three types of operators that act on $\mathcal{C}$

Second type: "Encoded logical" operators.

- There are two other types of loops on a torus: *non-contractible* loops c_1 and c_2 .



- Let $\bar{Z}_1$ and $\bar{Z}_2$ refer to the two types of products of Z operators along loops of type c_1 and c_2 .
- Let $\bar{X}_1$ and $\bar{X}_2$ refer to the two types of products of X operators along loops of type c_1 and c_2 .

Types c_1 and c_2 loop operators are called "encoded logical" operators:

- They act on codewords in $\mathcal{C}$ and transform them into other codewords.
- They are not "local" operators (in the sense that they are associated with non-contractible loops).

Aside: Non-contractible loop operators map vectors in $\mathcal{C}$ to other vectors in $\mathcal{C}$

Claim 1. Any operator D that maps $\mathcal{C}$ to $\mathcal{C}$ must commute with all A_v and B_p operators.

Proof. Recall $\mathcal{C} = \{|\phi\rangle : \mathcal{O}|\phi\rangle = |\phi\rangle, \text{ for } \mathcal{O} = A_v \text{ or } B_p\}$.

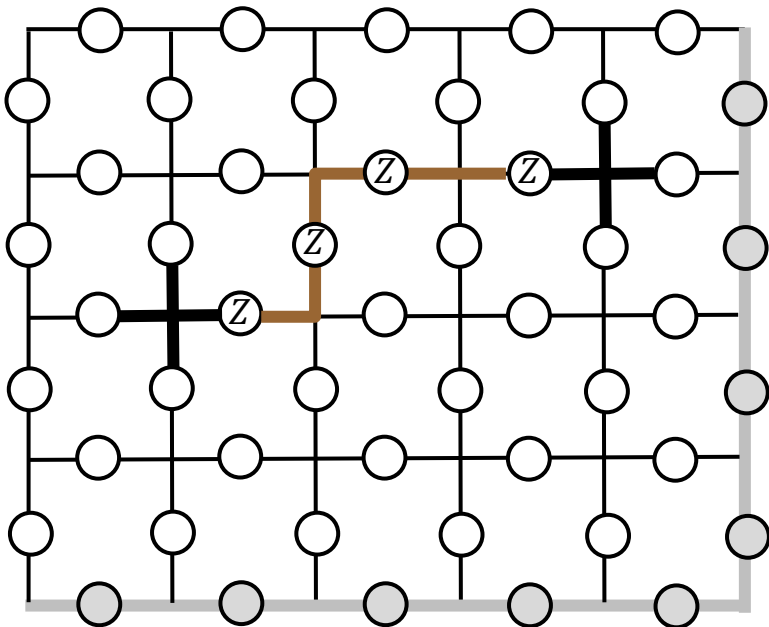
- Let D be an operator such that $D|\psi\rangle \in \mathcal{C}$, for any $|\psi\rangle \in \mathcal{C}$.
- Suppose $D\mathcal{O} = -\mathcal{O}D$ (D anticommutes with $\mathcal{O}$).
 - Then for any $|\psi\rangle \in \mathcal{C}$, $D|\psi\rangle = D\mathcal{O}|\psi\rangle = -\mathcal{O}D|\psi\rangle$.
 - Or: $\mathcal{O}D|\psi\rangle = -D\mathcal{O}|\psi\rangle = -D|\psi\rangle$.
 - So: $\mathcal{O}(D|\psi\rangle) = -(D|\psi\rangle) \neq D|\psi\rangle$.
 - But: $\mathcal{O}$ is the identity on $\mathcal{C}$!
 - So: $D|\psi\rangle \notin \mathcal{C}$ (contradiction!)
- Hence: D must commute with $\mathcal{O}$.

Claim 1 (reworded).

(maps $\mathcal{C}$ to $\mathcal{C}$) $\Rightarrow$ (commutes with all A_v, B_p)

Aside: Non-contractible loop operators map vectors in $\mathcal{C}$ to other vectors in $\mathcal{C}$

Claim 2. Any operator formed from an open path of X 's or Z 's will anticommute with two A_v 's or two B_p 's.



- Consider an operator $S^Z(t)$ which is the identity on all qubits except for an open path t of Z 's.
- $S^Z(t)$ commutes with all B_p 's.
 - We only need to consider B_p 's with Z 's that overlap the Z 's in $S^Z(t)$.
 - Those B_p 's commute with $S^Z(t)$, since Z commutes with itself.
- $S^Z(t)$ commutes with all A_v 's, except for the two at the endpoints of t .
 - We only need to consider A_v 's with X 's that overlap the Z 's in $S^Z(t)$.
 - Those A_v 's that are not at the endpoints of t commute with $S^Z(t)$, since each one has 2 X 's that overlap 2 Z 's in $S^Z(t)$, and Z anticommutes with X .
- $S^Z(t)$ anticommutes with the two A_v 's at the endpoints of t .
 - Each of these A_v 's has 1 X that overlaps 1 Z in $S^Z(t)$, and Z anticommutes with X .

Claim 2 (reworded).
(open path operator) $\Rightarrow$
 $\neg$ (commutes with all A_v, B_p)

Aside: Non-contractible loop operators map vectors in $\mathcal{C}$ to other vectors in $\mathcal{C}$

Claim 1 (reworded).

(maps $\mathcal{C}$ to $\mathcal{C}$) $\Rightarrow$ (commutes with all A_v, B_p)

Claim 2 (reworded).

(open path operator) $\Rightarrow \neg$ (commutes with all A_v, B_p)

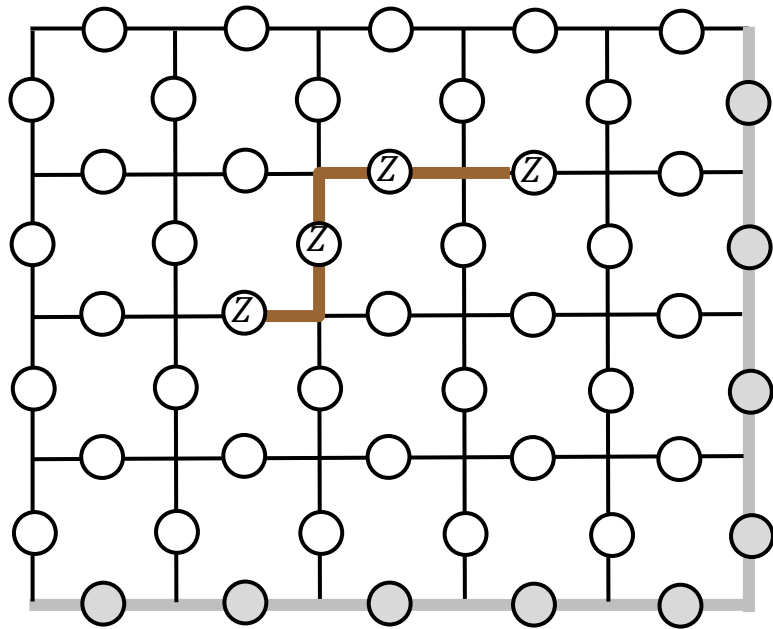
- Claims 1 & 2 entail:
(maps $\mathcal{C}$ to $\mathcal{C}$) $\Rightarrow \neg$ (open path operator)
- So: A code space operator is either an individual A_v or B_p , or a loop operator.
- And: The identity on $\mathcal{C}$ is either an individual A_v or B_p , or a contractible loop operator.
 - Moreover: A non-contractible loop operator cannot be constructed by a product of A_v 's or B_p 's (such a product is the boundary of a set of adjacent plaquettes or vertices).

So: A non-contractible loop operator is a code space operator that is not the identity on $\mathcal{C}$.

Three types of operators that act on $\mathcal{C}$

Third type: Error operators.

- By definition, error operators act on codewords and corrupt them (transform them into states not in $\mathcal{C}$).



- Error operators can't be associated with products of Z 's or X 's on loops: There are only three types, and each type transforms codewords to codewords.
- What about "open path" products of Z 's or X 's?

Claim: Open path products of Z 's or X 's transform codewords in $\mathcal{C}$ out of $\mathcal{C}$.

Proof: We've just seen that open path products of Z 's or X 's anticommute with two A_v 's or two B_p 's, and hence transform codewords out of $\mathcal{C}$.

- "Open path" operators are "local" (in the sense that they are associated with *contractible* line segments).

Summary: Three types of operators that act on $\mathcal{C}$

1. Stabilizer operators (*local*).

$$S^Z(c) = \bigotimes_{j \in c} Z_j$$

$$S^X(c') = \bigotimes_{j \in c'} X_j$$

$c, c' =$ contractible
loops

2. Encoded logical operators (*non-local*).

$$\bar{Z}_1 = \bigotimes_{j \in \gamma_1} Z_j \quad \bar{Z}_2 = \bigotimes_{j \in \gamma_2} Z_j$$

$$\bar{X}_1 = \bigotimes_{j \in \gamma'_1} X_j \quad \bar{X}_2 = \bigotimes_{j \in \gamma'_2} X_j$$

$\gamma_1, \gamma'_1 =$ non-contractible
loops of type c_1

$\gamma_2, \gamma'_2 =$ non-contractible
loops of type c_2

3. Error operators (*local*).

$$S^Z(t) = \bigotimes_{j \in t} Z_j$$

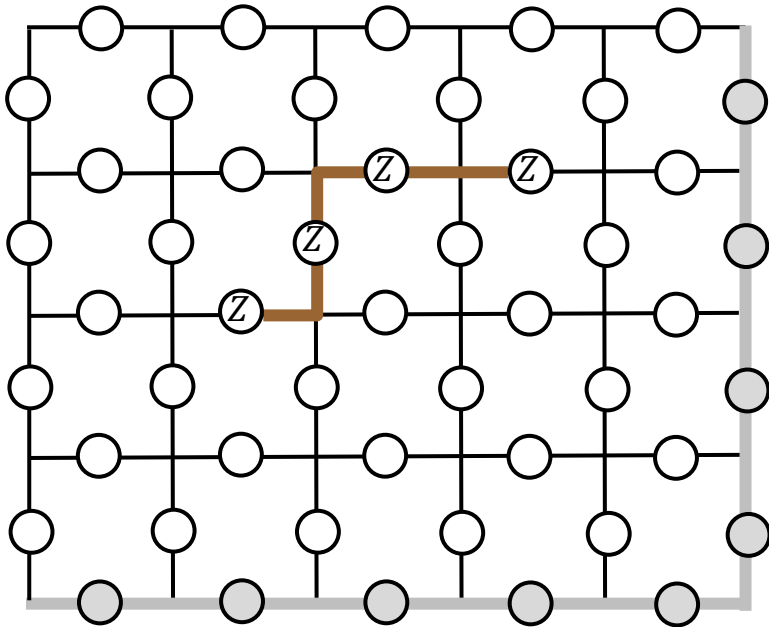
$$S^X(t') = \bigotimes_{j \in t'} X_j$$

$t, t' =$ contractible open
paths

Now: Check to see if the KL Condition holds for the toric code.

- Does $\mathcal{C}$ correct the error set $\mathcal{E} = \{S^Z(t), S^X(t') : \text{for all } t, t'\}$?
 - Is it the case that $\langle \psi_i | E_k^\dagger E_l | \psi_j \rangle = c_{kl} \delta_{ij}$, for any $E_k, E_l \in \mathcal{E}$, and $\psi_i, \psi_j \in \mathcal{C}$?

Yes!

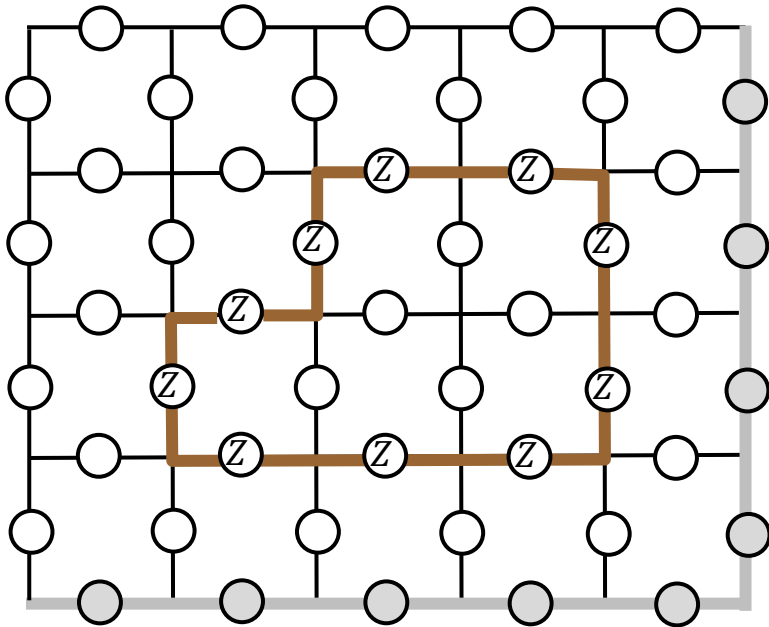


- For any open-path operator E_l between two endpoints...

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Yes!

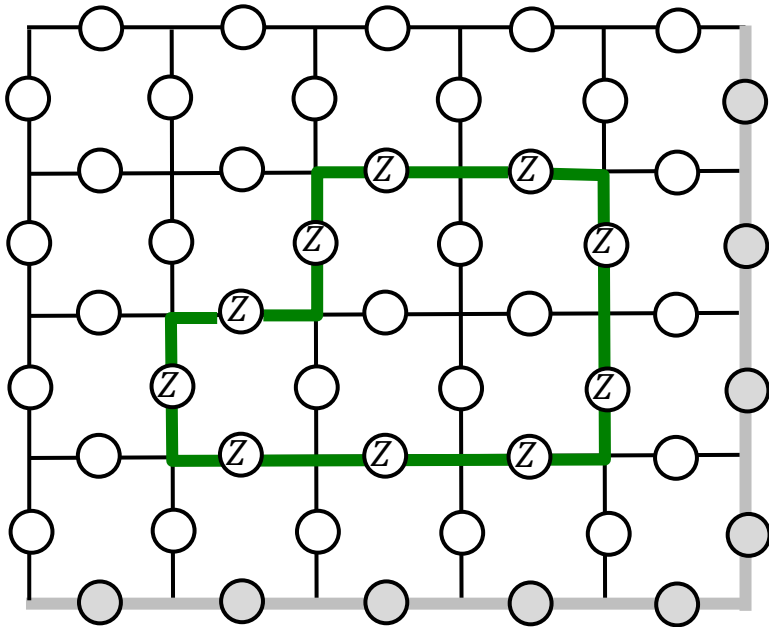


- For any open-path operator E_l between two endpoints...
- ... there is always another E_k with the same endpoints such that $E_k^\dagger E_l$ is a "type-c" loop operator; *i.e.*, a stabilizer operator.

Now: Check to see if the KL Condition holds for the toric code.

- Does $\mathcal{C}$ correct the error set $\mathcal{E} = \{S^Z(t), S^X(t') : \text{for all } t, t'\}$?
 - Is it the case that $\langle \psi_i | E_k^\dagger E_l | \psi_j \rangle = c_{kl} \delta_{ij}$, for any $E_k, E_l \in \mathcal{E}$, and $\psi_i, \psi_j \in \mathcal{C}$?

Yes!



- For any open-path operator E_l between two endpoints...
- ... there is always another E_k with the same endpoints such that $E_k^\dagger E_l$ is a "type-c" loop operator; *i.e.*, a stabilizer operator.
- And: Stabilizer operators act as the identity on $\mathcal{C}$.

Upshot: We've encoded information "non-locally" in $\mathcal{C}$ in such a way that local errors can be detected and corrected.

Two senses of "non-locality" in the Toric Code

- Entanglement non-locality: The codewords (elements of $\mathcal{C}$) are entangled states.
 - *Entanglement non-locality = Einstein non-locality + Bell non-locality*

Recall:

- Einstein non-locality occurs when two systems are correlated and the correlation cannot be explained by a direct cause that travels from one system to the other.
- Bell non-locality occurs when two systems are correlated and the correlation cannot be explained by a common cause

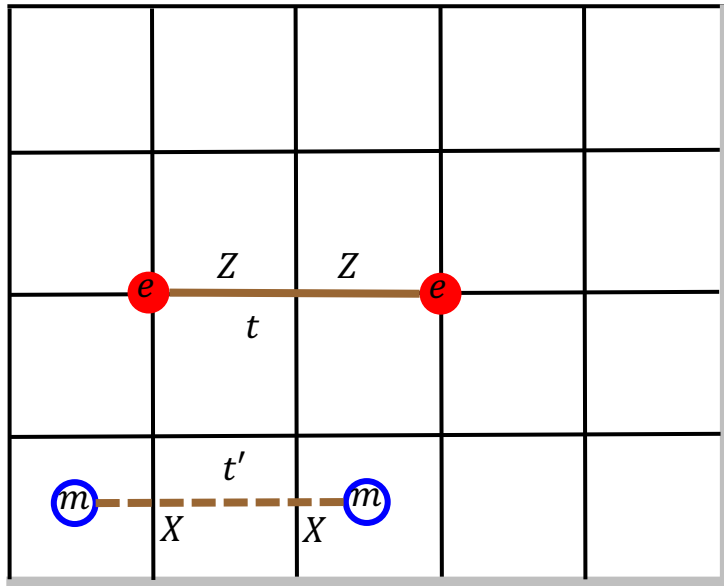
- Topological non-locality: The operators that act on codewords are non-contractible loop operators.

Suppose: Topological non-locality occurs when a quantity is not localized to a contractible region of space.

Open Question: Under what conditions does entanglement non-locality entail topological non-locality and/or vice-versa?

Let's add some (slightly more concrete) physics...

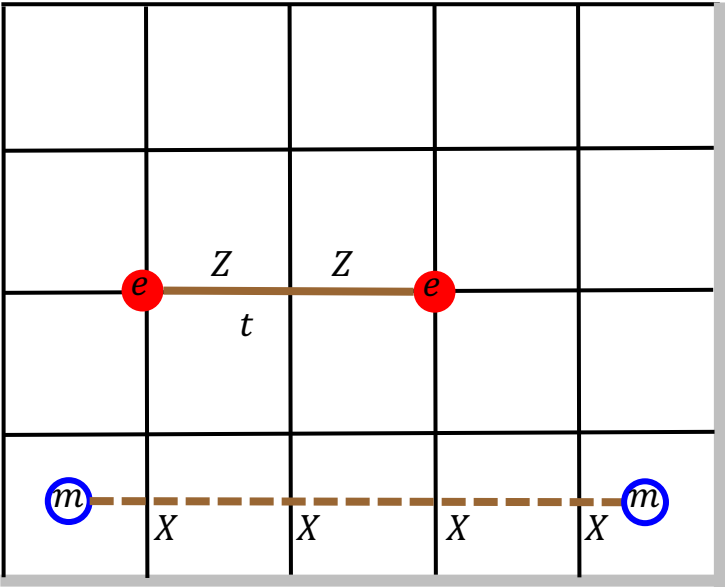
- Interpret the code space $\mathcal{C}$ as the space of ground-states $|q\rangle$ (states of lowest energy) of a physical system.



- Interpret a Z (or X) error operator as acting on a ground-state to produce a pair of " e " (or " m ") "quasiparticle" excitations at the ends of the open path.
- What happens when we move an m around an e ?
- $|\Psi_{initial}\rangle = S^Z(t)S^X(t')|q\rangle$

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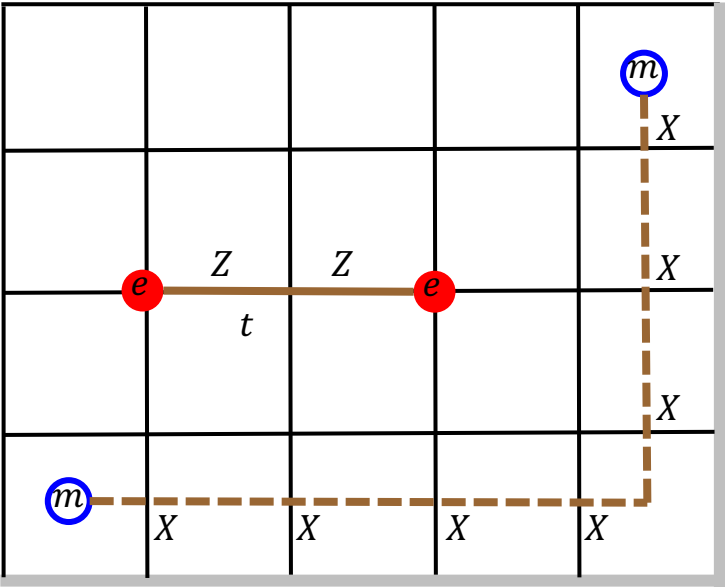
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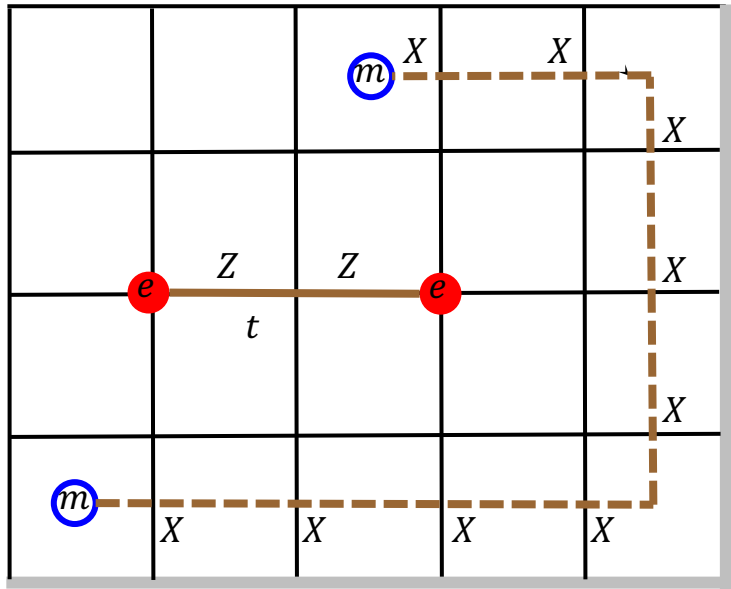
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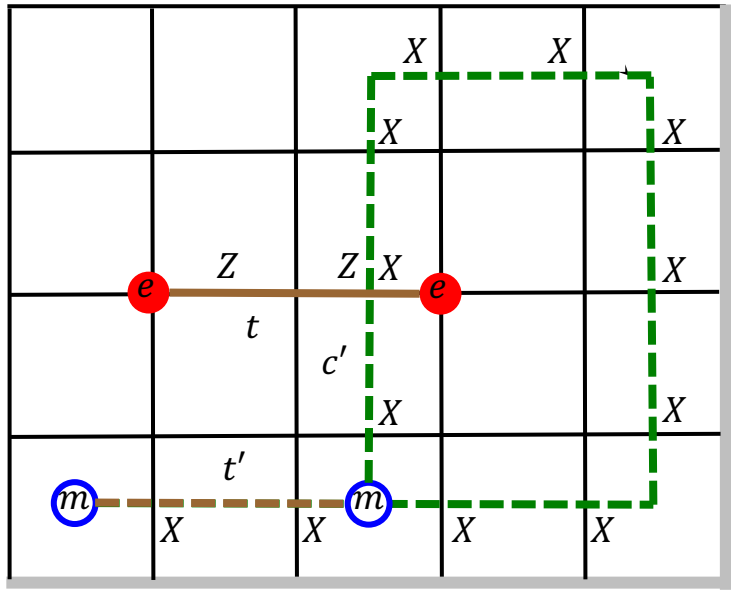
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- $$\begin{aligned}
 |\Psi_{final}\rangle &= S^X(c')S^Z(t)S^X(t')|q\rangle \\
 &= -S^Z(t)S^X(c')S^X(t')|q\rangle \\
 &= -S^Z(t)S^X(t')S^X(c')|q\rangle \\
 &= -|\Psi_{initial}\rangle
 \end{aligned}$$

$S^Z(t)$ and $S^X(c')$ anticommute

$S^X(c')$ and $S^X(t')$ commute

$S^X(c')$ acts like the identity on $\mathcal{C}$

- So: Moving an m quasiparticle completely around an e quasiparticle changes the phase of the initial 4-particle state by -1 .

In general: When two particles are exchanged in a multiparticle system, the multiparticle state $|\Psi\rangle$ picks up a phase $|\Psi\rangle \rightarrow e^{i\theta}|\Psi\rangle$.

- Taking one particle around another is equivalent to two exchanges; so $|\Psi\rangle \rightarrow e^{2i\theta}|\Psi\rangle$.
- So: Taking an m quasiparticle around an e quasiparticle produces the phase $e^{2i\theta} = -1$, or $\theta = \pi/2$.
- So: One exchange of an m quasiparticle and an e quasiparticle produces the phase $|\Psi\rangle \rightarrow e^{i\pi/2}|\Psi\rangle$.

$$e^{2i\theta} = \cos 2\theta + i \sin 2\theta$$

Bosons: Particle exchange phase $\theta = 0$.

Fermions: Particle exchange phase $\theta = \pi$.

Anyons: Particle exchange phase $\theta \in (0, \pi)$.

Upshot: m and e quasiparticles are anyons!
(They obey "fractional statistics".)

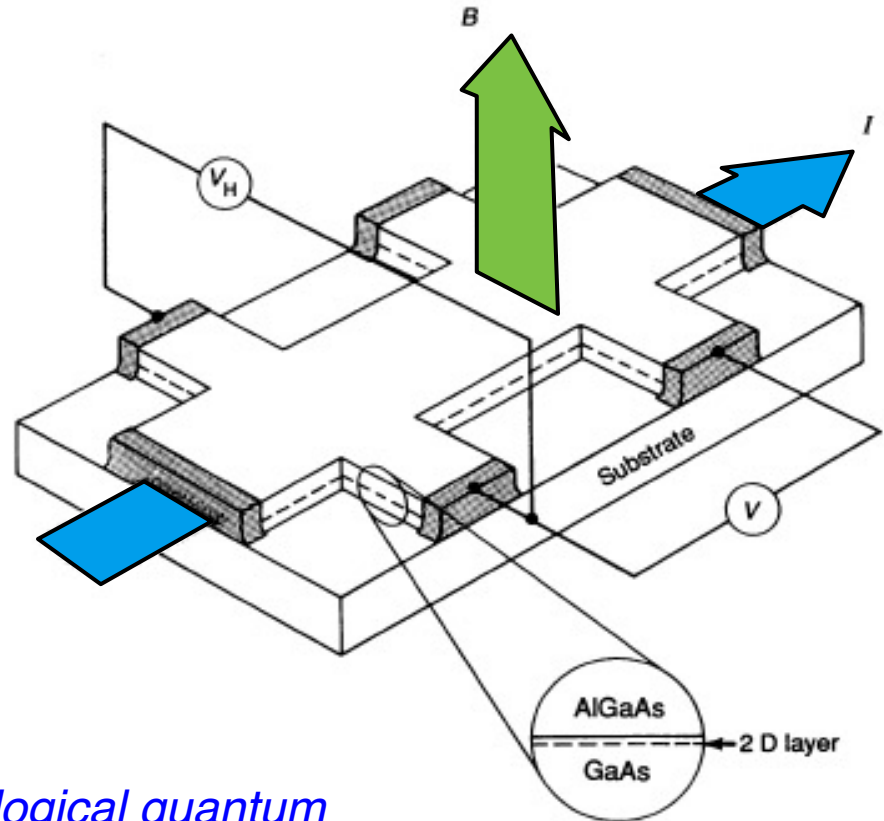
Physical significance: There are physical systems that exhibit characteristics of the toric code!



1998 Nobel Prize in Physics

- Fractional quantum Hall system:

- 2-dim conductor in external magnetic field B .
- At low temps, longitudinal resistance vanishes, and transverse (Hall) resistance becomes quantized.
- Prediction: Low-energy anyonic excitations.



Open Question: Can we build a topological quantum computer out of a fractional quantum Hall system?