

12b. Relativistic Spacetimes and Computability

1. Supertasks
2. Malament-Hogarth Spacetimes
3. Non-Turing Computability

1. Supertasks

- Recall: The decision problem for 1st-order arithmetic is *Turing unsolvable*.
- Which means: No TM that halts after a finite number of steps can determine if a given statement in arithmetic is a theorem.

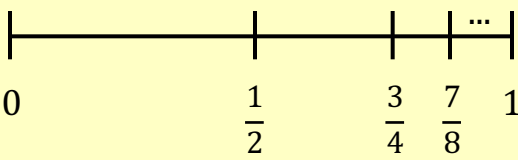
What if we allow the TM to perform an infinite number of steps?

Initial question: Should we allow such an infinity TM a finite or an infinite amount of time to do this?

Option A: An infinity TM that performs an infinite number of steps in a *finite* amount of time.

Goldbach's Conjecture: Every even integer greater than 2 can be expressed as the sum of two primes.

- Begin at $t = 0$.
- At $t = 1/2$, check 4.
- At $t = 3/4$, check 6.
- At $t = 7/8$, check 8.
- *etc....*
- At $t = 1$, all even integers greater than 2 will have been checked!



• Advantage: Puny finite humans can access the infinity TM's output in a finite amount of time.

But: Is such a "supertask" conceptually possible?

Thomson's Lamp:

- At $t = 0$, switch lamp on.
- At $t = 1/2$, switch lamp off.
- At $t = 3/4$, switch lamp on.
- *etc....*
- At $t = 1$, is lamp on or off?



• Thomson (1954): The lamp's operation requires:

- (i) For any time t , $0 < t < 1$, if the lamp is *off* at t , then there's another time t' , $t < t' \leq 1$, such that the lamp is *on* at t' .
- (ii) For any time t , $0 < t < 1$, if the lamp is *on* at t , then there's another time t' , $t < t' \leq 1$, such that the lamp is *off* at t' .

- (i) implies that the lamp is on at $t = 1$.
- (ii) implies that the lamp is off at $t = 1$.

} *Contradiction! So the lamp is not possible.*

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- *etc....*
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• Thomson (1954): The lamp's operation requires:

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- For any time t , $0 < t < 1$, if the lamp is *on* at t , then there's another time t' , $t < t' \leq 1$, such that the lamp is *off* at t' .

Benacerraf's (1962) response:

- The limit of a sequence is not a member of the sequence.
- So: Any properties of a sequence cannot necessarily be attributed to its limit.
- Thus: Since 1 is not a member of $(0, 1/2, 3/4, 7/8, \dots)$, any properties of the latter cannot be attributed to the former.
- In particular: The " $\leq$ " in (i) and (ii) is unjustified.

Option B: An infinity TM that performs an infinite number of steps in an *infinite* amount of time.

Goldbach's Conjecture: Every even integer greater than 2 can be expressed as the sum of two primes.

Use an infinity TM to check:

- At $t = 1$, check 4; at $t = 2$, check 6; *etc.*
- At $t = \infty$, all integers greater than 2 will have been checked!

But: How can puny humans access the data from such an infinity TM?

Answer: In a Malament-Hogarth spacetime!

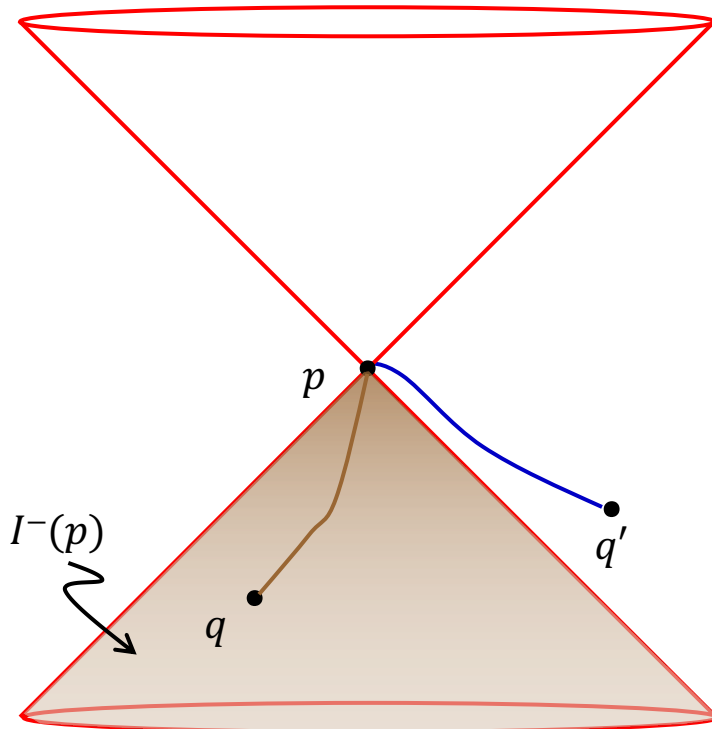
2. Malament-Hogarth Spacetimes

- Idea: We want the infinity TM's worldline to be infinitely long, *and* to be *accessible* by a causal signal from the programmer's worldline.

A Malament-Hogarth spacetime = a spacetime with a worldline of *infinite proper length* that lies in the *past* of some point.

Let's try to make this more precise...

- Recall: In a relativistic spacetime, there is a lightcone at every point p .



Def. 1. The *chronological past* $I^-(p)$ of p consists of all points q that can be connected to p by a timelike worldline.

$q \in I^-(p)$
 $q' \notin I^-(p)$

All points in $I^-(p)$ are accessible from p

Def. 2. The *proper length* of a timelike worldline γ is the sum of all infinitesimal intervals ds along it: (*proper length of γ*) $= \int_{\gamma} ds$

Def. 3. A *timelike half-worldline* is a timelike worldline with a past endpoint.

- Idea: We want the infinity TM's worldline γ to have a point, call it q , in its past that coincides with a point of the programmer's worldline.

← Think of q as the event at which the programmer programs the infinity TM.

A Malament-Hogarth spacetime = a 4-dim collection of points with a metric $g_{\mu\nu}$, a timelike half worldline γ , and a point p such that

(MH1) The proper length of γ is infinite: $\int_{\gamma} ds = \infty$

(MH2) γ lies completely in the chronological past of p : $\gamma \subset I^-(p)$.

Example: Toy M-H spacetime.

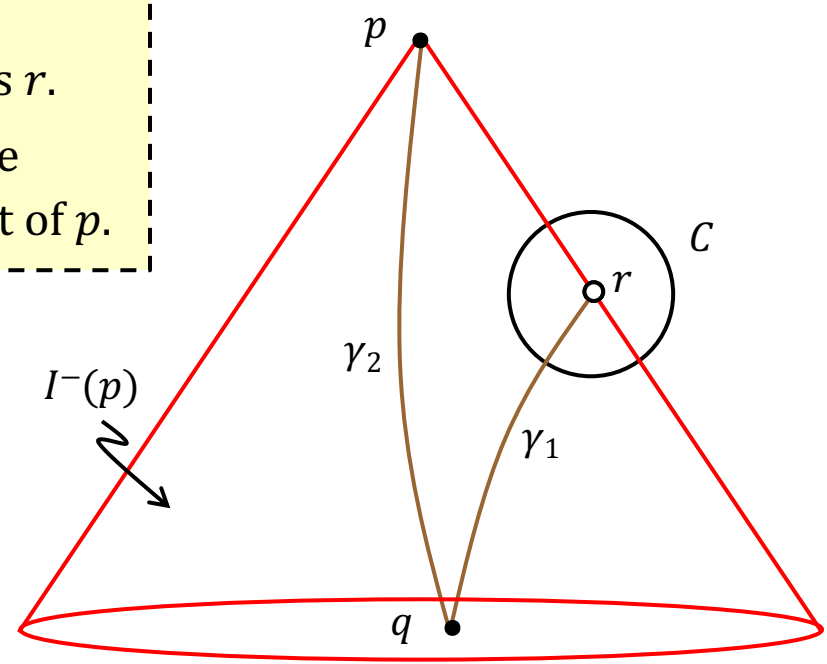
1. Start with Minkowski spacetime.
2. Add scalar field Ω that is the identity outside of a region C , and goes to ∞ as the point $r \in C$ is approached.
3. Remove r .

Claim. Resulting spacetime with metric $\Omega^2 \eta_{\mu\nu}$ is M-H.

Proof (by construction):

- Let γ_1 be a timelike worldline with future endpoint r and that passes through point q .
- Let p be a point whose past lightcone intersects r .
- Then γ_1 is a timelike half worldline with infinite proper length that lies in the chronological past of p .

- γ_1 is the TM worldline.
- γ_2 is the programmer worldline.
- At p , programmer has access to output of TM after it has performed ∞ steps.



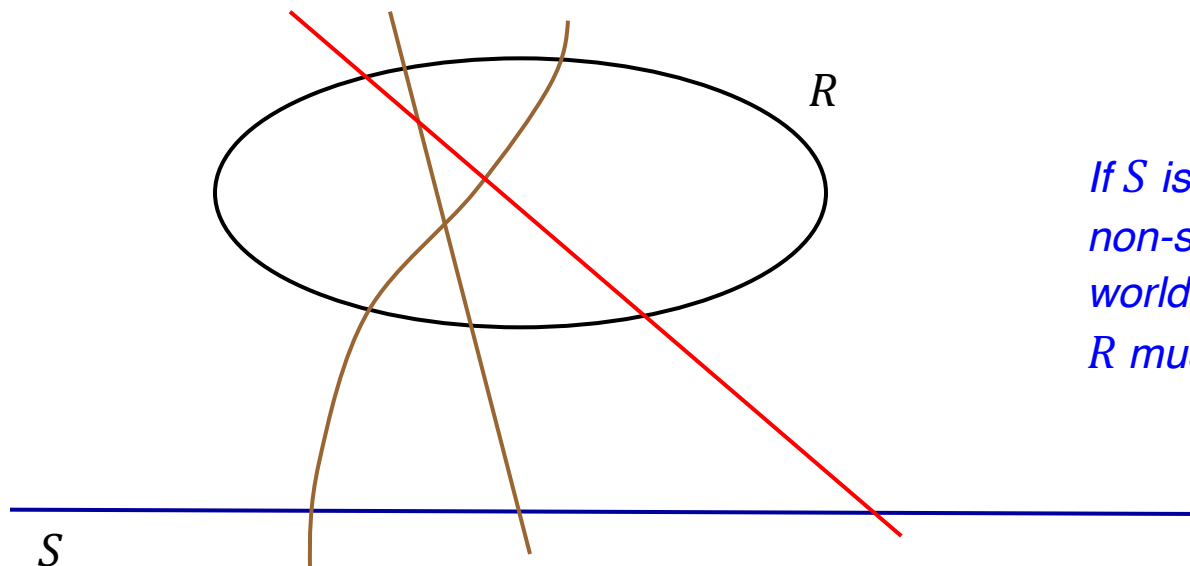
Are M-H spacetimes physically possible?

(1) *M-H spacetimes are not globally hyperbolic.*

Def. 4. A globally hyperbolic spacetime is a spacetime that admits a Cauchy surface.

A spacelike surface S such that every non-spacelike worldline without endpoints intersects S exactly once.

- Why is this important?* Cauchy surfaces serve as initial data surfaces and thus provide a basis for *predictability* in relativistic spacetimes.



If S is Cauchy then all non-spacelike (causal) worldlines interacting in R must register on S .

Claim. M-H spacetimes are not globally hyperbolic.

Proof: Suppose there is an M-H spacetime that is globally hyperbolic.

- Select a Cauchy surface S that contains the M-H point p .
- Extend the M-H worldline γ maximally into the past; call it γ' .
- Since γ' now has no endpoints and is timelike, it must intersect S .
- Since γ' is in $I^-(p)$, all points on it can be connected to p via timelike worldlines.
- In particular, the point p' where γ' intersects S can be connected to p via a timelike worldline.
- But both p and p' are in S , and S is spacelike.

Response:

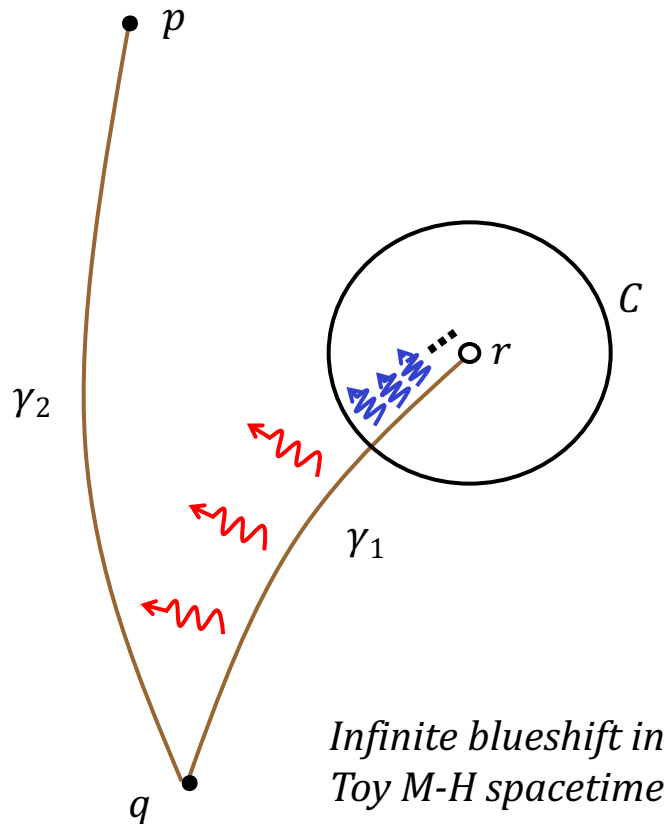
- Some solutions to the Einstein equations are M-H spacetimes, and thus not globally hyperbolic:
- Open question: Is the spacetime that describes our universe globally hyperbolic?



- anti-de Sitter spacetime
- Reissner-Nordstrom spacetime (charged black hole)
- Kerr spacetime (charged, rotating black hole)

(2) *M-H spacetimes may entail infinite blueshifts.*

- TM must send an infinite number of signals to the programmer.
- Programmer must receive these signals in a finite amount of her proper time.
- So: She must receive them in ever *decreasing* intervals = ever *increasing* frequencies = ever increasing energies!



More precisely:

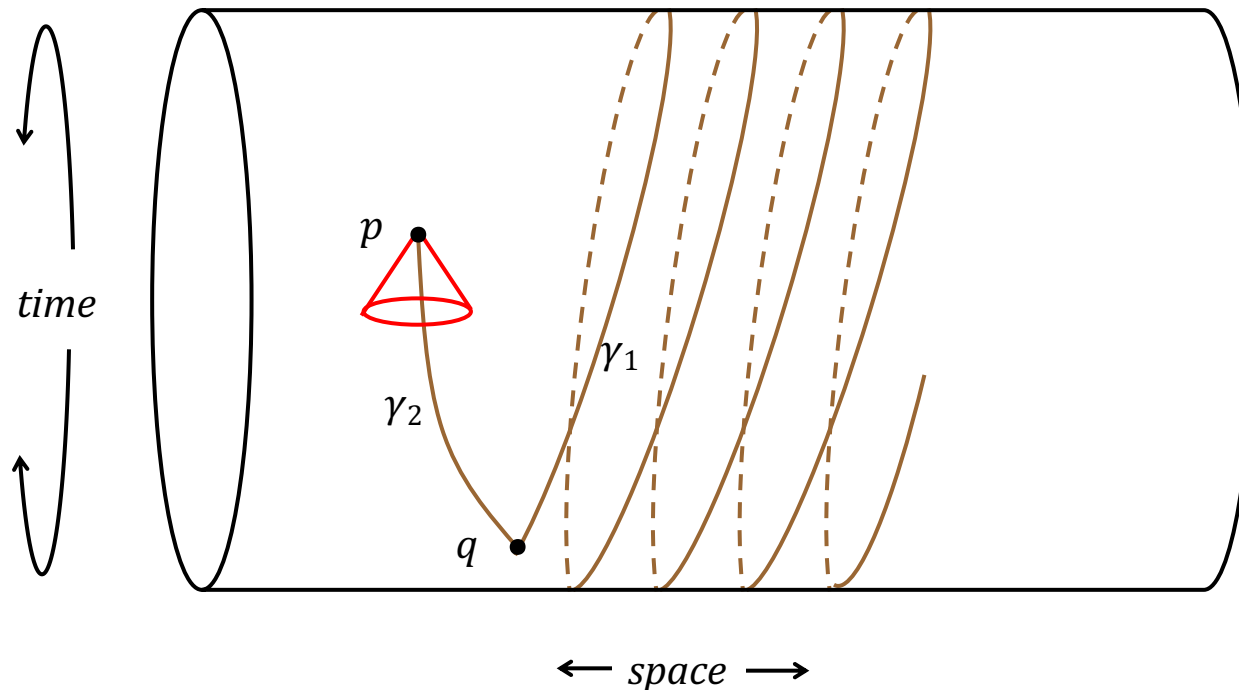
The frequency ω_1 of a signal sent by γ_1 is related to the frequency ω_2 of a signal received by γ_2 by:

$$\int_{\gamma_1} \omega_1 ds = \int_{\gamma_2} \omega_2 ds$$

Since $\int_{\gamma_1} ds = \infty$ and $\int_{\gamma_2} ds < \infty$ it must be the case that $\int_{\gamma_2} \omega_2 ds = \infty$

Response: Not all M-H spacetimes face this problem.

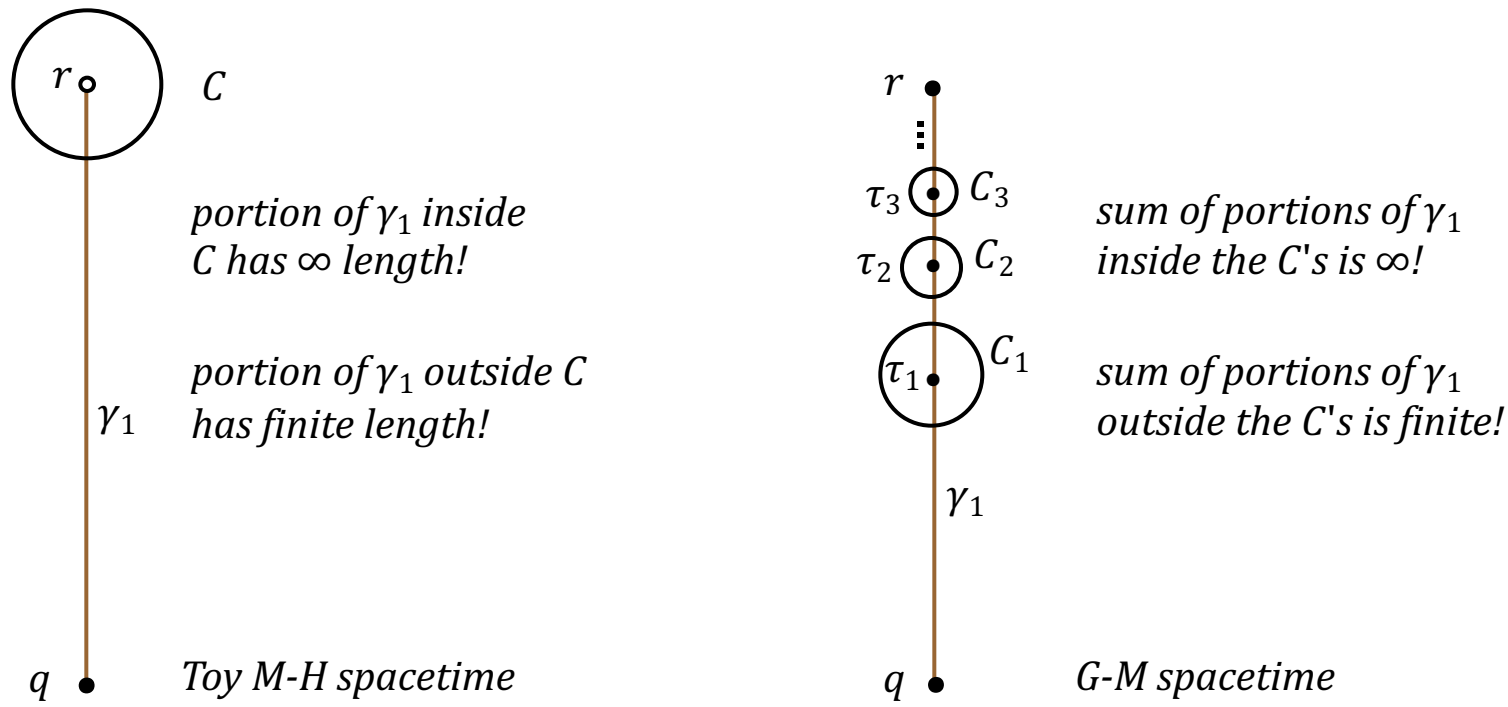
(a) *Rolled-up Minkowski spacetime*



- γ_1 's proper length is infinite.
- Chronological past of p is the entire spacetime!
- No infinite blueshifts: signals from γ_1 to γ_2 need not increase in frequency.
- But: Rolled-up Minkowski spacetime has closed timelike curves (time travel!).
Are such spacetimes physically realistic?

(b) *Geroch-Malament (G-M) spacetime*

- Recall: In Toy M-H spacetime γ_1 is guaranteed to have ∞ length due to the scalar field Ω in the interior of the sphere C centered at the missing point r .
- But: The frequencies of signals sent from γ_1 inside C blow up as r is approached.
- Strategy: Guarantee that γ_1 has ∞ length by constructing scalar fields for spheres centered at an infinite sequence of points $\tau_1, \tau_2, \tau_3, \dots$ along γ_1 . Signals sent from points *not* in this sequence will not blow up!



How to construct G-M spacetime.

- Step 1. Let $(\tau_1, \tau_2, \dots)$ be an infinite sequence of points on γ_1 given by $\tau_n = 1 - (3/4)(1/2)^n$.
- Step 2. At each point τ_n , construct a sphere C_n of radius $r_n = (1/2)^{n+3}$.
- Step 3. In the interior of each sphere C_n , construct a scalar field Ω_n such that
- (i) Ω_n goes to 1 as r_n is approached, and reaches its maximum value as τ_n is approached; and
 - (ii) the length of γ_1 inside C_n is equal to 1.

Consequences:

- The total length of the portion of γ_1 in all the spheres is ∞ .
- The total length of the portion of γ_1 outside the spheres is finite.
- Signals sent from the points of γ_1 where its length is finite will *not* be blueshifted.
- In particular, signals sent from the sequence of points $(1/2, 3/4, 7/8, \dots)$ will not be blueshifted.
- But: How physically realistic is a G-M spacetime?

(3) *M-H spacetimes may violate the Cosmic Censorship Hypothesis (CCH).*

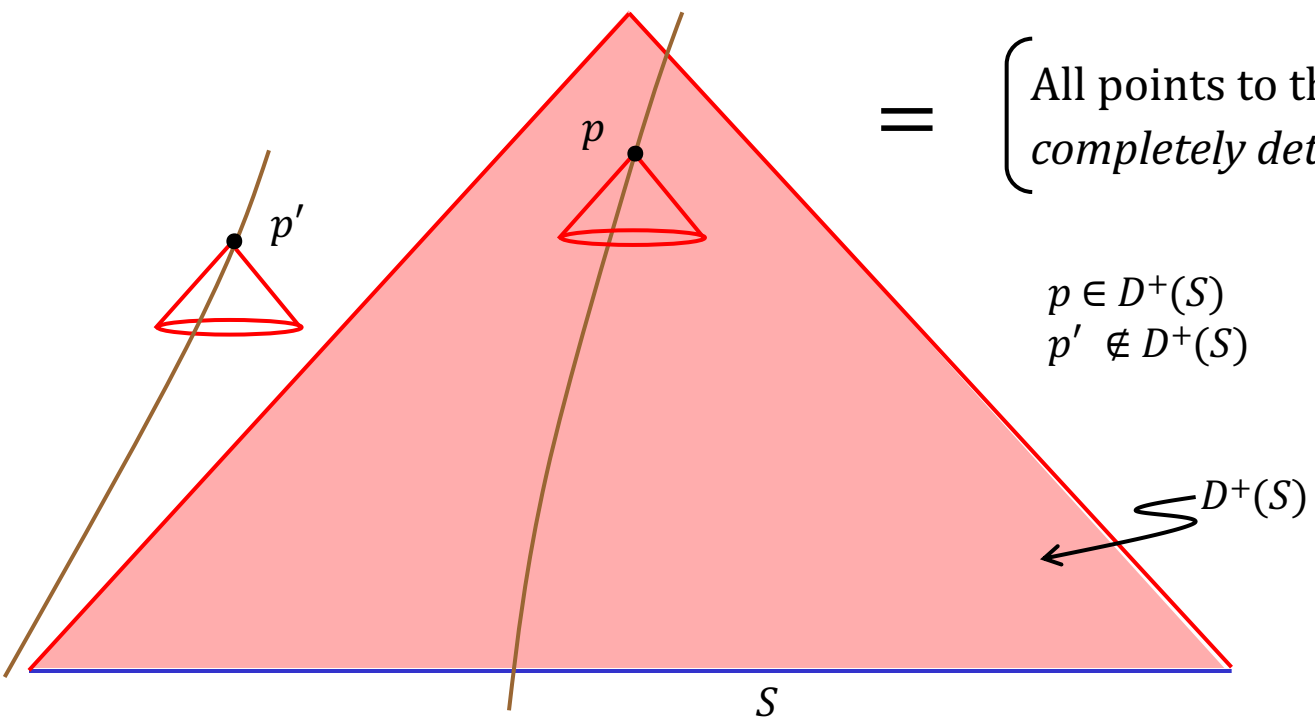
CCH. "Naked" singularities do not develop in physically reasonable solutions to the Einstein equations.

(Penrose 1974)

- Naked singularities entail a breakdown of *determinism*.
- Let S be a spacelike surface.

$\left(\text{The future domain of dependence } D^+(S) \text{ of } S \right) = \left(\text{All points } p \text{ such that every non-space-like worldline through } p \text{ with no future endpoint intersects } S \text{ exactly once} \right)$

$= \left(\text{All points to the future of } S \text{ that are completely determined by data on } S. \right)$

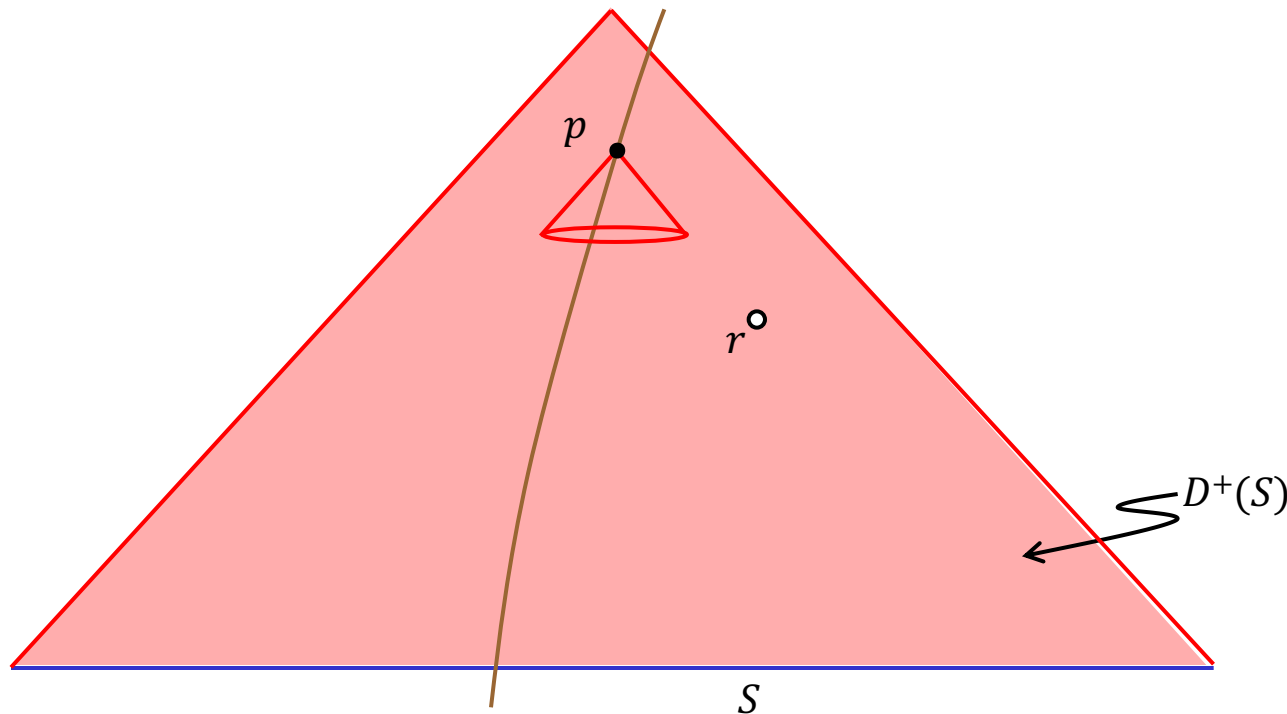


(3) *M-H spacetimes may violate the Cosmic Censorship Hypothesis (CCH).*

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- Remove r . $\leftarrow$ *Changes $D^+(S)$!*

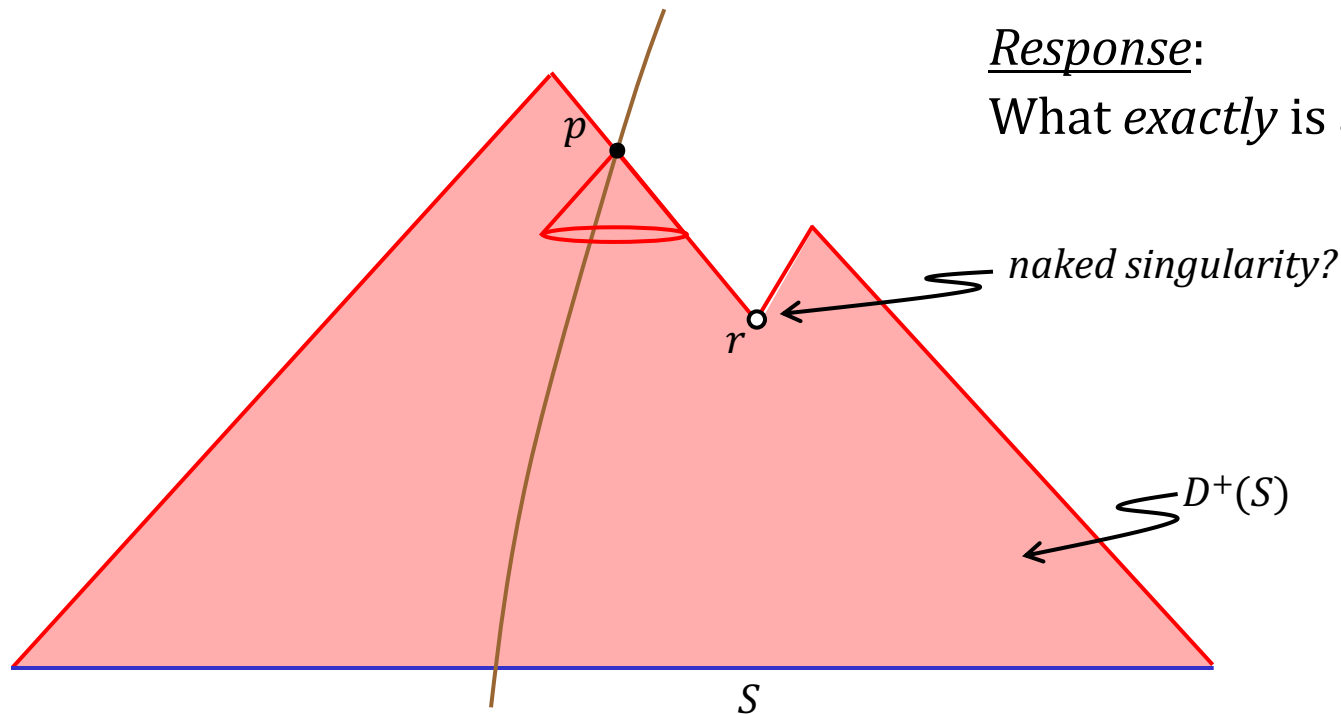


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CCH. "Naked" singularities do not develop in physically reasonable solutions to the Einstein equations.

(Penrose 1974)

- Naked singularities entail a breakdown of *determinism*.
- Remove r . $\leftarrow$ *Changes $D^+(S)$!*
- p is no longer in $D^+(S)$!
- Causal influences that do not register on S can now affect p .



Response:

What *exactly* is a naked singularity?

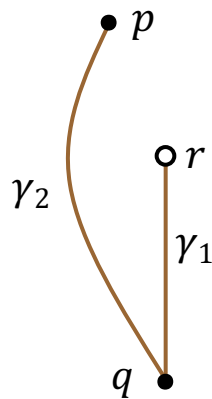
3. Non-Turing Computability

Def. 5. A *simple infinity TM* (TM_{∞}) is a TM that can complete an infinite number of steps.

- A TM following a M-H worldline is a TM_{∞} .

The Big Difference between a regular TM and a TM_{∞}

- All TMs either halt or fail to halt.
- If a regular TM fails to halt, this could mean either that it may or may not halt.
- If a TM_{∞} fails to halt, this *definitely* means it will never halt.



At M-H point p , programmer knows whether TM has halted or failed to halt. If the latter, this definitely means the TM will never halt.

Toy M-H spacetime

What can be solved by a TM_{∞}

(i) *The halting problem.*

- Does arbitrary $TM T_t$ halt on input n ?
- Run T_t on a TM_{∞} :
 - If the TM_{∞} halts, T_t halts.
 - If the TM_{∞} fails to halt, T_t does not halt.

(ii) *The decision problem for purely existential and purely universal statements in arithmetic.*

Ex: Fermat's Last Theorem.

- "For $n \geq 3$, there are no x, y, z , such that $x^n + y^n = z^n$."
- Or: $\forall x \forall y \forall z \forall n \neg F(x, y, z, n)$, where $F(x, y, z, n)$ means " x, y, z, n are natural numbers, and $n \geq 3$, and $x^n + y^n = z^n$ ".

- Note: Any statement (in prenex form) with two adjacent quantifiers of the same type is equivalent to a statement with a single quantifier of that type in place of these two.
- Thus: Any purely existential/universal statement can be rewritten as a statement with a *single* quantifier.

TM_∞ program that decides purely universal statements $\forall xF(x)$

- (i) For each number x , determine if F holds.
- (ii) If F holds, then go to $x + 1$.
- (iii) If F does not hold, halt.

Result:

- If TM_∞ halts, then $\forall xF(x)$ is false.
- If TM_∞ fails to halt, then $\forall xF(x)$ is true.

TM_∞ program that decides purely existential statements $\exists xF(x)$

- (i) For each number x , determine if F holds.
- (ii) If F holds, then halt.
- (iii) If F does not hold, then go to $x + 1$.

Result:

- If TM_∞ halts, then statement is true.
- If TM_∞ fails to halt, then statement is false.

What about mixed quantifier statements?

- Ex: $\exists x \forall y R(x, y)$  "There's an ultimate number x which stands in the relation R to all numbers."

Attempt at a TM_{∞} program to decide mixed quantifier statements

- (i) For each x , check to see if there's an y for which R doesn't hold.
- (ii) If so, then go to $x + 1$ and start checking y 's again.

Result:

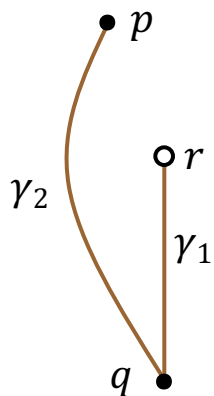
- The TM_{∞} will fail to halt, but what does this mean?
 - *Either the x currently being checked against all numbers is the ultimate (so the statement is true).*
 - *Or the TM_{∞} is still checking candidates for x (so the statement may be false).*
 - Thus: The truth of the statement cannot be determined.
- So: It looks like a *single* TM_{∞} cannot solve the decision problem for *arbitrary* statements in arithmetic.

What about using an infinite number of TM_{∞} 's?

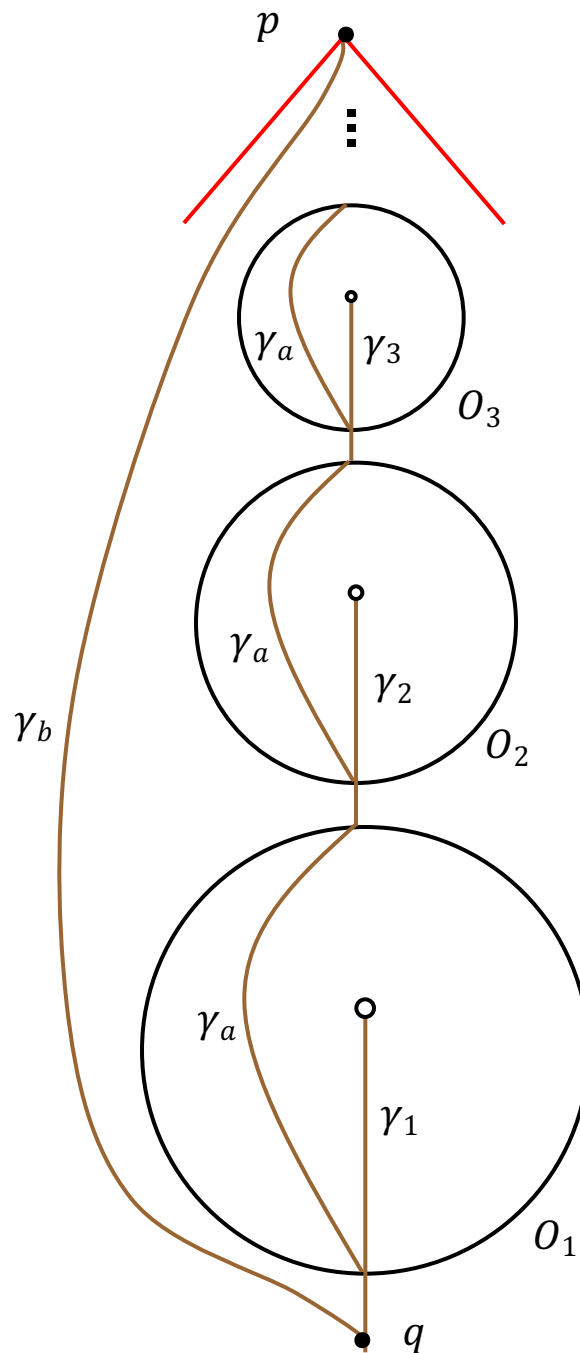
- Idea: To decide the truth of $\exists x \forall y R(x, y)$,
 - TM^1_∞ checks $\forall y R(1, y)$, TM^2_∞ checks $\forall y R(2, y)$, ...
 - Collect results using another TM_∞ .

Is there a relativistic spacetime that can support such a computation?

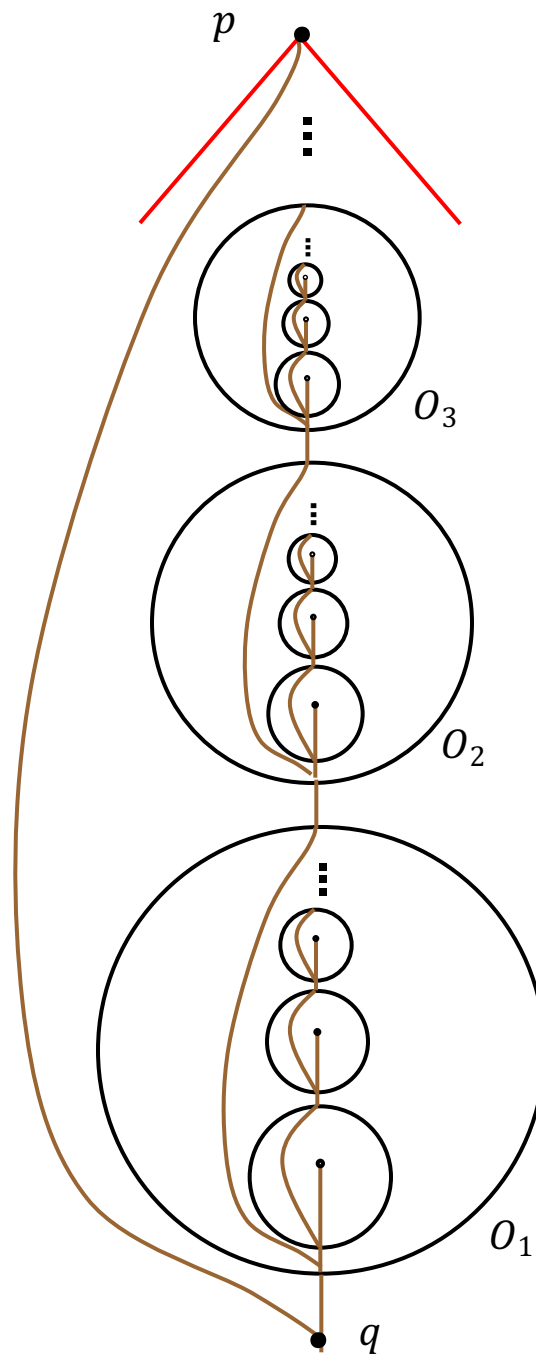
Yes!



- M-H spacetime.
- γ_1 is a single TM_∞ .
- γ_2 is the programmer.
- Decides *singly*-quantified statements $\exists x R(x), \forall x R(x)$.



- Each region O_i is an M-H spacetime such that, for all i
 - (a) $O_i \subset I^-(O_{i+1})$
 - (b) $O_i \subset I^-(p)$
- Each γ_i is a TM_∞ that determines $\forall y R(i, y)$.
- γ_a is another TM_∞ that collects these results to determine $\exists x \forall y R(x, y)$.
- γ_b is the programmer.
- Decides *doubly-mixed* quantifier statements $\exists x \forall y R(x, y), \forall x \exists y R(x, y)$.
- Call this a SAD_2 (*2nd-order arithmetical sentence deciding*) spacetime.



- Each region O_i is a SAD_2 spacetime such that, for all i
 - (a) $O_i \subset I^-(O_{i+1})$
 - (b) $O_i \subset I^-(p)$
- Decides *triply*-mixed quantifier statements

$$\forall x \exists y \forall z R(x, y, z),$$

$$\exists x \forall y \exists z R(x, y, z).$$
- Call this a SAD_3 spacetime.

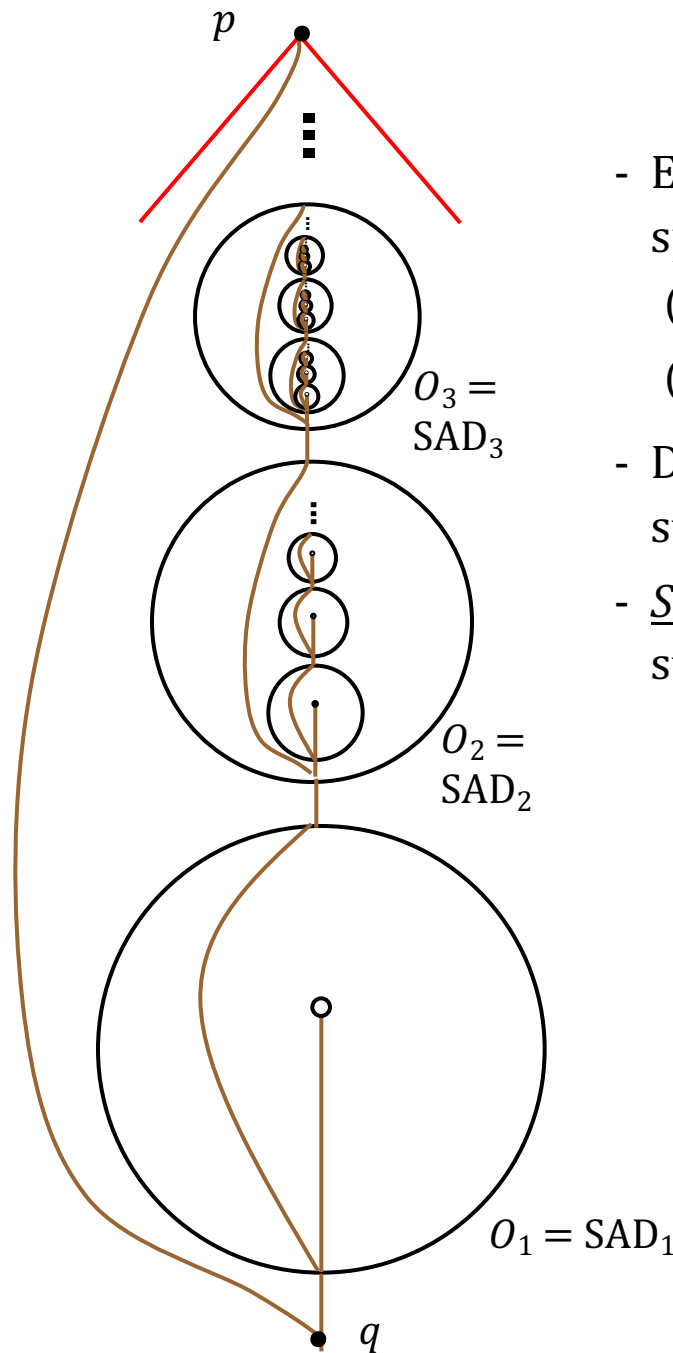
A SAD_n (n th-order arithmetical sentence deciding) spacetime = a 4-dim collection of points such that:

- (1) If $n = 1$, the spacetime is M-H.
- (2) If $n > 1$, the spacetime admits a *string of SAD_{n-1} spacetimes*: a collection of non-intersecting open regions $O_i, i = 1, 2, \dots$, such that
 - (a) Each O_i is in the chronological past of the next O_{i+1} .
 - (b) There is a point p such that all regions are in the chronological past of p .
 - (c) Each region O_i is a SAD_{n-1} spacetime.

An AD (*arithmetical sentence deciding*) spacetime = a 4-dim collection of points that admits a string of open regions $O_i, i = 1, 2, \dots$, such that:

- (a) Each O_i is in the chronological past of the next O_{i+1} .
- (b) There is a point p such that all regions are in the chronological past of p .
- (c) For each $n \geq 1$, O_n is a SAD_n spacetime.

An AD spacetime



- Each region O_i is a SAD_i spacetime such that, for all i
 - (a) $O_i \subset I^-(O_{i+1})$
 - (b) $O_i \subset I^-(p)$
- Decides mixed quantifier statements of all orders.
- So: Decides truth of arbitrary statements in arithmetic!

Recap:

- A TM operating in a classical spacetime cannot decide (any) quantifier statements in arithmetic.
- A TM_{∞} operating in a M-H spacetime can decide purely existential/universal statements in arithmetic.
- A configuration of TM_{∞} 's operating in a SAD_n spacetime, $n > 1$, can decide n th-order mixed quantifier statements in arithmetic.
- A configuration of TM_{∞} 's operating in an AD spacetime can decide all types of quantifier statements (of arbitrary mixed order) in arithmetic.

Thus: The decision problem for 1st-order arithmetic is solvable in an AD spacetime!

Consequences for the Concept of Computability

The Church-Turing Thesis

Turing computability = effective computability.

Question 1: What is "Turing computability"?

- | | | |
|----------------------------|---|--|
| - TM computable | | (computability in classical spacetimes) |
| - TM_{∞} computable | } | three types of computability
in relativistic spacetimes |
| - SAD_n computable | | |
| - AD computable | | |

Do these correspond to distinct types of computing devices?

Or are they all simply regular TMs operating in different types of spacetimes?

Question 2: How do physical constraints affect the concept of computability?

- Do physical constraints rule in favor of TM computability?

Possible claim: TM_{∞} -, SAD_n -, and AD-computability are only possible in highly idealized relativistic spacetimes.

- But: Ordinary TMs are themselves ideal concepts: they assume an infinite amount of memory!

The Physical Church-Turing Thesis

Turing computability = an *upper bound* on effective computability.

Should this idealized upper bound be replaced by one associated with relativistic spacetimes?

- Practical response: The spacetime arena in which real computers (designed by computer engineers) operate is, for all intents and purposes, a classical spacetime.
- But: If the issue involves formulating a *theoretical* notion of "computability" (what can and cannot be computed in-principle) then practical issues hold lesser sway.