

12a. Classical and Relativistic Spacetimes

Motivation: Turing solvable problems require a TM to halt after a finite number of steps with a given output.

- What if we allow TMs to perform an *infinite* number of steps?
 - Then: Some Turing unsolvable problems may become solvable!

How could puny finite humans access the output of a TM that performs an infinite number of steps?

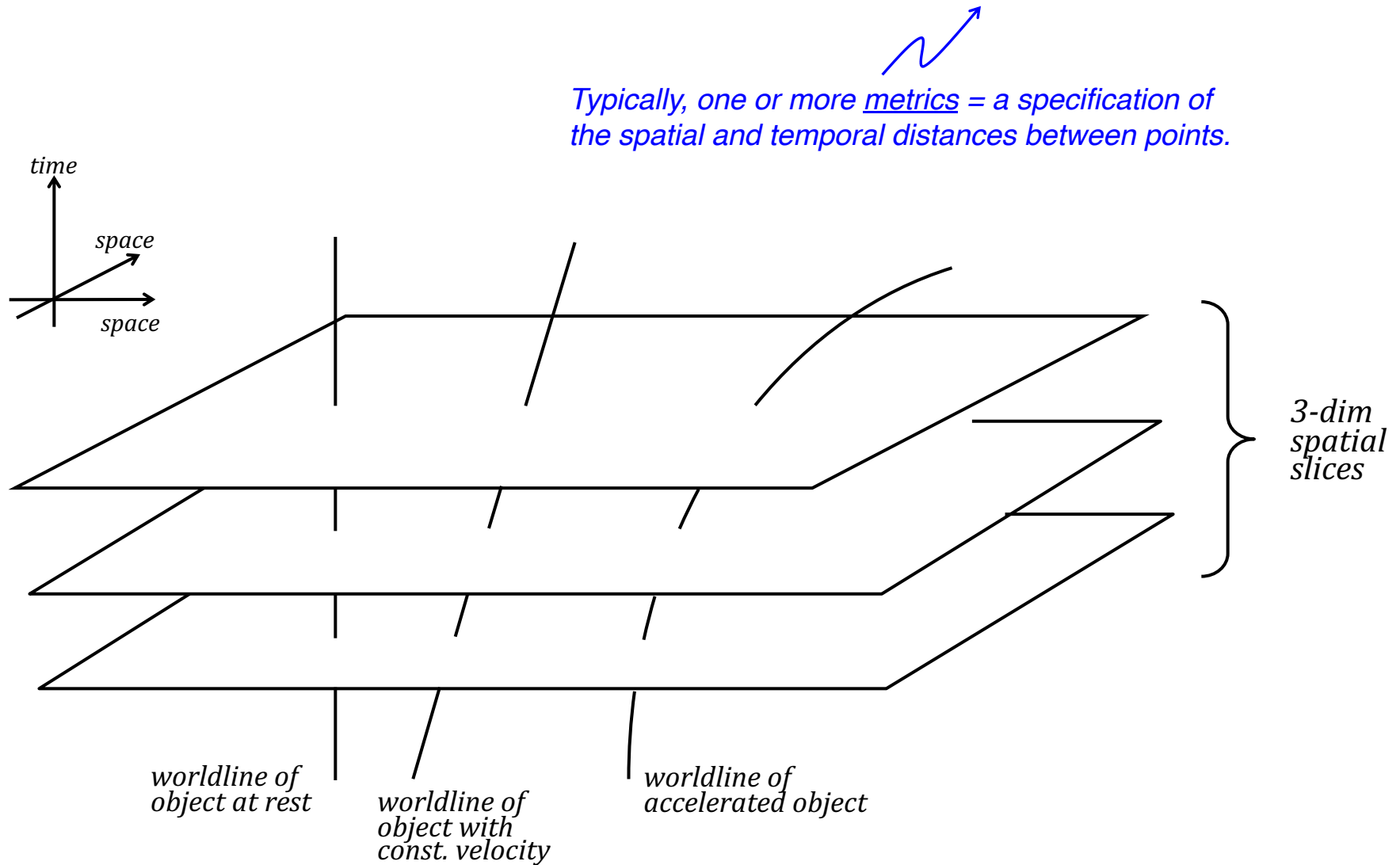
Answer to come: Place the puny human in a spacetime that:

- (a) Allows the TM to live to infinity in its rest frame; and
- (b) Allows the human to access the TM's output in a finite amount of time.

- Mathematically: A matter of determining the appropriate curved geometry for the given spacetime.
- Physically: Are such appropriately curved spacetimes physically possible?

1. Types of Spacetimes

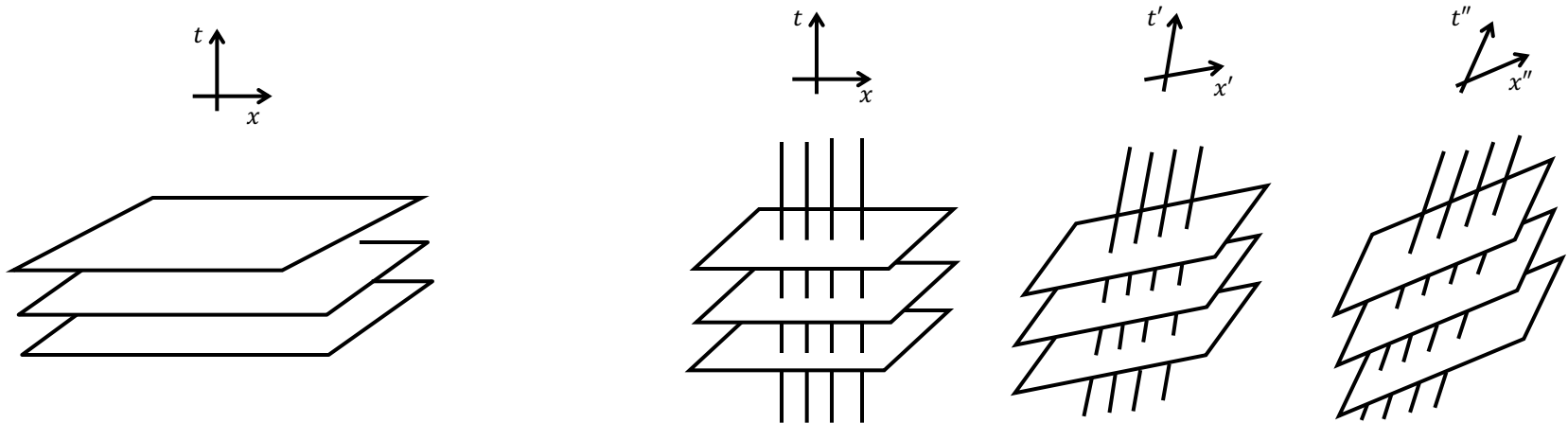
- A *spacetime* is a 4-dim collection of points with additional structure.



Two ways spacetimes can differ:

(1) Different ways of specifying distances between points yield different types of spacetimes.

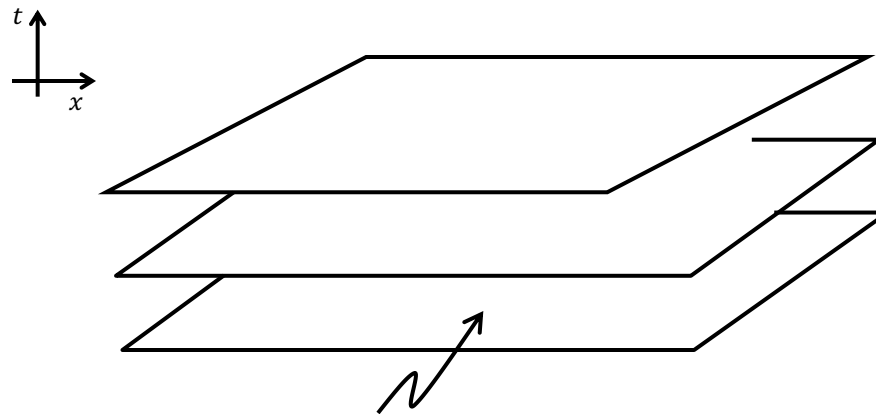
- *Classical spacetimes* have *separate* spatial and temporal metrics: only one way to split time from space (spatial and temporal distances are *absolute*).
- *Relativistic spacetimes* have a *single* spatiotemporal metric, and how it gets split into spatial and temporal parts depends on one's inertial reference frame (spatial and temporal distances are *relative*).



Classical spacetime: only one way to split time from space.

Relativistic spacetime: many ways to split time from space.

- (2) Metrics can be *flat* or *curved*: how one specifies the distance between points encodes the curvature of the spacetime.
- *Classical spacetimes* can be flat or curved.
 - *Relativistic spacetimes* can be flat (Minkowski spacetime) or curved (general relativistic spacetimes).
- Two ways curvature can manifest itself:



The spatial slices can be flat or curved.

How the spatial slices are "rigged" together can be flat or curved.

2. Classical Spacetimes

Newtonian spacetime is a 4-dim collection of points such that:

(N1) Between any two points p, q , with coordinates (t, x, y, z) and (t', x', y', z') , there is a definite temporal interval $T(p, q) = t' - t$.

(N2) Between any two points p, q , with coordinates (t, x, y, z) and (t', x', y', z') , there is a definite *Euclidean distance*

$$R(p, q) = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}$$

(N1) and (N2) entail:

(a) All worldlines have a definite *absolute velocity*.

For worldline γ , and any two points p, q on γ , the *absolute velocity* of γ with respect to p, q can be defined by $R(p, q)/T(p, q)$.

(b) There is a privileged collection of worldlines defined by $R(p, q)/T(p, q) = 0$.

(c) All worldlines have a definite *absolute acceleration*.

For worldline γ , and points p, q on γ , the *absolute acceleration* of γ with respect to p, q can be defined by $d/dt\{R(p, q)/T(p, q)\}$.

 Newton's
absolute
space!

But absolute space and absolute velocity are unobservable!

Galilean spacetime is a 4-dim collection of points such that:

(G1) Between any two points p, q , with coordinates (t, x, y, z) and (t', x', y', z') , there is a definite temporal interval $T(p, q) = t' - t$.

(G2) Between any two simultaneous points p_t, q_t , with coordinates (t, x, y, z) and (t, x', y', z') , there is a definite Euclidean distance,

$$R(p_t, q_t) = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}$$

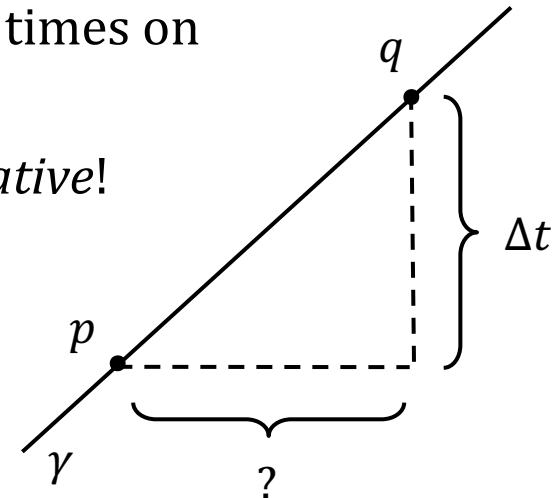
(G3) Any worldline γ through point p has a definite curvature $S(\gamma, p)$.

(G2) entails:

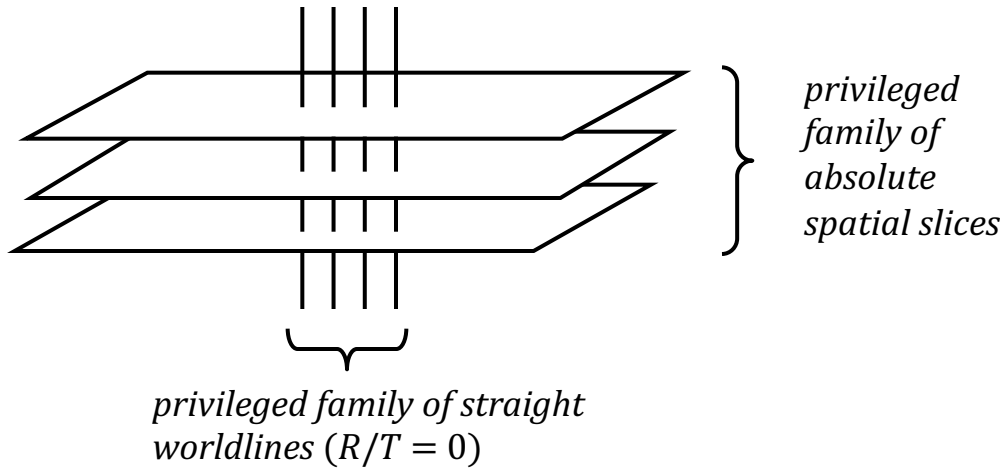
- No absolute spatial distance between points at different times on any worldline γ .
- So: No absolute velocity for any worldline: *velocity is relative!*
- So: No single privileged inertial reference frame.

(G3) entails: acceleration remains absolute!

For worldline γ and point p on γ , the *absolute acceleration* of γ with respect to p is given by $S(\gamma, p)$.

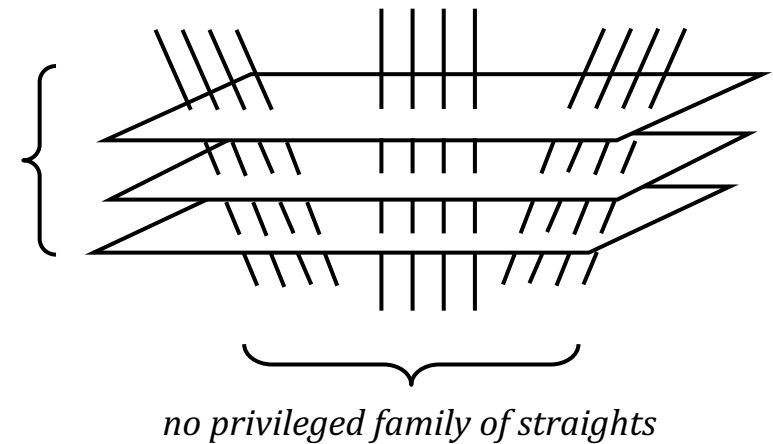


Newtonian Spacetime



1. Single, privileged inertial frame.
2. Velocity is absolute.
3. Acceleration is absolute.
4. Simultaneity is absolute.

Galilean Spacetime



1. Many inertial frames; none privileged.
2. Velocity is relative.
3. Acceleration is absolute.
4. Simultaneity is absolute.

- Both Newtonian spacetime and Galilean spacetime have absolute temporal metrics: *Everyone agrees on what time it is.*
- Relativistic spacetimes have no absolute temporal metric: *What time it is depends on your inertial reference frame.*

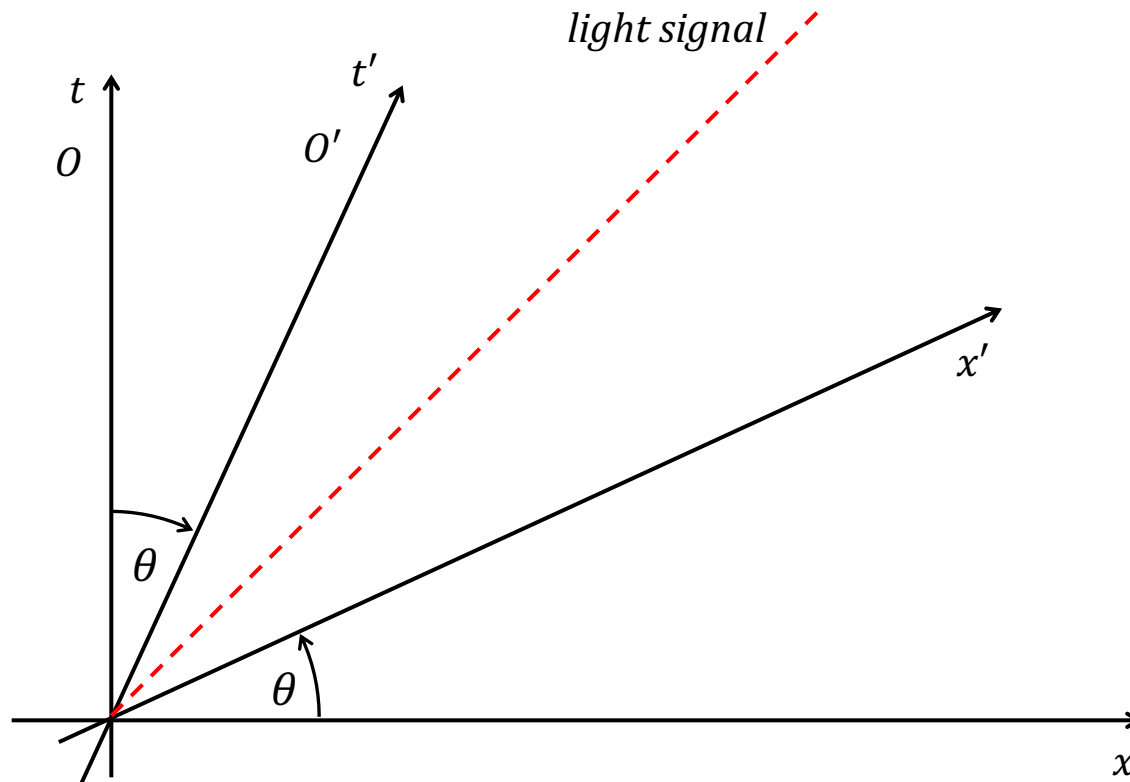
3. Relativistic Spacetimes

- Light Postulate of Special Relativity entails:

The speed of light c is the same in all inertial reference frames.



Albert Einstein
(1879-1955)



$$c = \frac{\Delta x}{\Delta t} = \frac{\Delta x'}{\Delta t'}$$

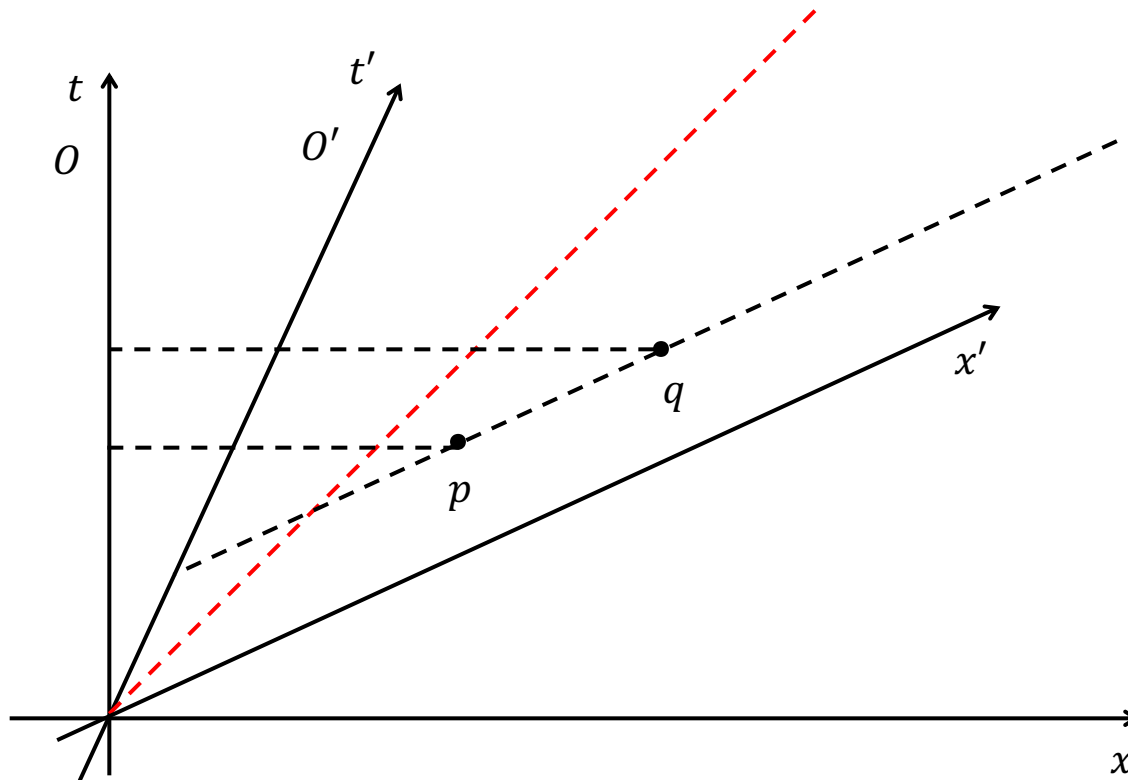
value of c for O value of c for O'

- Object O' is moving at constant velocity with respect to object O .
- O and O' must measure same speed c for light signal.
- So: The x' axis must be inclined by the same amount θ from the x axis as the t' axis is inclined from the t axis.
- Thus: O and O' must disagree on spatial and temporal measurements!

3. Relativistic Spacetimes

- Light Postulate of Special Relativity entails:

The speed of light c is the same in all inertial reference frames.



- O and O' make different judgements of simultaneity (*relativity of simultaneity*).
- p and q are simultaneous according to O' .
- p happens before q according to O .

Spacetime of Special Relativity = Minkowski spacetime

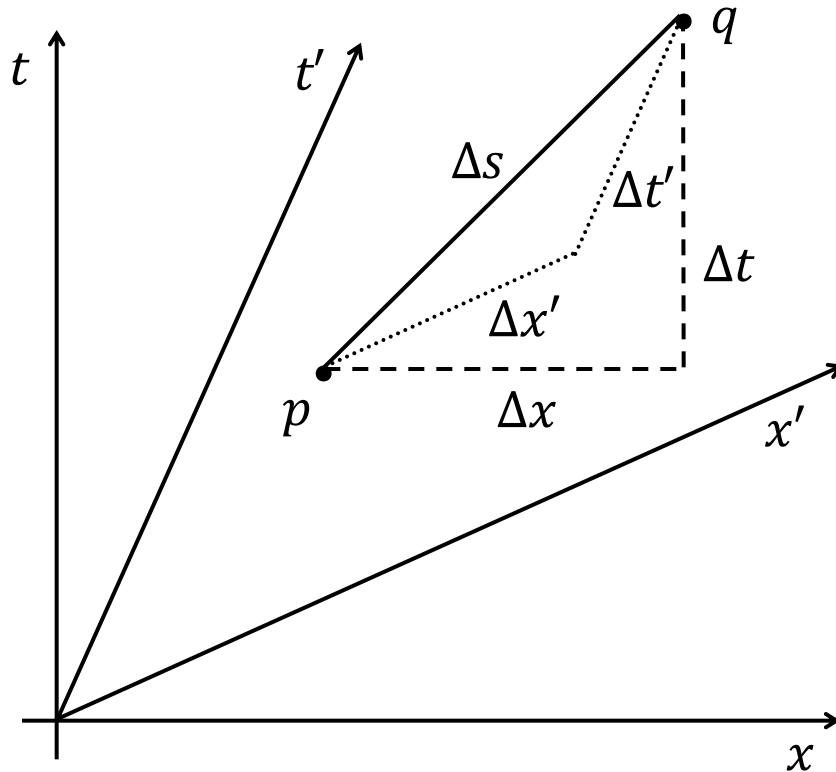


Hermann
Minkowski
(1864-1909)

Minkowski spacetime is a 4-dim collection of points such that between any two points p, q with coordinates (t, x, y, z) and $(t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z)$, there is a definite spacetime interval given by

$$\Delta s = \sqrt{-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

- Similar to Euclidean *spatial* interval $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$.
- But: Includes the time coordinate difference, too! And it's *negative*!
 - Idea: All inertial frames will agree on the spatiotemporal distance Δs between any points p and q .
 - But they will disagree on how Δs gets split into a temporal part and a spatial part: they will disagree on measurements of time and measurements of space.



$$\begin{aligned}\Delta s &= \sqrt{-(c\Delta t)^2 + (\Delta x)^2} \\ &= \sqrt{-(c\Delta t')^2 + (\Delta x')^2}\end{aligned}$$

- All inertial frames agree on the *spacetime* distance between any two events P and Q .
- They disagree on the *temporal* distance between p and q (time dilation) and on the *spatial* distance (length contraction).
 - They disagree on how they split Δs into temporal and spatial parts.

The Minkowski spacetime interval is encoded in the *Minkowski metric* $\eta_{\mu\nu}$.

$$\begin{aligned}(\Delta s)^2 &= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\&= \eta_{00}\Delta x^0 \Delta x^0 + \eta_{01}\Delta x^0 \Delta x^1 + \cdots + \eta_{33}\Delta x^3 \Delta x^3 \\&= \sum_{\mu,\nu=0}^3 \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu\end{aligned}$$

$$\text{where } \begin{cases} \Delta x^0 = c\Delta t \\ \Delta x^1 = \Delta x \\ \Delta x^2 = \Delta y \\ \Delta x^3 = \Delta z \end{cases} \quad \text{and} \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Infinitesimally, we can write (suppressing the summation sign):

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

The Minkowski interval $(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ has three forms:

(a) *Timelike*. $(\Delta s)^2 < 0$, or: $\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{\Delta t} < c$

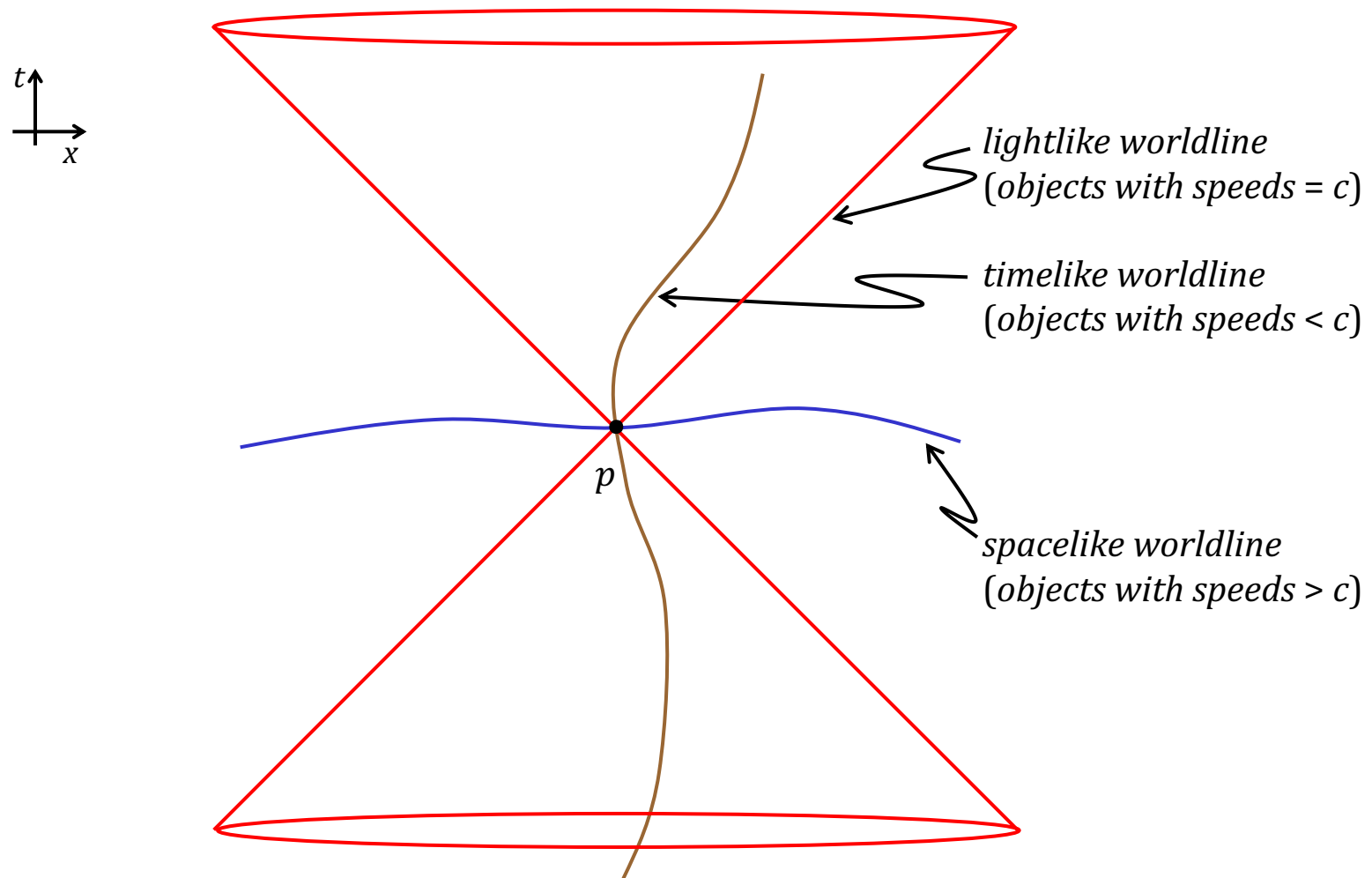
(b) *Lightlike*. $(\Delta s)^2 = 0$, or: $\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{\Delta t} = c$

(c) *Spacelike*. $(\Delta s)^2 > 0$, or: $\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{\Delta t} > c$

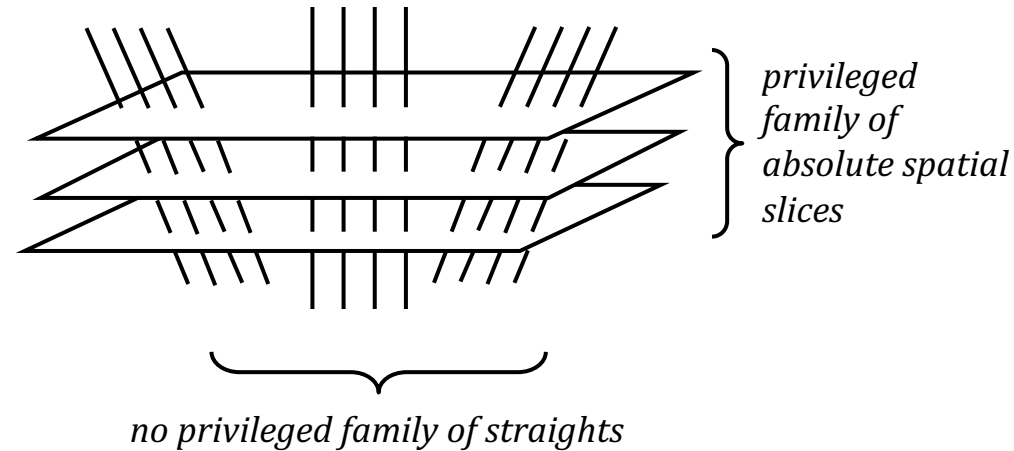
- The worldlines of physical objects have negative length!
 - *Only strange from a Euclidean perspective.*
- The worldlines of light signals have zero length!
 - *Only strange from a Euclidean perspective.*

Three different
types of worldline
in Minkowski
spacetime!

Hence: The Minkowski metric defines a *lightcone* at any point p :

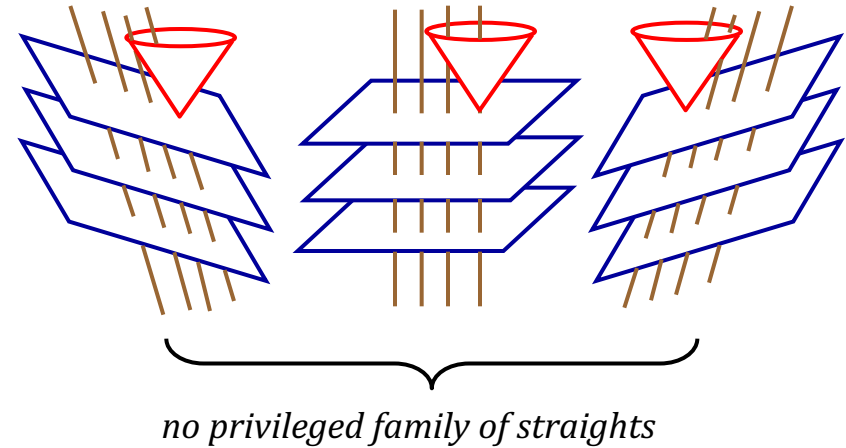


Neo-Newtonian Spacetime



1. Many inertial frames; none privileged.
2. Velocity is relative.
3. Acceleration is absolute.
4. Simultaneity is absolute.

Minkowski Spacetime



1. Many inertial frames; none privileged.
2. Velocity is relative.
3. Acceleration is absolute.
4. Simultaneity is relative.
5. Invariant light-cone structure at each point.

Problem: Special relativity does not account for the gravitational force.

- To include gravity...

Geometricize it! Make it a feature of spacetime geometry.



Two requirements:

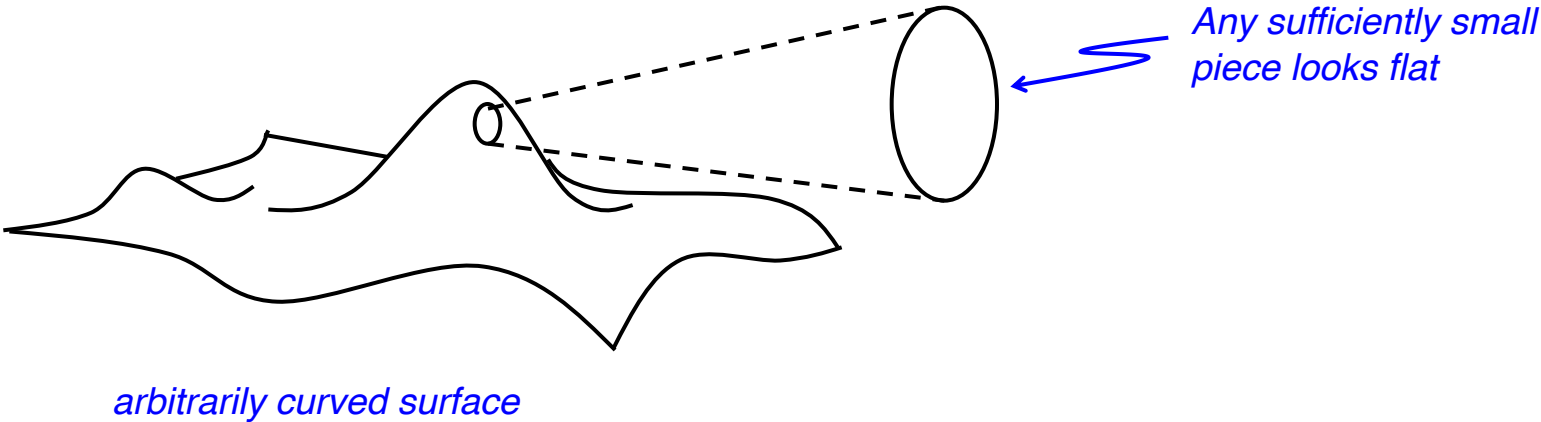
(1) New theory ("general relativity") must reduce to special relativity in sufficiently flat regions of spacetime:

- Replace $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ with $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

flat Minkowski metric

non-flat metric

- Require $g_{\mu\nu}$ to reduce to $\eta_{\mu\nu}$ in small regions of spacetime.



Problem: Special relativity does not account for the gravitational force.

- To include gravity...

Geometricize it! Make it a feature of spacetime geometry.



Two requirements:

(2) Curvature of spacetime must be related to matter density:

- The Einstein equations (1916):

$$G_{\mu\nu}(g_{\mu\nu}) = \kappa T_{\mu\nu}$$

*Einstein tensor = encodes
curvature of spacetime as a
function of $g_{\mu\nu}$*

*Stress-energy tensor =
encodes matter density*

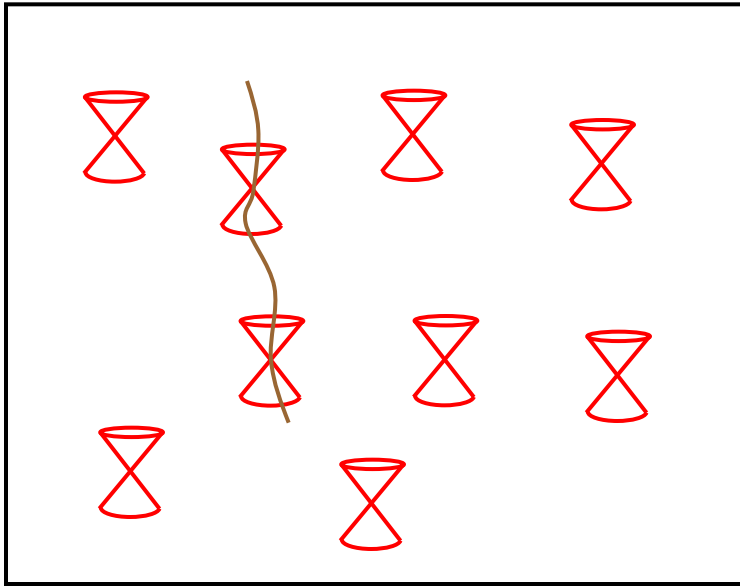
- Consequence: The Minkowski metric is the solution for zero curvature $G_{\mu\nu}(\eta_{\mu\nu}) = 0$ (i.e., spatiotemporal flatness).

A general relativistic spacetime is a 4-dim collection of points such that between any two (infinitesimally close) points, there is a definite spacetime interval given by $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, where $g_{\mu\nu}$ is a Lorentzian metric that satisfies the Einstein equations.



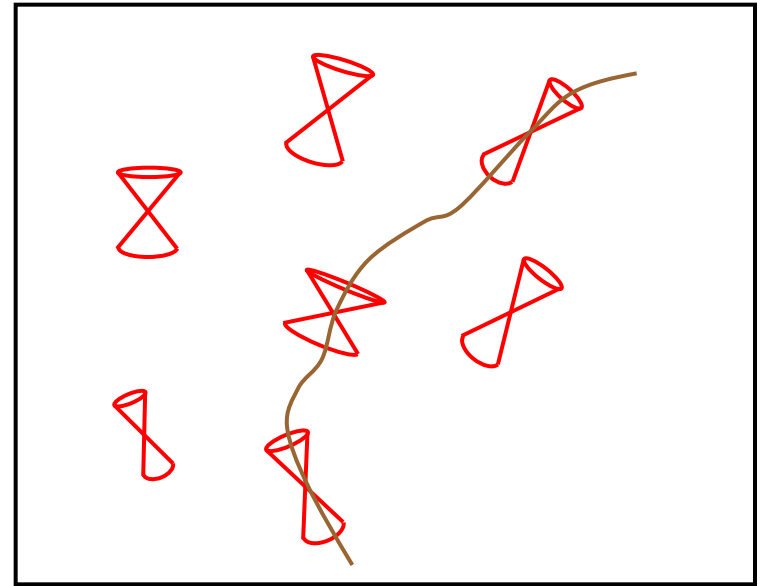
*"reduces to the Minkowski
metric at any point"*

Minkowski Spacetime



Invariant light-cone structure at each point: light-cones all have same size and orientation.

Arbitrary General Relativistic Spacetime



Variable light-cone structure at each point: light-cones can twist and turn due to curvature.

- Idea: The light-cone structure constrains the motion of physical objects (traveling on timelike worldlines).
- And: In an arbitrary general relativistic spacetime, the matter density determines the light-cone structure.