

1. QECs
2. Single-Qubit Flip Code
3. Three-Qutrit Erasure Protection Code

10. Quantum Information: Protection

1. Quantum Error Correction Codes (QECs)

- Classical Error Correction: Redundantly encode Shannon information in a larger classical system in such a way that errors can be detected and corrected.

Ex: Repetition code. To send a zero bit "0", copy it five times and transmit "00000".

- Some of the bits may be corrupted in transit (i.e., flipping to "1").
- Receiver does a "majority vote" on the bits she receives to recover "0".
- Note: Assumes some prior knowledge of the type of error that we expect (i.e., it assumes a majority of the bits don't get corrupted).

- Quantum Error Correction: Redundantly encode quantum information in the *entanglement structure* of a larger quantum system in such a way that errors can be detected and corrected.

Task: To detect and correct errors *without* decohering the relevant entangled qubits.

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- No-Cloning theorem prevents a qubit version of the repetition code.
 - But perhaps entanglement offers just as good, if not better, coding possibilities!

Goal: To transmit a qubit $|Q\rangle = a|0\rangle + b|1\rangle$ that gets corrupted in transmission, and then reconstruct it at the receiver.

Step 1. Encoding protocol. Encode the "logical" $|Q\rangle$ in a multi-qubit "physical" **codeword** vector state.

- Do this by performing appropriate transformations on the single-qubit basis states $|0\rangle, |1\rangle$.  *The type of transformations depends on the type of errors we expect to occur.*
- The multi-qubit basis states span a vector space called the **code space** $\mathcal{H}_C$.
- Complete the set of basis states to form a larger space called the **coding space** $\mathcal{H}$.  *$\mathcal{H}_C$ is a subspace of $\mathcal{H}$*

Example: We might transform the single-qubit basis states into three-qubit basis states:

$$|0\rangle \rightarrow |000\rangle$$

$$|1\rangle \rightarrow |111\rangle$$

- The **codeword** is then $a|000\rangle + b|111\rangle$.
- The **code space** $\mathcal{H}_C$ is the space spanned by $\{|000\rangle, |111\rangle\}$, which is a 2-dim subspace of the larger 8-dim three-qubit **coding space** space $\mathcal{H}$ spanned by $\{|000\rangle, |001\rangle, |010\rangle, |100\rangle, |110\rangle, |101\rangle, |011\rangle, |111\rangle\}$.

Step 2. Represent **errors** by multi-qubit operators constructed from the single-qubit operators I, X, Y, Z .

- Errors "corrupt" the basis states of $\mathcal{H}_C$, and hence the codeword, projecting it out of $\mathcal{H}_C$.

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- An **error** might be represented by the operator $X \otimes I \otimes I$.
- This would produce a corrupted codeword $a|100\rangle + b|011\rangle$, which is an element of $\mathcal{H}$ but not of $\mathcal{H}_C$.

Step 3. *Recovery protocol.* Devise an appropriate operation that acts on a corrupted codeword in $\mathcal{H}$ and projects it back into $\mathcal{H}_C$ (thereby "correcting" it).

Necessary and sufficient condition for error correction

- Let $\mathcal{H}_C = \text{span}\{|\psi_1\rangle, ..., |\psi_p\rangle\}$, for some number p of basis states.
Let $\mathcal{E} = \{E_1, ..., E_q\}$ be a set of q error operators.

Knill-Laflamme (KL) Condition: A code space $\mathcal{H}_C = \text{span}\{|\psi_1\rangle, ..., |\psi_p\rangle\}$ corrects the error set $\mathcal{E} = \{E_1, ..., E_q\}$ if and only if

(i) $\langle \psi_i | E_k^\dagger E_l | \psi_j \rangle = 0$

Corrupted basis states $E_l|\psi_j\rangle, E_k|\psi_i\rangle$ are orthogonal, and hence distinguishable from each other

(ii) $\langle \psi_i | E_k^\dagger E_l | \psi_i \rangle = \langle \psi_j | E_k^\dagger E_l | \psi_j \rangle, \quad i \neq j$

Measurements made to determine the error will not give any information about the codeword (and thereby possibly decohere it).

- Constraints (i) & (ii) together entail:

$$\langle \psi_i | E_k^\dagger E_l | \psi_j \rangle = c_{kl} \delta_{ij}$$

The projection of an error operator and its "reverse" onto the code space is a multiple of the identity

where c_{kl} are arbitrary constants and δ_{ij} is the identity

Intuition: Errors can be corrected if we can reverse their damage; i.e., if for any error E_l , there is a reverse error $E_k^\dagger$.

2. Single-Qubit Flip Code

Goal: To transmit a "logical" qubit $|Q\rangle = a|0\rangle + b|1\rangle$ in the presence of noise that flips single-qubit basis states.

Step 1. Encoding protocol. Encode $|Q\rangle$ in codeword $|\Phi\rangle = a|000\rangle + b|111\rangle$.

- $|\Phi\rangle$ is an element of the 2-dim code space $\mathcal{H}_c = \text{span}\{|000\rangle, |111\rangle\}$.
- $|\Phi_{\text{corrupt}}\rangle = a|_{-}\rangle_1|_{-}\rangle_2|_{-}\rangle_3 + b|_{-}\rangle_1|_{-}\rangle_2|_{-}\rangle_3$ can take one of four forms:

$$\begin{aligned} &a|000\rangle + b|111\rangle \\ &a|100\rangle + b|011\rangle \\ &a|010\rangle + b|101\rangle \\ &a|001\rangle + b|110\rangle \end{aligned}$$

$|\Phi_{\text{corrupt}}\rangle$ is an element of the 8-dim three-qubit space $\mathcal{H} = \text{span}\{|000\rangle, |001\rangle, |010\rangle, |100\rangle, |110\rangle, |101\rangle, |011\rangle, |111\rangle\}$

Step 2. Represent single-qubit flip errors by 4 three-qubit operators:

$$\mathcal{E} = \{I \otimes I \otimes I, X \otimes I \otimes I, I \otimes X \otimes I, I \otimes I \otimes X\}$$


does nothing


flips 1st qubit


flips 2nd qubit


flips 3rd qubit

Step 3. Recovery protocol.

(a) Attach two "empty register" qubits $|00\rangle$ to $|\Phi_{\text{corrupt}}\rangle$:

$$|\Phi_{\text{corrupt}}\rangle|00\rangle = \{a|-\rangle_1|-\rangle_2|-\rangle_3 + b|-\rangle_1|-\rangle_2|-\rangle_3\}|0\rangle_4|0\rangle_5$$

(b) Error detection:

- Perform *XOR* on qubits 1 and 2 and store result in qubit 4.
- Perform *XOR* on qubits 1 and 3 and store result in qubit 5.

0	XOR	0	=	0
0	XOR	1	=	1
1	XOR	0	=	1
1	XOR	1	=	0

(b) Error correction: Measure qubits 4 and 5 to determine form of $|\Phi_{\text{corrupt}}\rangle$ and what three-qubit operator to use to correct it.

<u>Corrupted codeword/register</u>	<u>Error detection</u>	<u>Error correction</u>
$\{a 000\rangle + b 111\rangle\} 00\rangle$	$\{a 000\rangle + b 111\rangle\} 00\rangle$	$I \otimes I \otimes I$
$\{a 100\rangle + b 011\rangle\} 00\rangle$	$\{a 100\rangle + b 011\rangle\} 11\rangle$	$X \otimes I \otimes I$
$\{a 010\rangle + b 101\rangle\} 00\rangle$	$\{a 010\rangle + b 101\rangle\} 10\rangle$	$I \otimes X \otimes I$
$\{a 001\rangle + b 110\rangle\} 00\rangle$	$\{a 001\rangle + b 110\rangle\} 01\rangle$	$I \otimes I \otimes X$

Both detection and correction protocols do not decohere $|\Phi_{\text{corrupt}}\rangle$!

Does the KL Condition hold?

- Does $\mathcal{H}_c = \text{span}\{|000\rangle, |111\rangle\}$ correct the error set $\mathcal{E} = \{III, XII, IXI, IIX\}$?

Yes!

- Do the following constraints hold, for any $E_k, E_l \in \mathcal{E}$:
 - (i) $\langle 000 | E_k^\dagger E_l | 111 \rangle = 0$
 - (ii) $\langle 000 | E_k^\dagger E_l | 000 \rangle = \langle 111 | E_k^\dagger E_l | 111 \rangle$
- Note: $I^\dagger = I, II = I, X^\dagger = X, XX = I$.
- Also: $(A \otimes B \otimes C)(D \otimes E \otimes F) = (AD) \otimes (BE) \otimes (CF)$
 - So, e.g., $(XII)^\dagger(IIX) = (XI) \otimes (II) \otimes (IX) = XIX$
- In general: In all combinations of $E_k^\dagger E_l$, there will be at most two X 's.
- So: Any combination of $E_k^\dagger E_l$ will fail to convert $|111\rangle$ into $|000\rangle$ or vice-versa.
- Thus: In all cases of both (i) and (ii), the inner products will vanish.

3. Three-Qutrit Erasure Protection Code

Goal: To transmit a single "logical" qutrit using three "physical" qutrits in the presence of noise that *erases* one of the latter.

*Qutrit (state sense):
a state of a 3-state
quantum system!*

Step 1. Encoding protocol. Encode a single-qutrit state $|\psi\rangle = C_0|0\rangle + C_1|1\rangle + C_2|2\rangle$ in a three-qutrit codeword state $|\tilde{\psi}\rangle = C_0|\tilde{0}\rangle + C_1|\tilde{1}\rangle + C_2|\tilde{2}\rangle$.

Encoding map

$$|\tilde{i}\rangle = (U_{12} \otimes I_3)(|i\rangle_1 \otimes |\chi\rangle_{23})$$

where

$$|\chi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$

$$U: \begin{array}{lll} |00\rangle \rightarrow |00\rangle & |11\rangle \rightarrow |20\rangle & |22\rangle \rightarrow |10\rangle \\ |01\rangle \rightarrow |11\rangle & |12\rangle \rightarrow |01\rangle & |20\rangle \rightarrow |21\rangle \\ |02\rangle \rightarrow |22\rangle & |10\rangle \rightarrow |12\rangle & |21\rangle \rightarrow |02\rangle \end{array}$$

To get a three-qutrit basis state $|\tilde{i}\rangle$, $i = 0, 1, 2$, of the codespace, take a single-qutrit basis state $|i\rangle$ of the logical qutrit, attach an ancillary 2-qutrit entangled state $|\chi\rangle$, and act on the result with a 3-qutrit operator $U \otimes I$, where U is a 2-qutrit operator.

Result

$$\left. \begin{array}{l} |\tilde{0}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) \\ |\tilde{1}\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle) \\ |\tilde{2}\rangle = \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle) \end{array} \right\}$$

$$|\tilde{\psi}\rangle = (U_{12} \otimes I_3)(|\psi\rangle_1 \otimes |\chi\rangle_{23})$$

*3-qutrit
codeword
to be sent.*

Or: To encode our single-qutrit $|\psi\rangle$, attach the ancillary 2-qutrit entangled state $|\chi\rangle$, and act on the result with $U \otimes I$.

Step 2. Recovery protocol: To recover the single-qutrit state $|\psi\rangle$ from the three-qutrit codeword $|\tilde{\psi}\rangle = (U_{12} \otimes I_3)(|\psi\rangle_1 \otimes |\chi\rangle_{23})$, all we need to know is which of the physical qutrits has been erased:

<u>Qutrit lost</u>	<u>Act on $\tilde{\psi}\rangle$ with:</u>	<u>Result</u>
3	$(U_{12}^\dagger \otimes I_3)$	$ \psi\rangle_1 \otimes \chi\rangle_{23}$
2	$(U_{31}^\dagger \otimes I_2)$	$ \psi\rangle_3 \otimes \chi\rangle_{12}$
1	$(U_{23}^\dagger \otimes I_1)$	$ \psi\rangle_2 \otimes \chi\rangle_{13}$

Does the KL Condition hold?

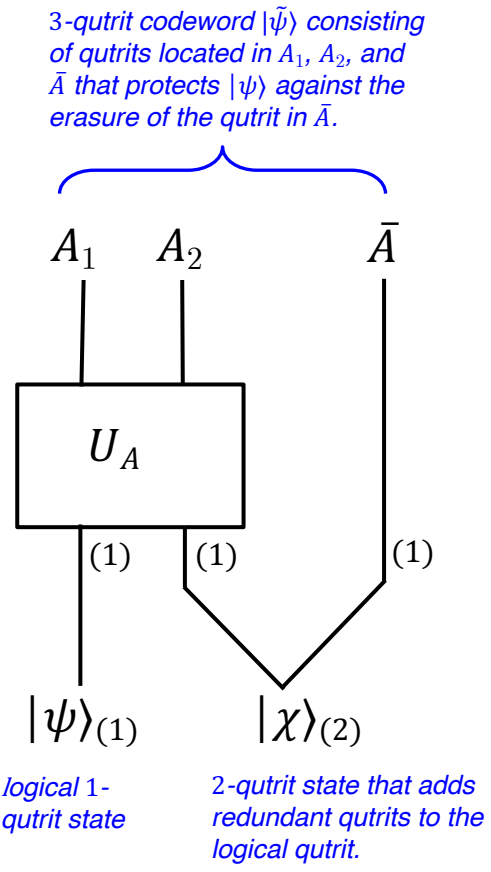
Yes!

- The physical 27-dim three-qutrit coding space can be represented by $\mathcal{H}^{(3)} = \mathcal{H}_A^{(2)} \otimes \mathcal{H}_{\bar{A}}^{(1)}$.
- The three-qutrit codespace $\mathcal{H}_C$ is a 9-dim subspace of $\mathcal{H}^{(3)}$.
- The encoding map requires that the 2-qutrit subspace $\mathcal{H}_A^{(2)}$ be decomposed into two further single-qutrit subspaces $\mathcal{H}_A^{(2)} = \mathcal{H}_{A_1}^{(1)} \otimes \mathcal{H}_{A_2}^{(1)}$, with the ancillary entangled state $|\chi\rangle \in \mathcal{H}_{A_2}^{(1)} \otimes \mathcal{H}_{\bar{A}}^{(1)}$.

• One can now prove:

For any operator $O_{\bar{A}}$ that acts on $\mathcal{H}_{\bar{A}}^{(1)}$,

$$\langle \tilde{i} | O_{\bar{A}} | \tilde{j} \rangle = c \delta_{ij}$$



- KL Condition!
- Think of $O_{\bar{A}}$ as an error operator and its reverse.
 - $O_{\bar{A}}$ is any operator that acts on the erased qutrit: it doesn't matter what it does, and this is reflected in the KL requirement that it acts like the identity on $\mathcal{H}_C$.