

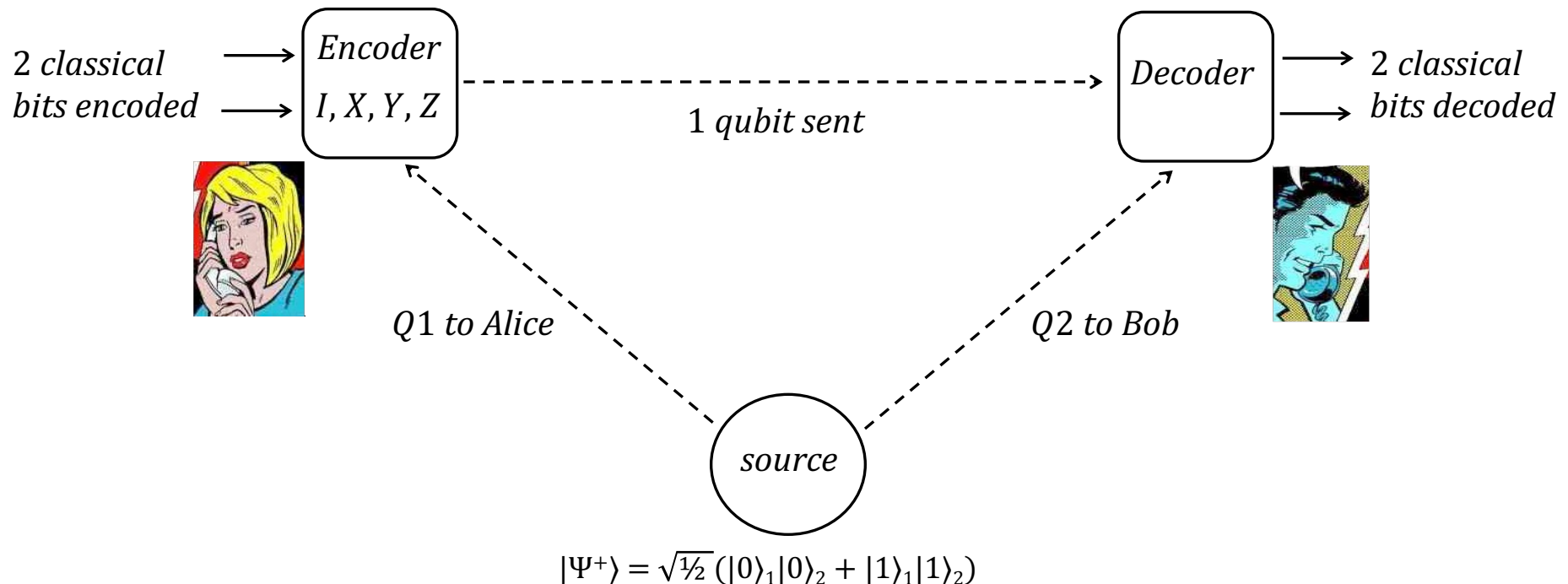
09. Quantum Information: Transmission.

1. Quantum Dense Coding

- Goal: To use one qubit to transmit two classical bits.
- But: One qubit (supposedly) only contains one classical bit's worth of (Shannon) information!
- So: How can we send 2 classical bits using just one qubit?
- Answer: Use entangled states!

Set-Up:

- Prepare two qubits $Q1, Q2$ in an entangled state $|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2)$
- Alice gets $Q1$, Bob gets $Q2$.
- Alice manipulates her $Q1$ so that it steers Bob's $Q2$ into a state from which he can read off the 2 classical bits Alice desires to send. All he needs to do this is the post-manipulated $Q1$ that Alice sends to him.



Protocol

1. Alice has a pair of classical bits: either 00, 01, 10, or 11.

She first encodes the pair in $Q1$ by acting on $Q1$ with one of $\{I, X, Y, Z\}$ according to:

<u>pair</u>	<u>transform</u>	<u>new state</u>
00	$(I_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2} (0\rangle_1 0\rangle_2 + 1\rangle_1 1\rangle_2) = \Psi^+\rangle$
01	$(X_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2} (1\rangle_1 0\rangle_2 + 0\rangle_1 1\rangle_2) = \Phi^+\rangle$
10	$(Y_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2} (- 1\rangle_1 0\rangle_2 + 0\rangle_1 1\rangle_2) = \Phi^-\rangle$
11	$(Z_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2} (0\rangle_1 0\rangle_2 - 1\rangle_1 1\rangle_2) = \Psi^-\rangle$

- Let $Q1$ and $Q2$ be electrons in Hardness states.
- Let $|0\rangle$ be $|\text{soft}\rangle$ and $|1\rangle$ be $|\text{hard}\rangle$.

2. Alice now sends $Q1$ to Bob.

3. After reception of $Q1$, Bob first applies a C_{NOT} transformation to both $Q1$ and $Q2$:

<u>pair</u>	<u>transform</u>	<u>new state</u>	<u>Apply C_{NOT}</u>
00	$(I_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2} (0\rangle_1 0\rangle_2 + 1\rangle_1 1\rangle_2) = \Psi^+\rangle$	$\sqrt{1/2} (0\rangle_1 + 1\rangle_1) 0\rangle_2$
01	$(X_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2} (1\rangle_1 0\rangle_2 + 0\rangle_1 1\rangle_2) = \Phi^+\rangle$	$\sqrt{1/2} (1\rangle_1 + 0\rangle_1) 1\rangle_2$
10	$(Y_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2} (- 1\rangle_1 0\rangle_2 + 0\rangle_1 1\rangle_2) = \Phi^-\rangle$	$\sqrt{1/2} (- 1\rangle_1 + 0\rangle_1) 1\rangle_2$
11	$(Z_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2} (0\rangle_1 0\rangle_2 - 1\rangle_1 1\rangle_2) = \Psi^-\rangle$	$\sqrt{1/2} (0\rangle_1 - 1\rangle_1) 0\rangle_2$

According to the Eigenvalue-Eigenvector Rule, $Q1$ still has no definite value, but $Q2$ now does!

Protocol

4. Bob now applies a Hadamard transformation to $Q1$:

<u>pair</u>	<u>transform</u>	<u>new state</u>	<u>Apply C_{NOT}</u>	<u>Apply $\mathfrak{S}_1$</u>
00	$(I_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2}(0\rangle_1 0\rangle_2 + 1\rangle_1 1\rangle_2) = \Psi^+\rangle$	$\sqrt{1/2}(0\rangle_1 + 1\rangle_1) 0\rangle_2$	$ 0\rangle_1 0\rangle_2$
01	$(X_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2}(1\rangle_1 0\rangle_2 + 0\rangle_1 1\rangle_2) = \Phi^+\rangle$	$\sqrt{1/2}(1\rangle_1 + 0\rangle_1) 1\rangle_2$	$ 0\rangle_1 1\rangle_2$
10	$(Y_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2}(- 1\rangle_1 0\rangle_2 + 0\rangle_1 1\rangle_2) = \Phi^-\rangle$	$\sqrt{1/2}(- 1\rangle_1 + 0\rangle_1) 1\rangle_2$	$ 1\rangle_1 1\rangle_2$
11	$(Z_1 \otimes I_2) \Psi^+\rangle$	$\sqrt{1/2}(0\rangle_1 0\rangle_2 - 1\rangle_1 1\rangle_2) = \Psi^-\rangle$	$\sqrt{1/2}(0\rangle_1 - 1\rangle_1) 0\rangle_2$	$ 1\rangle_1 0\rangle_2$

According to the EE Rule, $Q1$ and $Q2$ now both have definite values.

5. Bob now measures $Q1$ and $Q2$ to determine the number Alice sent!

(a) $(Q1 = 0, Q2 = 0) \Rightarrow 00$

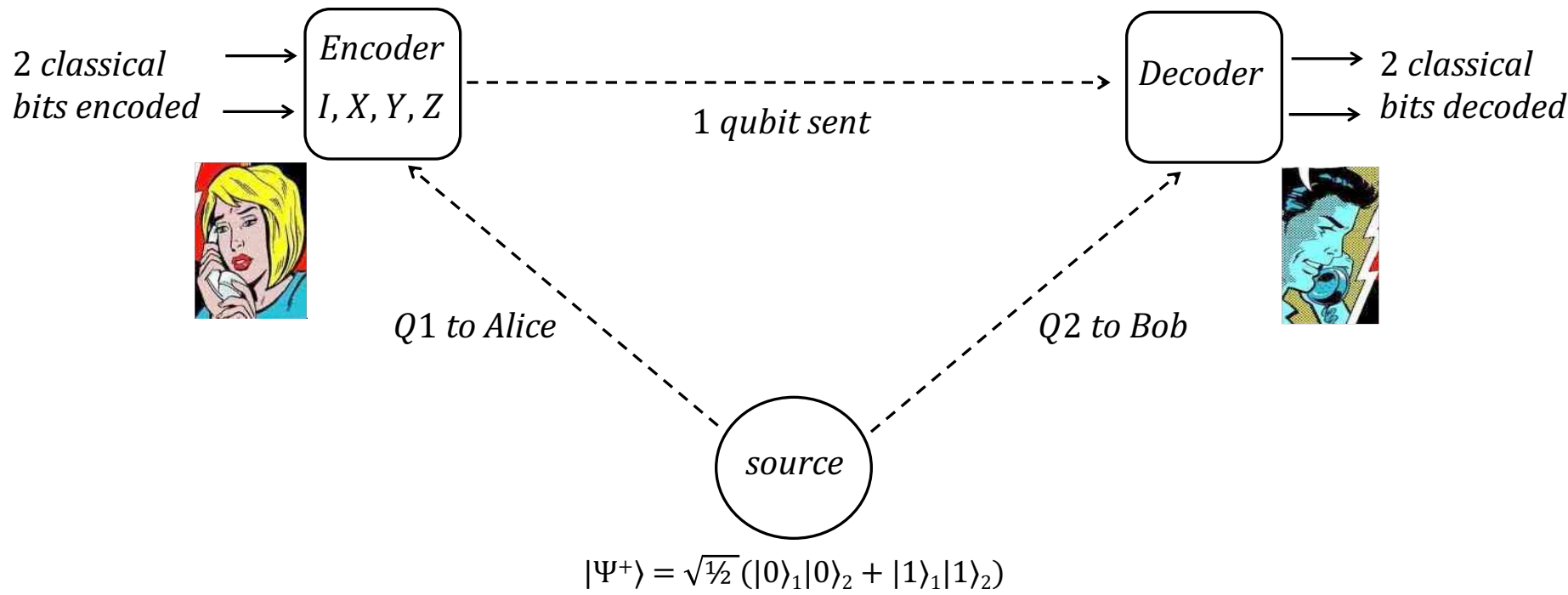
(b) $(Q1 = 0, Q2 = 1) \Rightarrow 01$

(c) $(Q1 = 1, Q2 = 1) \Rightarrow 10$

(d) $(Q1 = 1, Q2 = 0) \Rightarrow 11$

Question: How are the 2 classical bits transferred from Alice to Bob?

- *Not* transferred *via* the single qubit.
- Transferred by the *correlations* present in the 2-qubit entangled state $|\Psi^+\rangle$.
- Claim: In order to convey information (quantum or Shannon) between Alice and Bob, it need *not* be physically transported from Alice to Bob across the intervening spatial distance.
- The *only* thing required to convey information (quantum or Shannon) is to set up a correlation between the sender's data and the receiver's data.

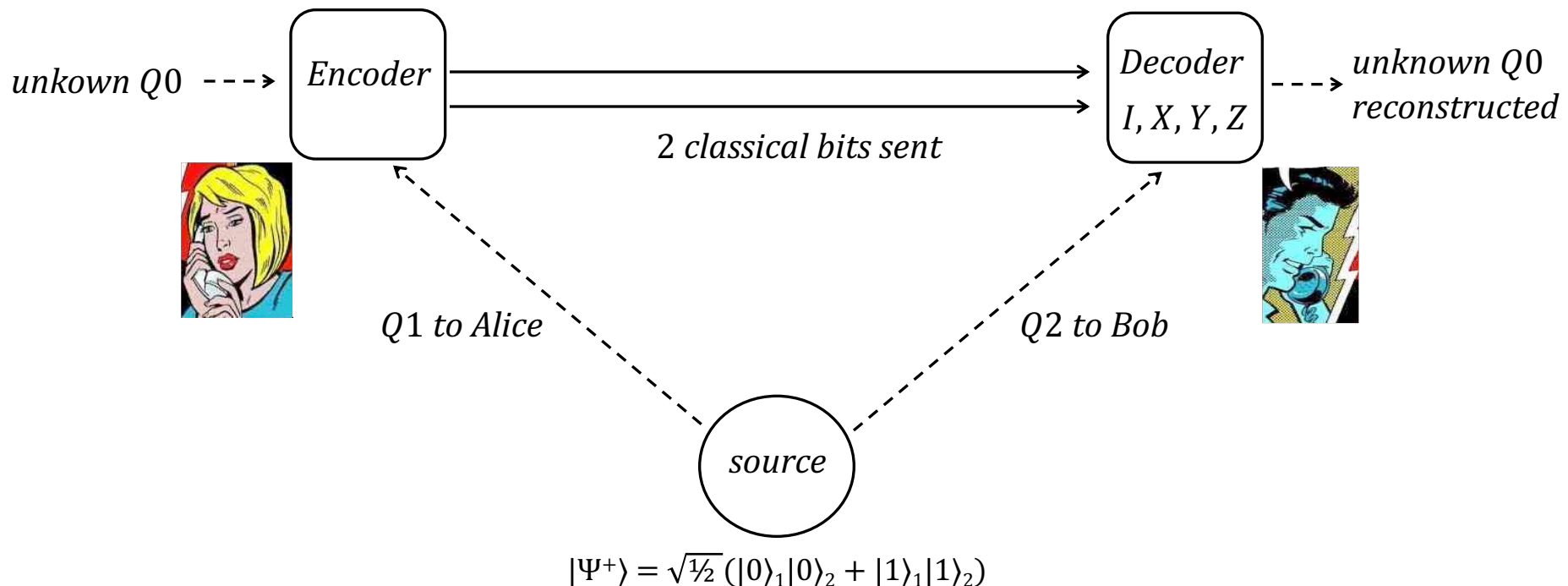


2. Quantum Teleportation

- Goal: To transmit an unknown quantum state using classical bits and to reconstruct the exact quantum state at the receiver.
- But: How can this avoid the No-Cloning Theorem?
- Answer: Use entangled states!

Set-Up:

- Alice has an unknown $Q0$, $|Q\rangle_0 = a|0\rangle_0 + b|1\rangle_0$, and wants to send it to Bob.
- $Q1$ and $Q2$ are prepared in an entangled state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2)$. Alice gets $Q1$, Bob gets $Q2$.
- Alice manipulates $Q0$ and $Q1$ so that they steer Bob's $Q2$ into a form from which he can reconstruct the unknown state of $Q0$. All Bob needs to do this are 2 classical bits sent by Alice.



Protocol

1. Alice starts with a 3-qubit system ($Q0, Q1, Q2$) in the state:

$$|Q\rangle_0|\Psi^+\rangle = \sqrt{1/2} [a|0\rangle_0|0\rangle_1|0\rangle_2 + a|0\rangle_0|1\rangle_1|1\rangle_2 + b|1\rangle_0|0\rangle_1|0\rangle_2 + b|1\rangle_0|1\rangle_1|1\rangle_2]$$

Alice now applies C_{NOT} on $Q0$ & $Q1$, and then a Hadamard transformation on $Q0$:

First C_{NOT} on $Q0$ & $Q1$:

$$(C_{NOT_{01}} \otimes I_2)|Q\rangle_0|\Psi^+\rangle = \sqrt{1/2} [a|0\rangle_0|0\rangle_1|0\rangle_2 + a|0\rangle_0|1\rangle_1|1\rangle_2 + b|1\rangle_0|1\rangle_1|0\rangle_2 + b|1\rangle_0|0\rangle_1|1\rangle_2]$$

Then $\mathfrak{H}$ on $Q0$:

$$\begin{aligned} (\mathfrak{H}_0 \otimes I_1 \otimes I_2) \{ \sqrt{1/2} [a|0\rangle_0|0\rangle_1|0\rangle_2 + a|0\rangle_0|1\rangle_1|1\rangle_2 + b|1\rangle_0|1\rangle_1|0\rangle_2 + b|1\rangle_0|0\rangle_1|1\rangle_2] \} \\ = \dots \end{aligned}$$

$$(\mathfrak{H}_0 \otimes I_1 \otimes I_2) \{ \sqrt{1/2} [a|0\rangle_0 |0\rangle_1 |0\rangle_2 + a|0\rangle_0 |1\rangle_1 |1\rangle_2 + b|1\rangle_0 |1\rangle_1 |0\rangle_2 + b|1\rangle_0 |0\rangle_1 |1\rangle_2] \}$$

$$= \sqrt{1/2} [a\sqrt{1/2} (|0\rangle_0 + |1\rangle_0) |0\rangle_1 |0\rangle_2 + a\sqrt{1/2} (|0\rangle_0 + |1\rangle_0) |1\rangle_1 |1\rangle_2 \\ + b\sqrt{1/2} (|0\rangle_0 - |1\rangle_0) |1\rangle_1 |0\rangle_2 + b\sqrt{1/2} (|0\rangle_0 - |1\rangle_0) |0\rangle_1 |1\rangle_2]$$

$$= 1/2 [a|0\rangle_0 |0\rangle_1 |0\rangle_2 + a|1\rangle_0 |0\rangle_1 |0\rangle_2 + a|0\rangle_0 |1\rangle_1 |1\rangle_2 + a|1\rangle_0 |1\rangle_1 |1\rangle_2 \\ + b|0\rangle_0 |1\rangle_1 |0\rangle_2 - b|1\rangle_0 |1\rangle_1 |0\rangle_2 + b|0\rangle_0 |0\rangle_1 |1\rangle_2 - b|1\rangle_0 |0\rangle_1 |1\rangle_2]$$

$$= 1/2 |0\rangle_0 |0\rangle_1 (a|0\rangle_2 + b|1\rangle_2) + 1/2 |0\rangle_0 |1\rangle_1 (a|1\rangle_2 + b|0\rangle_2) \\ + 1/2 |1\rangle_0 |0\rangle_1 (a|0\rangle_2 - b|1\rangle_2) + 1/2 |1\rangle_0 |1\rangle_1 (a|1\rangle_2 - b|0\rangle_2)$$

Protocol

1. Alice starts with a 3-qubit system ($Q0, Q1, Q2$) in the state:

$$|Q\rangle_0|\Psi^+\rangle = \sqrt{1/2} (a|0\rangle_0|0\rangle_1|0\rangle_2 + a|0\rangle_0|1\rangle_1|1\rangle_2 + b|1\rangle_0|0\rangle_1|0\rangle_2 + b|1\rangle_0|1\rangle_1|1\rangle_2)$$

Alice now applies C_{NOT} on $Q0$ & $Q1$, and then a Hadamard transformation on $Q0$:

First C_{NOT} on $Q0$ & $Q1$:

$$(C_{NOT_{01}} \otimes I_2)|Q\rangle_0|\Psi^+\rangle = \sqrt{1/2} (a|0\rangle_0|0\rangle_1|0\rangle_2 + a|0\rangle_0|1\rangle_1|1\rangle_2 + b|1\rangle_0|1\rangle_1|0\rangle_2 + b|1\rangle_0|0\rangle_1|1\rangle_2)$$

Then $\mathfrak{H}$ on $Q0$:

$$\begin{aligned} (\mathfrak{H}_0 \otimes I_1 \otimes I_2) \{ \sqrt{1/2} [a|0\rangle_0|0\rangle_1|0\rangle_2 + a|0\rangle_0|1\rangle_1|1\rangle_2 + b|1\rangle_0|1\rangle_1|0\rangle_2 + b|1\rangle_0|0\rangle_1|1\rangle_2] \} \\ = 1/2|0\rangle_0|0\rangle_1(a|0\rangle_2 + b|1\rangle_2) + 1/2|0\rangle_0|1\rangle_1(a|1\rangle_2 + b|0\rangle_2) \\ + 1/2|1\rangle_0|0\rangle_1(a|0\rangle_2 - b|1\rangle_2) + 1/2|1\rangle_0|1\rangle_1(a|1\rangle_2 - b|0\rangle_2) \end{aligned}$$

2. Alice now measures $Q0$ and $Q1$:

<u>If measurement outcome is:</u>	<u>...$Q2$ is now in state:</u>
$ 0\rangle_0 0\rangle_1$	$a 0\rangle_2 + b 1\rangle_2$
$ 0\rangle_0 1\rangle_1$	$a 1\rangle_2 + b 0\rangle_2$
$ 1\rangle_0 0\rangle_1$	$a 0\rangle_2 - b 1\rangle_2$
$ 1\rangle_0 1\rangle_1$	$a 1\rangle_2 - b 0\rangle_2$

EE Rule: Each of the terms represents a state in which $Q0$ and $Q1$ have definite values, but $Q2$ does not.

Protocol

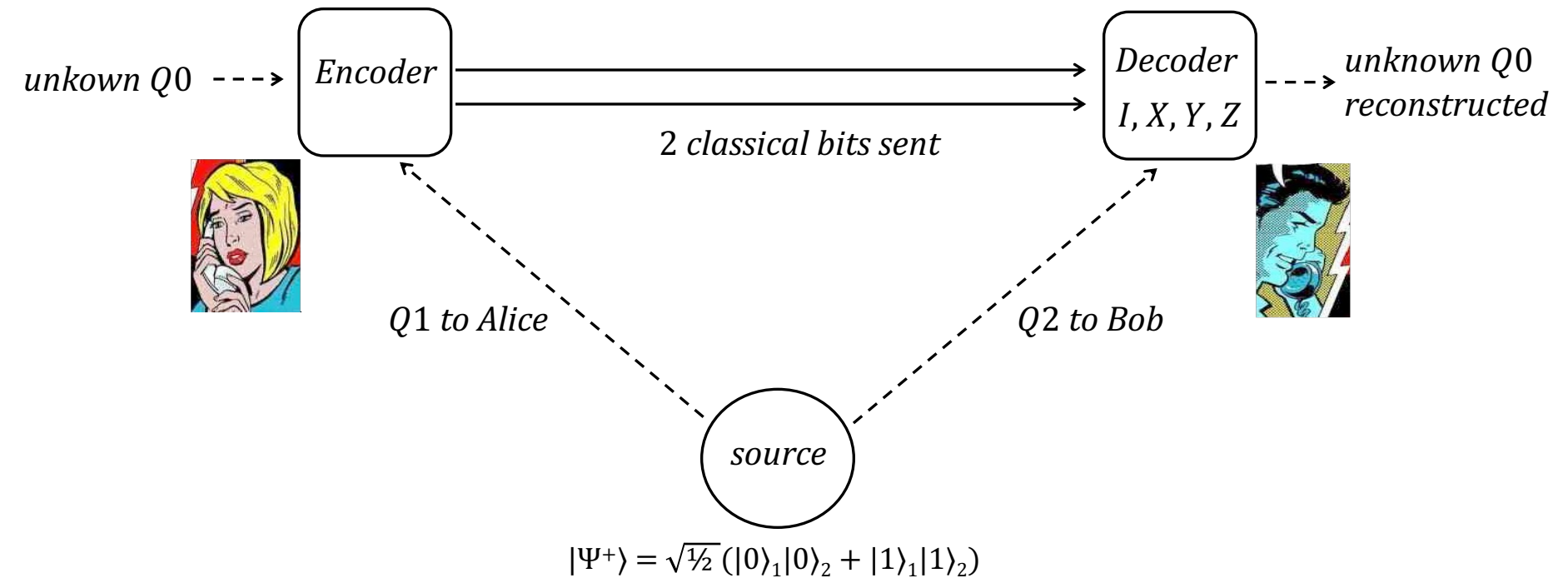
<u>If measurement outcome is:</u>	<u>...Q2 is now in state:</u>
$ 0\rangle_0 0\rangle_1$	$a 0\rangle_2 + b 1\rangle_2$
$ 0\rangle_0 1\rangle_1$	$a 1\rangle_2 + b 0\rangle_2$
$ 1\rangle_0 0\rangle_1$	$a 0\rangle_2 - b 1\rangle_2$
$ 1\rangle_0 1\rangle_1$	$a 1\rangle_2 - b 0\rangle_2$

- Alice sends the result of her measurement to Bob in the form of 2 classical bits: 00, 01, 10, or 11.
- Depending on what he receives, Bob performs one of $\{I, X, Y, Z\}$ on $Q2$. This allows him to turn it into (reconstruct) the unknown $Q0$.

<u>If bits received are</u>	<u>...then Q2 is now in state</u>	<u>...so to reconstruct Q0, use</u>
00	$a 0\rangle_2 + b 1\rangle_2$	I_2
01	$a 1\rangle_2 + b 0\rangle_2$	X_2
10	$a 0\rangle_2 - b 1\rangle_2$	Z_2
11	$a 1\rangle_2 - b 0\rangle_2$	Y_2

Question 1: Does Bob violate the *No-Cloning Theorem*? Doesn't he construct a copy of the unknown $Q0$?

- *No violation occurs.*
- Bob *does* construct a copy: $Q2$ has become an exact duplicate of $Q0$.
- But: After Alice is through transforming $Q0$ and $Q1$, the original $Q0$ has now collapsed to either $|0\rangle_0$ or $|1\rangle_0$!
 - *Alice destroys $Q0$ in the process of conveying the (quantum) information contained in it to Bob!*



Question 2: How does Bob reconstruct the unknown $Q0$ (that encodes an arbitrarily large amount of quantum information) from just 2 classical bits?

- The quantum info to reconstruct $Q0$ is transferred by the correlations present in the entangled state $|\Psi^+\rangle$, *in addition* to the 2 classical bits.
- The 2 classical bits are used simply to determine the appropriate transformation on $Q2$, *after* it has been "steered" into the appropriate state by Alice.

