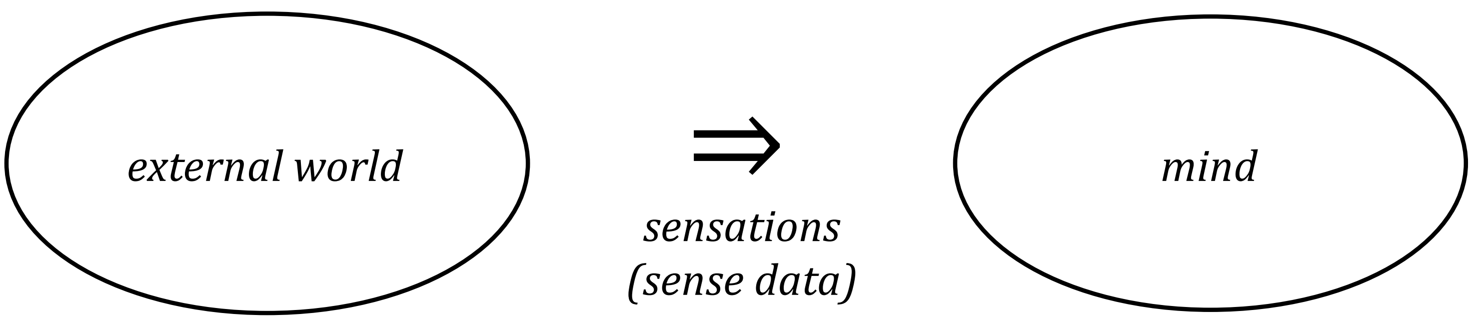
13. Kant and Geometry

- *Old topic*: What is the ontological status of space?
- *New topic*: How is knowledge of space possible?

1. Empiricism, Rationalism, Kantianism

> *Rationalism*: Knowledge originates in reason.
>
> *Empiricism*: Knowledge originates in experience.

Classical Empiricist View:



- *passively receives sense data*
- *initially "empty" (blank slate)*

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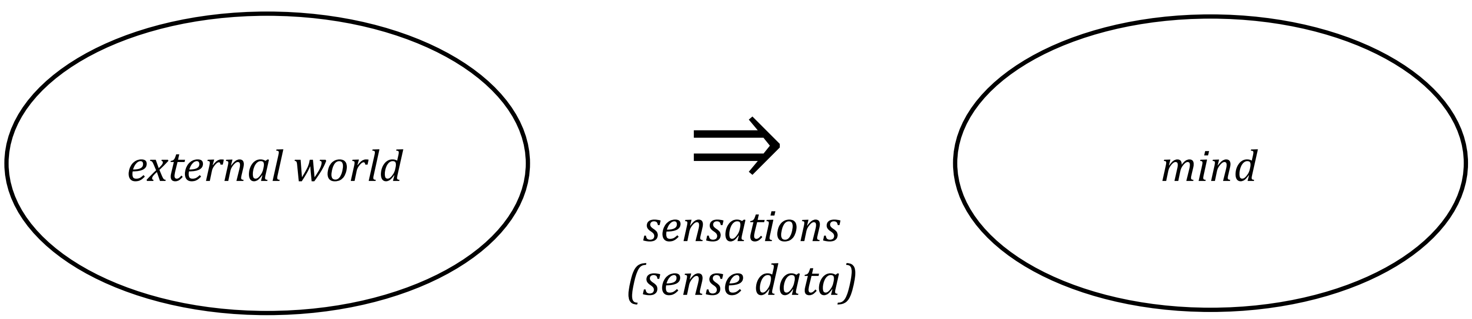
1. Empiricism, Rationalism, Kantianism

Rationalism: Knowledge originates in reason.

Empiricism: Knowledge originates in experience.

Rationalist View:



- *passively receives sense data*
- *initially non-empty: some knowledge of external world possible prior to experience*

Kant's View (The Critique of Pure Reason, 1781)


Critik
der
reinen Vernunft
von
Immanuel Kant
Professor in Königsberg.
Riga,
verlegts Johann Friedrich Hartknoch
1781.

> "That in which alone the sensations can be posited and ordered in a certain form, cannot itself be sensation; and therefore, while the matter of all appearance is given to us *a posteriori* only, its form must lie ready for the sensations *a priori* in the mind, and so must allow of being considered apart from all sensation."

- Knowledge consists of both a *form* and a *content*.

- The *form* of knowledge is conceptual and mind-dependent: a precondition of knowledge (necessary and "*a priori*").
- The *content* of knowledge is contributed by the external world (contingent and "*a posteriori*").

> "The pure form of sensible intuitions in general, in which all the manifold of intuition is intuited in certain relations, must be found in the mind *a priori*. This pure form of sensibility may also itself be called *pure intuition*."

<u>*On Space*</u>:

"Space is a necessary *a priori* representation, which underlies all outer intuitions. We can never represent to ourselves the absence of space, though we can quite well think it as empty of objects. It must therefore be regarded as the condition of the possibility of appearances, and not as a determination dependent on them."

"The apodictic certainty of all geometrical propositions, and the possibility of their *a priori* construction, is grounded in this *a priori* necessity of space. Were this representation of space a concept acquired *a posteriori*, and derived from outer experience in general, the first principles of mathematical determination would be nothing but perceptions. They would therefore all share in the contingent character of perception; that there should be only one straight line between two points would not be necessary, but only what experience always teaches."

- <u>*Recall*</u>: Euclidean geometry is a *consistent axiomatic* system.
  - *Given Euclid's 5 postulates, all other Euclidean claims can be derived, and no contradictory claims can be derived.*
- <u>*Kant's Claim*</u>: Euclidean geometry is *necessary* and *universal*.

Kant's View:

The "noumenal world"

- *unstructured*
- *unordered*

- *actively receives raw data*
- *distintion between content (raw data; initially empty) and form (data filters; initially present)*

- Raw data has no structure or order.
- All structure and order (including causal, temporal, and *spatial*) is imposed on raw data by filters ("pure forms of understanding and intuition") already present in the mind.
- Filtered data (structured, ordered) constitute experience (the "phenomenal world").
- *So*: Some claims about the external world can be known to be true prior to experience!

2. Synthetic *A Priori* Truths

<u>*Four Types of Statements (Judgements)*</u>

1. *analytic*
   - logical truth or definition.
   - predicate is contained in subject.
   - devoid of factual content.

<u>*Ex*</u>: All ravens are either black or non-black.
A bachelor is an unmarried man.
All bodies are extended.

2. *synthetic*
   - contingent (could be either true or false).
   - predicate not contained in subject.
   - contains factual content.

<u>*Ex*</u>: All ravens are black.
All bodies are heavy.

3. *a priori*
   - truth can be established without recourse to experience.
   - "necessary and in the strict sense universal".

<u>*Ex*</u>: $2 + 2 = 4$

4. *a posteriori*
   - truth can be established only with recourse to experience.

<u>*Ex*</u>: All ravens are black.

(1) and (2) are mutually exclusive.

(3) and (4) are mutually exclusive.

	a priori	*a posteriori*
analytic	✓	✗
synthetic	?	✓

- There can be analytic *a priori* statements.
  - *Ex. All ravens are either black or not black.*
- There cannot be analytic *a posteriori* statements.
  - *The truth of an analytic statement is a matter of logic or definition.*
- There can be synthetic *a posteriori* statements.
  - *Ex. All ravens are black.*
- Can there be synthetic *a priori* statements?
  - *Synthetic a priori statement = truth is established by reason alone (a priori) and contains factual content (synthetic).*

Kant's example: 7 + 5 = 12.

- Truth is established by reason (*a priori*).
- But not analytic: Concept of "12" is not contained in concepts of "7" and "5".
- The negation of "7 + 5 = 12" does not entail a contradiction.

Synthetic a priori truths: True claims about the world knowable through reason alone; necessary and universal.

- _Kant's primary examples_:
  - Causal statements.

    > *The concept of causality is a "pure form of understanding" that preconditions how we comprehend natural phenomena.*

  - Statements in arithmetic.

    > *Concepts in arithmetic (continuity and infinity) provide a "pure form of intuition" that preconditions our sensations of time.*

  - Statements in Euclidean geometry.

    > *Concepts in Euclidean geometry provide a "pure form of intuition" that preconditions our sensation of space.*

- The general form our experience takes (the phenomenal world) is the result of a number of sub-forms (forms of intuition and understanding) working in conjunction.
- _And_: These prefiguring forms include Euclidean geometry.

3. Euclidean Geometry as a Synthetic *A Priori* Truth?

Recap: Euclidean geometry is a necessary precondition for *experience*.

- Statements of Euclidean geometry are synthetic *a priori* truths

provide factual content about the physical world

necessary and universal

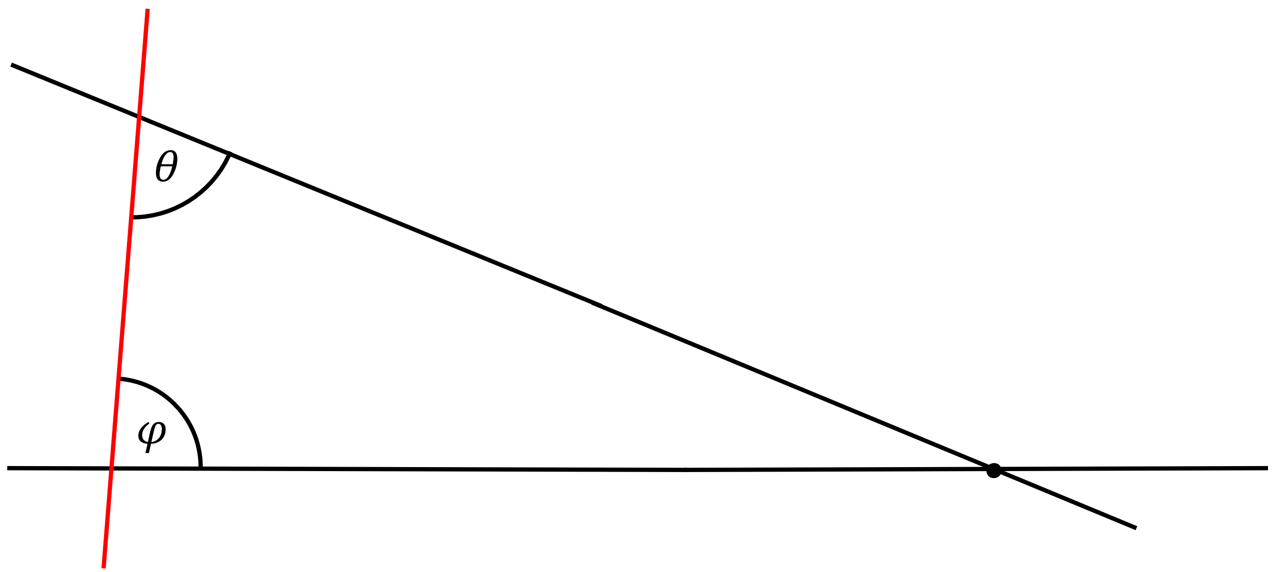
- Such statements:
  - *Are required in order to have experience.*
  - *Are necessarily present in all experience.*
  - *Precede all experience.*

Question: Is Euclidean geometry a necessary precondition for a consistent description of the spatial aspect of the physical world?

- *Prior to the 19th century, yes!*
- *Subsequent to the 19th century, no!*

<u>Recall Euclid's 5th Postulate:</u>

5. If a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the angles are together less than two right angles.



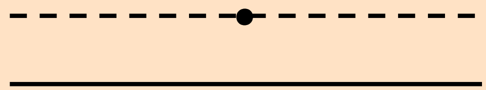
$$\theta + \varphi < 90^{o}$$

- Is 5th Postulate necessary? (1st cent.-19th cent.)
- *<u>Basic strategy</u>*: Attempt to show that replacing 5th Postulate with alternative leads to contradiction.

- Equivalent to 5th Postulate (Playfair 1795):

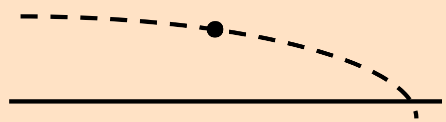
John Playfair
(1748-1819)

> $5'$. Through a given point, *exactly one* line can be drawn parallel to a given line (that does not contain the point).

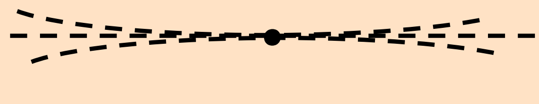


- Only two logically possible alternatives:

> 5^{none}. Through a given point, *no* lines can be drawn parallel to a given line.

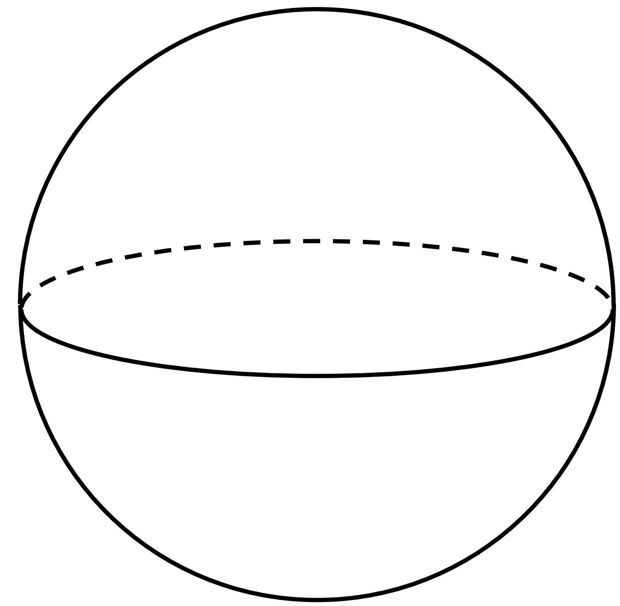


> 5^{many}. Through a given point, *more than one* line can be drawn parallel to a given line.



Case of 5^{none}

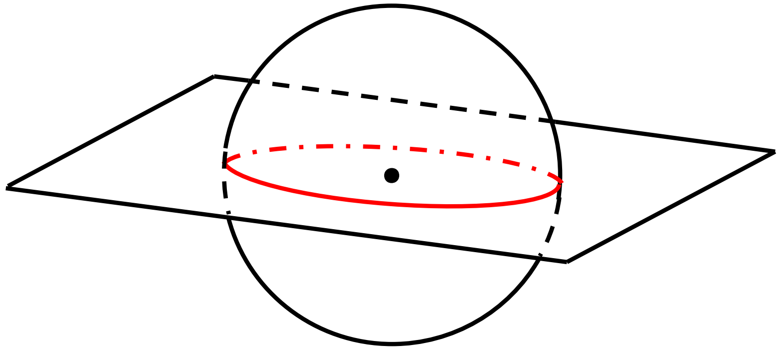
- 5^{none} geometry = *spherical geometry* = 2-dim geometry of surface of a 3-dim sphere.
- Generalized 5^{none} geometry = *elliptical* geometry.
- Euclidean geometry is "flat". *Elliptical* geometry is "positively curved".



On the surface of a sphere:

- There are no parallel straight lines.

Def. A *great circle* on a sphere of radius R and center C, is any circle with radius R and center C.



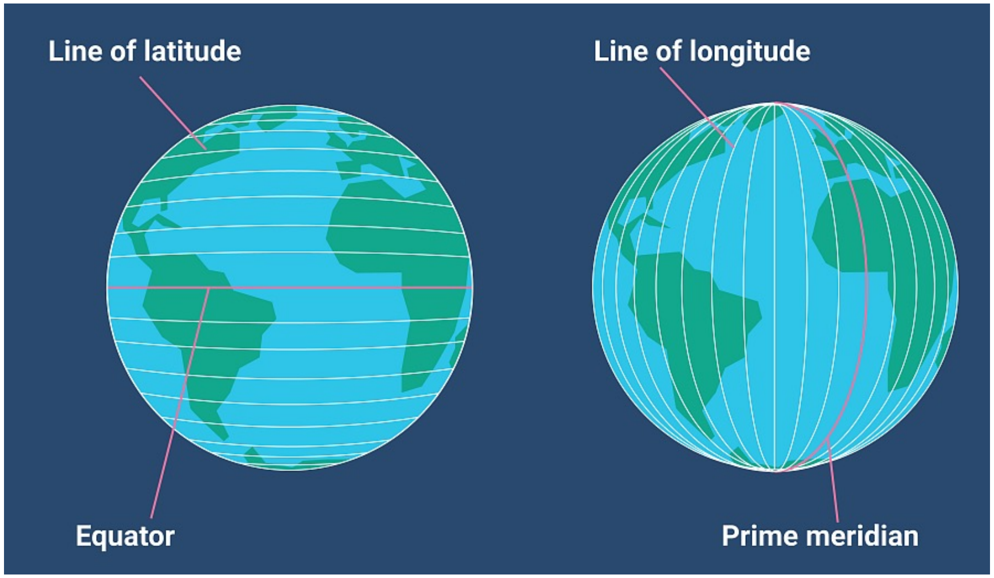
Pass plane through center: where it intersects sphere defines a great circle

Def. A *geodesic* is the shortest distance between two points.

<u>*Claim*</u>:

(a) On the surface of a sphere, the geodesics are given by great circles.

(b) Any two great circles on the surface of a sphere intersect at two points.



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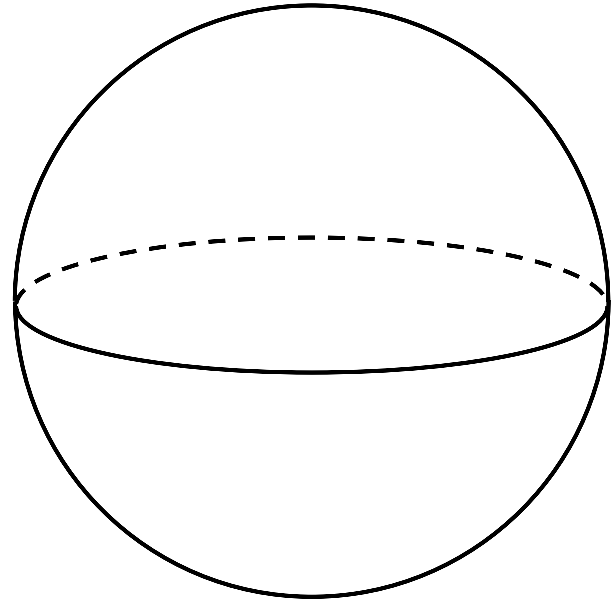
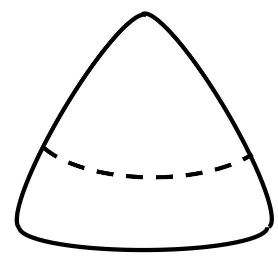


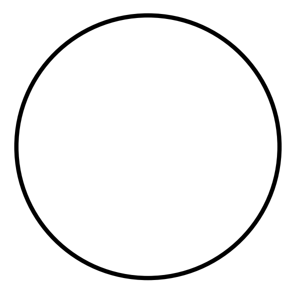
On the surface of a sphere:

- There are no parallel straight lines.
- The sum of angles of a triangle > 2 right angles.



- The *circumference* of any circle < $2\pi \times$ *radius*.

Tighten up a Euclidean circle -- remove wedges from it.



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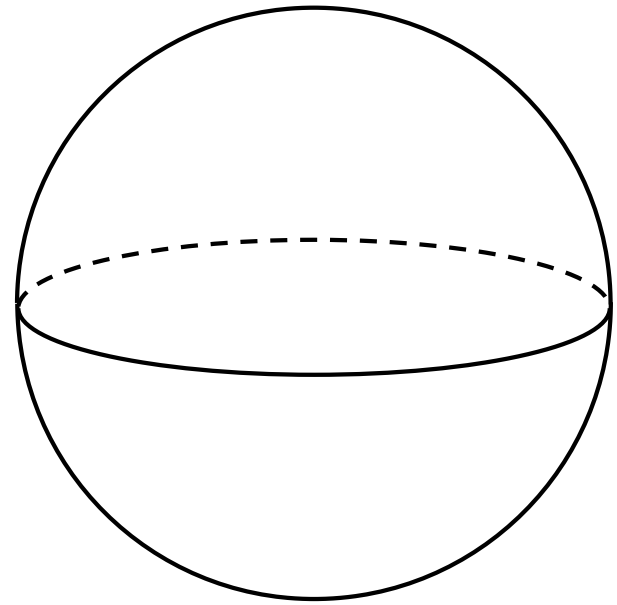

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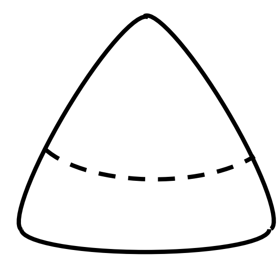

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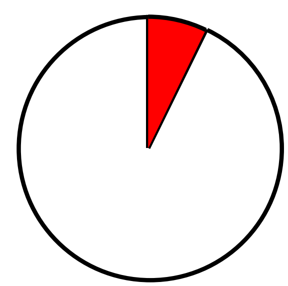

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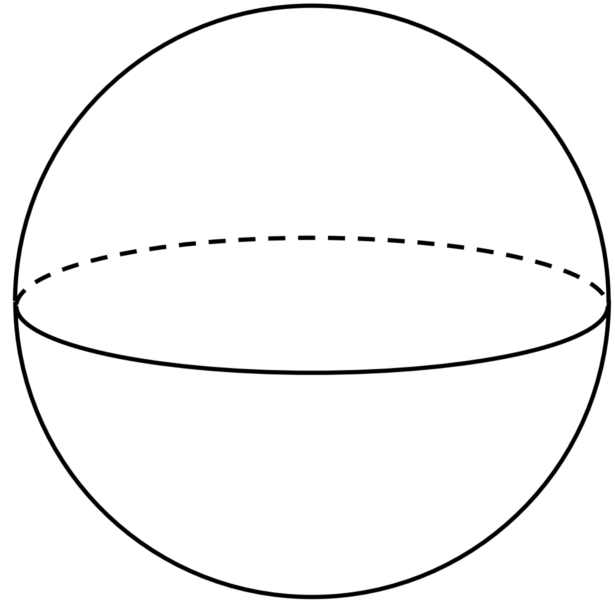

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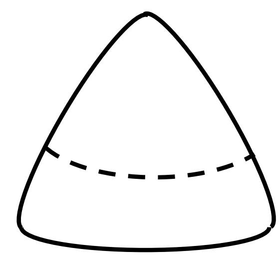

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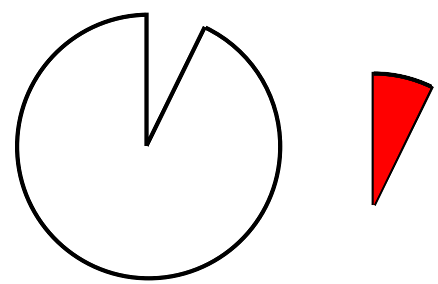

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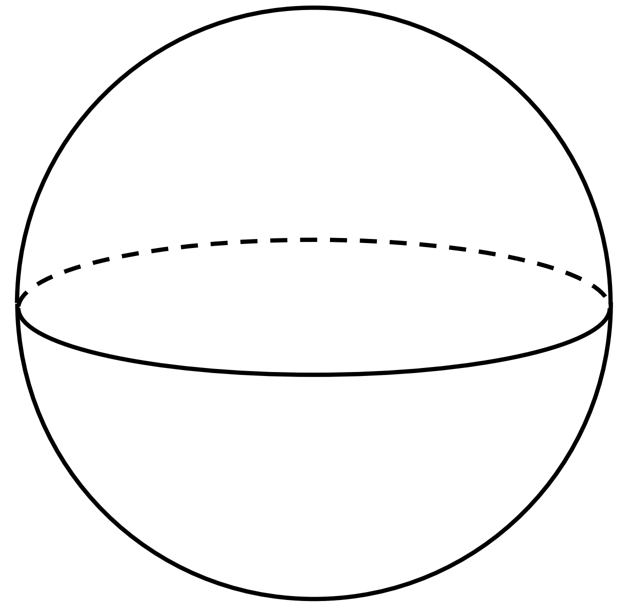

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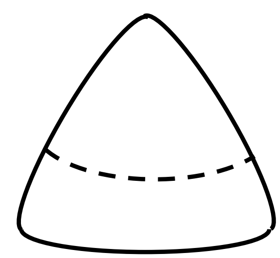

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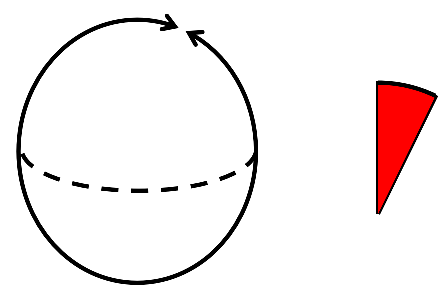
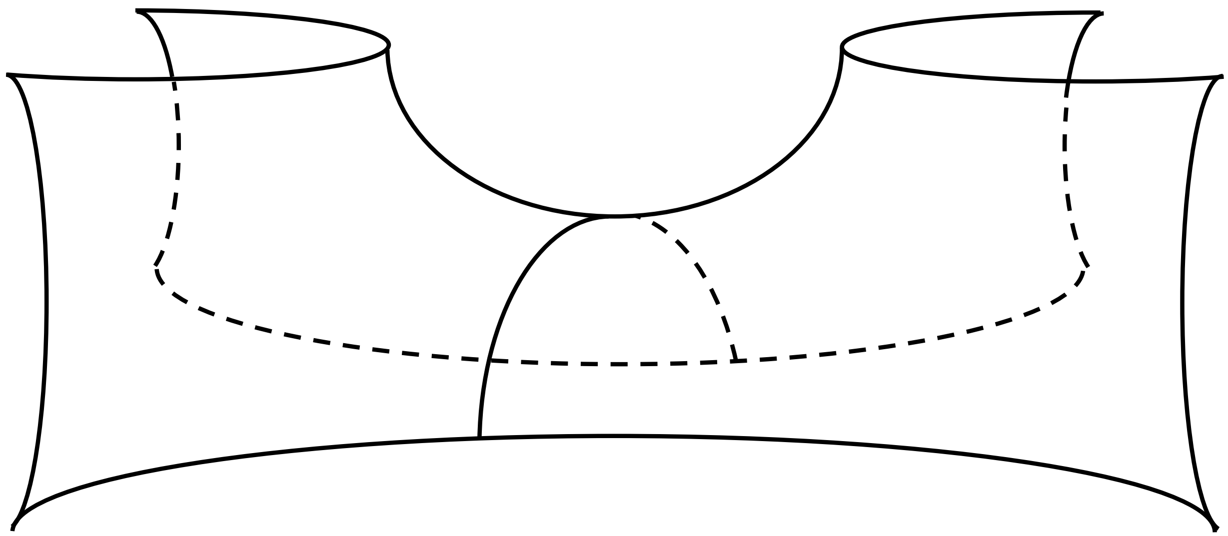


Tighten up a Euclidean circle -- remove wedges from it.

Case of 5^{many}

- 5^{many} geometry = *hyperbolic geometry* = geometry of surfaces of *negative* curvature.
- *Example*: surface of a saddle (hyperbolic paraboloid)



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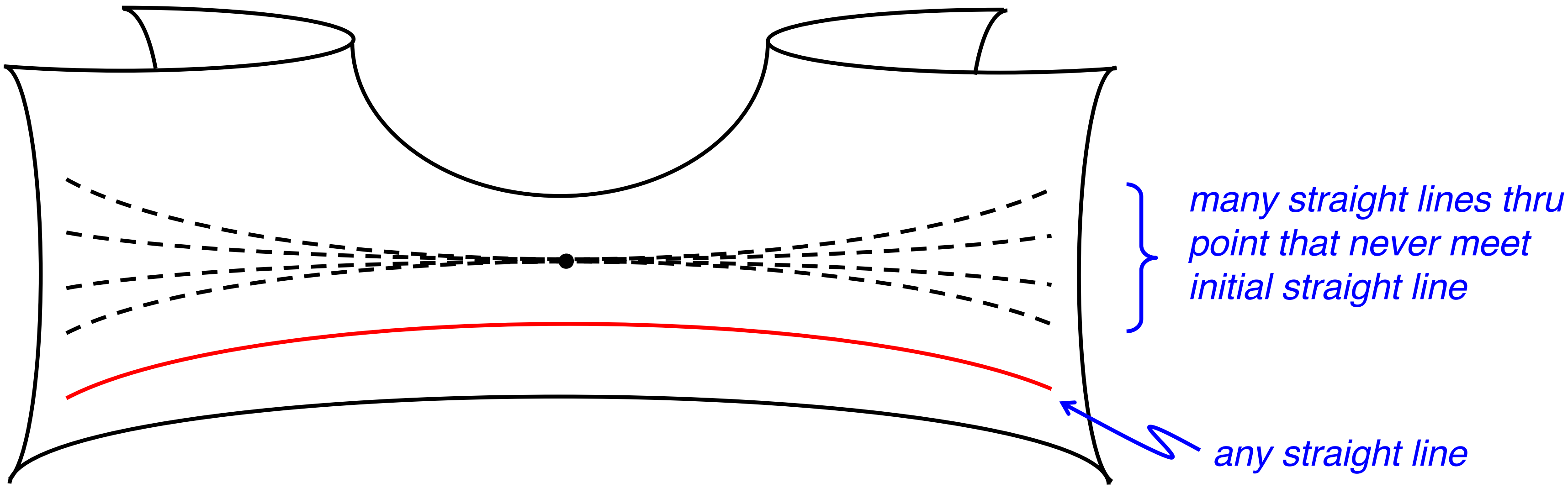
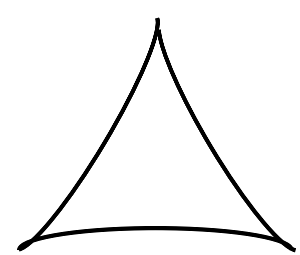


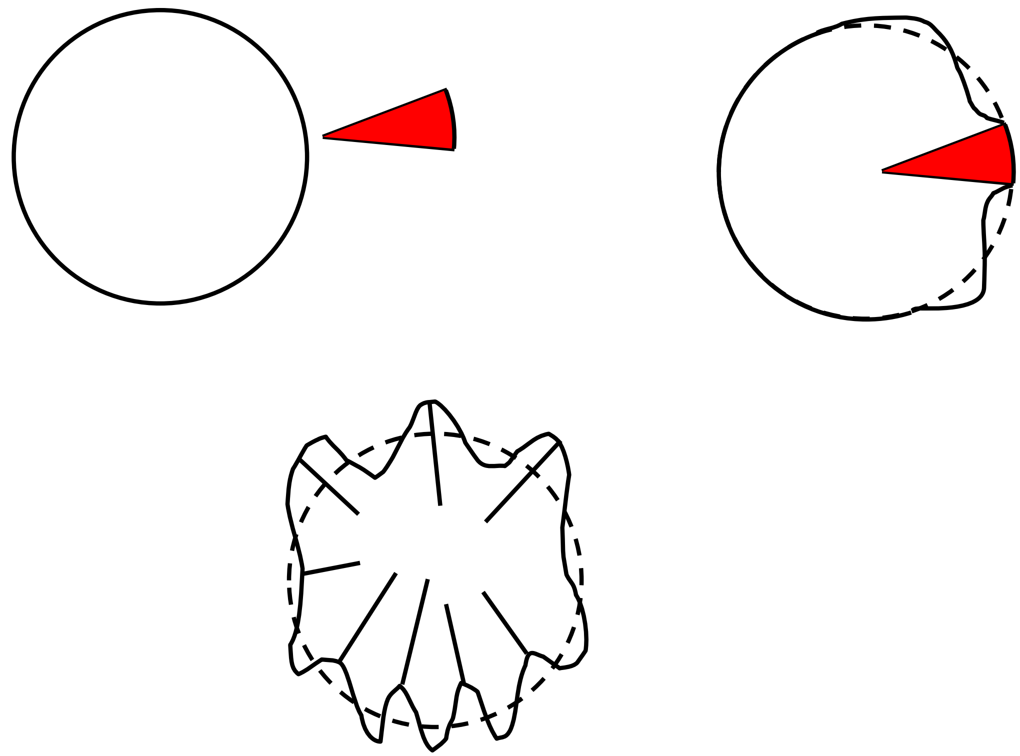
On the surface of a saddle:

- There are indefinitely many lines through a given point that are parallel to any given straight line.
- The sum of angles of a triangle < 2 right angles.



- The *circumference* of a circle $> 2\pi \times$ *radius*.

Loosen up a Euclidean circle --
add wedges to it.

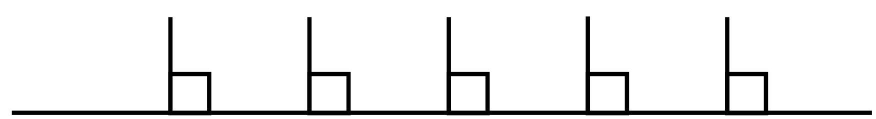


Euclidean circle (dotted line) with circumference $= 2\pi R$.

Wavey hyperbolic circle with circumference $> 2\pi R$.

Question: How could we determine what type of space we live in?

Answer: Use "geodesic deviation" to detect intrinsic curvature.



- Draw base line.
- Erect perpendiculars.
- Determine whether geodesics deviate in other regions of space.

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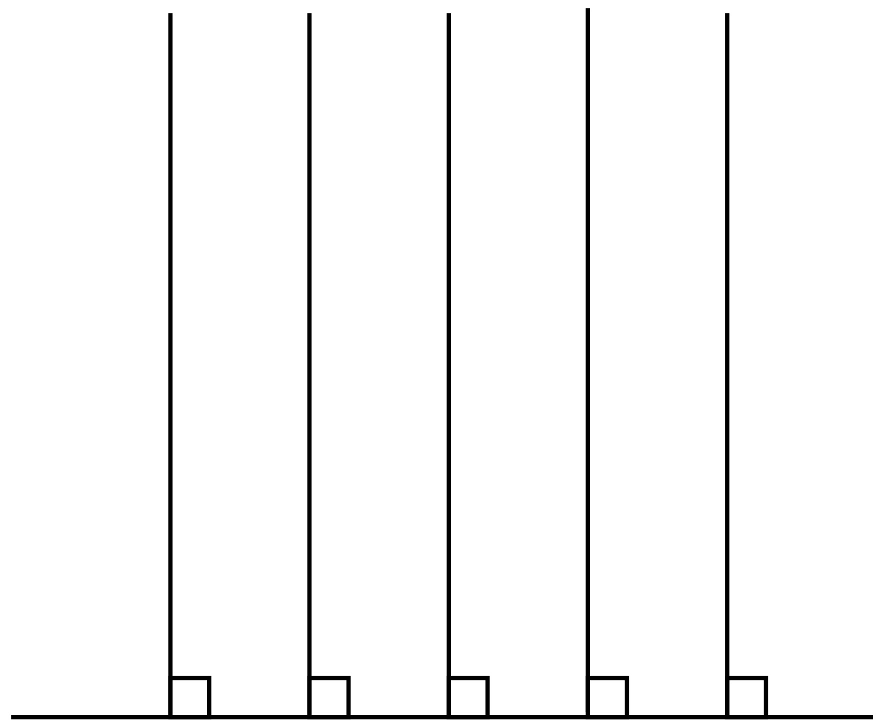

zero curvature

- Draw base line.
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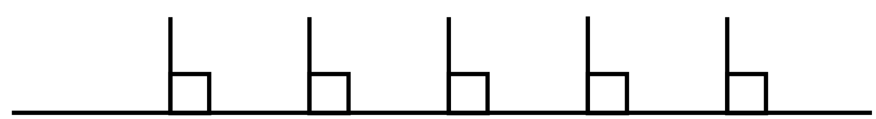

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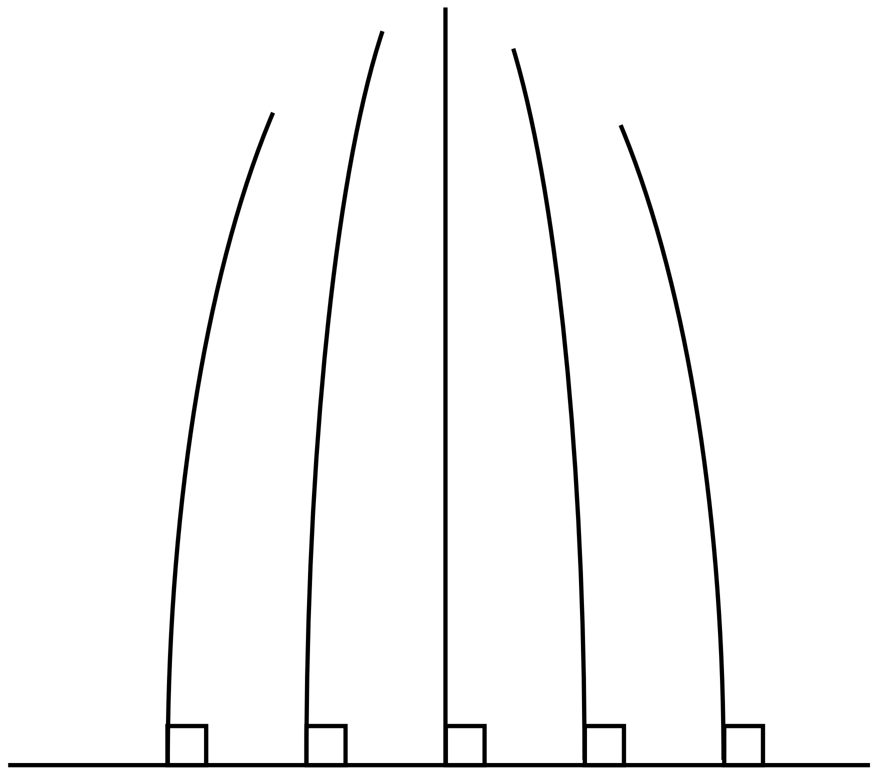

positive curvature

- Draw base line.
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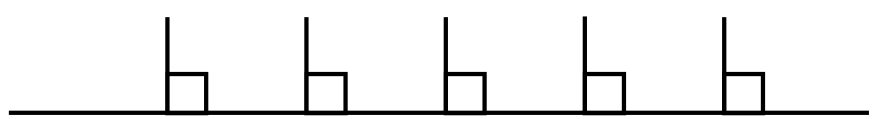

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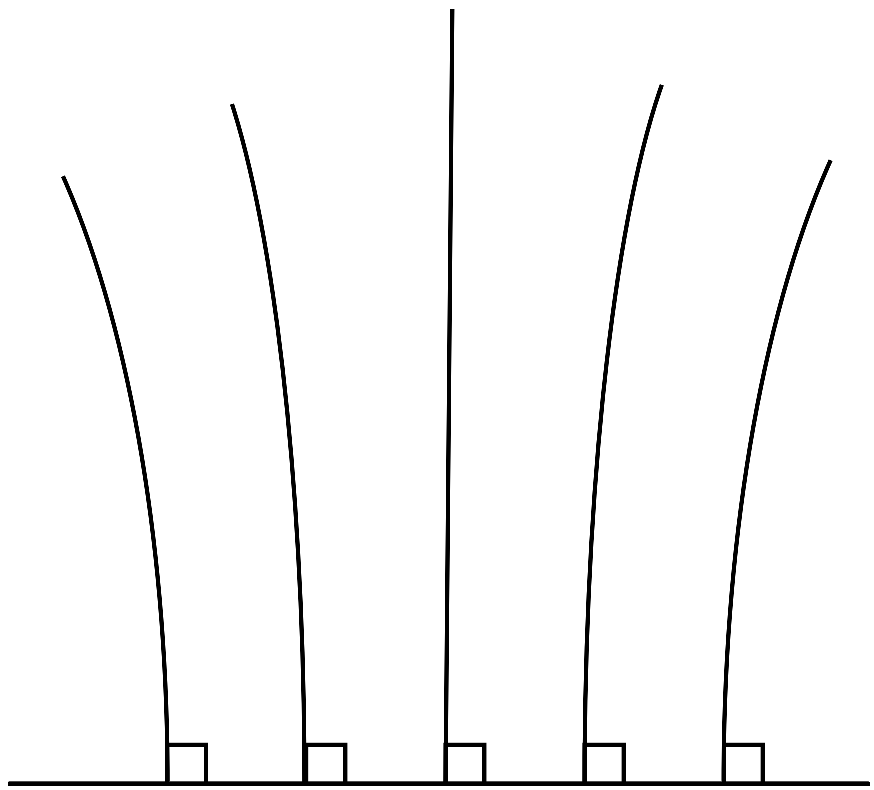

negative curvature

- Draw base line.
- Erect perpendiculars.
- Determine whether geodesics deviate in other regions of space.

1868: Eugenio Beltrami demonstrates that hyperbolic geometry is logically consistent.

1871: Felix Klein demonstrates that elliptical geometry is logically consistent.

Eugenio Beltrami
(1835-1900)

Felix Klein
(1849-1925)

- <u>*So*</u>: Euclidean geometry is *not* a necessary precondition for a *consistent description* of the spatial aspect of the physical world.
- <u>*Kant*</u>: Euclidean geometry is a necessary precondition for *experiencing* the world.
- Can this still be maintained?
  - *Perhaps humans are predisposed to perceive the world in Euclidean terms, even though the world might not be Euclidean.*
- <u>*But*</u>: Part of Kant's claim is stronger than this.

Statements in Euclidean geometry are *synthetic a priori*: they are necessary and universal truths about the world.

- *This* claim can no longer be the case:
  - *Is Euclid's 5th Postulate a necessary and universal truth?*
  - *Not if hyperbolic or elliptic geometry is true of the world.*

- What geometry is true of the world is now a matter of empirical inquiry, and no longer a matter of pure reason alone.
- One can thus distinguish between:
  (a) *pure geometery*: statements are analytic *a priori*.
  (b) *applied geometry*: statements are synthetic *a posteriori*.
- This distinction leaves no room for synthetic *a priori* statements.