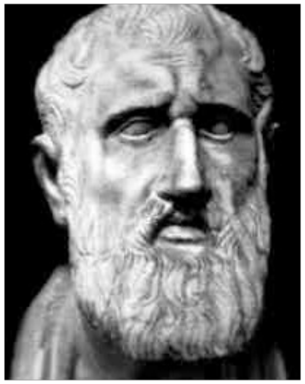
# 04. Zeno (5th century B.C.)

1. The Dichotomy
2. The Paradox of Plurality
3. The Arrow Paradox
4. Fragment 12: Chariots



- *Recall*: Euclid's theory of space:
  - Is it *consistent*?
  - Is it *true of the actual world*?

- *General Form of Zeno's Critique (reductio ad absurdum)*:
  - The following argument is valid:

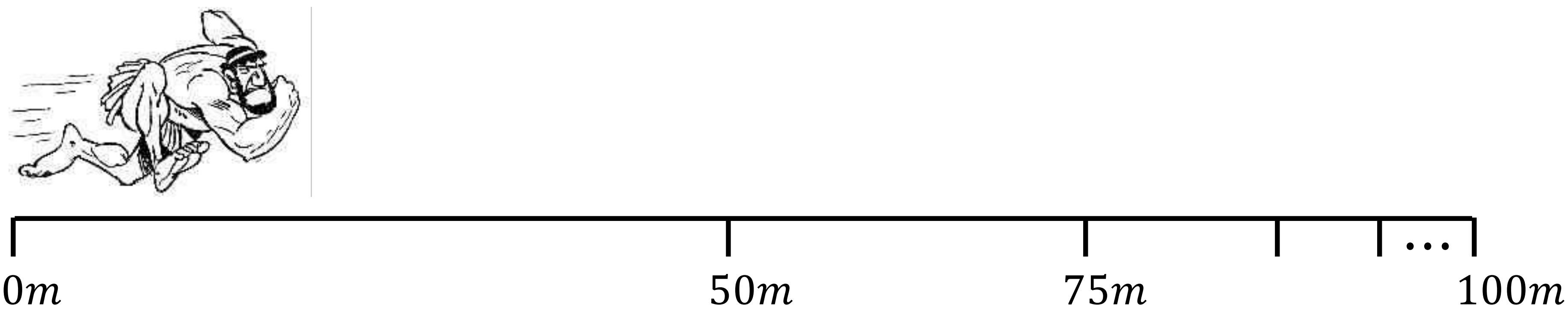
    Euclid's theory is true of the actual world.
    ----------------------
    ∴ Motion is impossible.

  - The conclusion is false!
  - So Euclid's theory must be wrong.

- *But*: More nuanced form questions *consistency* of Euclid's theory.
- Four specific arguments (Zeno's "Paradoxes").

## 1a. The Dichotomy Argument (Progressive Version)

- Consider a runner on a track




*Claim*: Achilles will never reach the end of the track in a finite time.

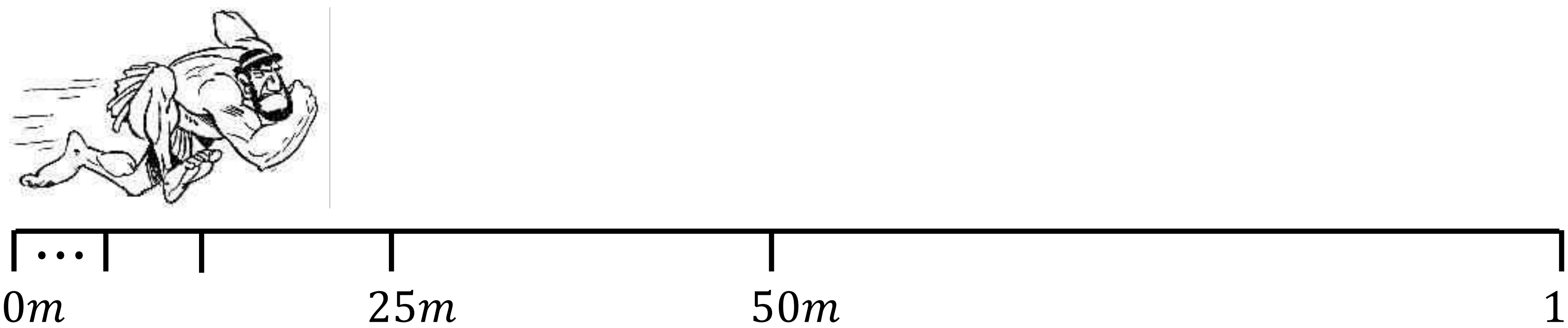
*Proof*:

1. To run full track, Achilles must run the initial half...
2. and then run the first half of the remainder...
3. and then run the first half of the remainder; *etc*.
4. Since any finite line segment can be divided in half, there will *always* be a remainder left to run!

- *In other words*: The track consists of an infinite number of finite segments.
- *And*: It is impossible to travel an infinite number of finite lengths in a finite time.

## 1b. The Dichotomy Argument (Regressive Version)

*Claim*: Achilles can never even get *started*.

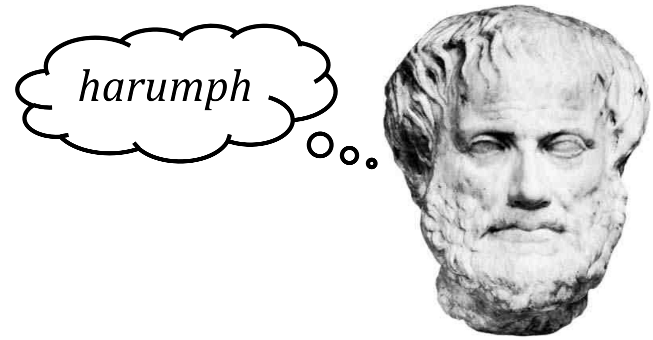



*Proof*:

1. To run first half, Achilles must run first *quarter*.
2. To run first quarter, Achilles must run first *eighth*.
3. To run first eighth, Achilles must run first *sixteenth*; *etc*.
4. Since any finite line segment can be divided in half, there will *always* be a length to run *before* running can begin!

- *Again*: The track consists of an infinite number of finite segments.
- *And*: It is impossible to travel an infinite number of finite lengths in a finite time.

## *Aristotle's Initial Response*

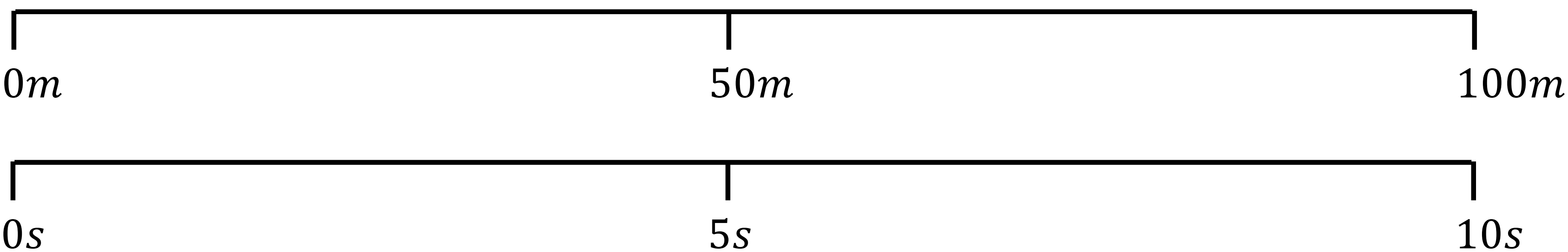



- Distinguish between:

  (a) Having an infinite number of parts.

  (b) Being infinitely large.

- *Claim*: One cannot traverse an infinite distance (b) in a finite time, but one *can* traverse a finite distance made up of an infinite number of parts (a).
- Construct 1-1 map between time of travel and distance of travel.




- *But*: Shouldn't the sum of an infinite number of finite time intervals be an infinite amount of time?
- *In general*: Shouldn't the sum of an infinite number of finite quantities be an infinite magnitude?

<u>*Root of paradox*</u>: A challenge to the Euclidean notion of finite line segment.

1. All segments can be divided into two segments. (*Euclidean assump.*)

-----------------------------------------------------------------

∴ C1′. All segments are composed of an infinite number of segments.

C1′.

2. All segments have finite length. (*Euclidean assump.*)

3. The length of any segment = the sum of its components. (*Euclidean assump.*)

-----------------------------------------------------------------

∴ C2. The length of any segment = an infinite sum of finite lengths.

C2.

4. All infinite sums of finite lengths are infinite.

-----------------------------------------------------------------

∴ All segments are infinitely long.

- <u>*Zeno's intent (Huggett)*</u>: Euclid's theory is inconsistent.
  - *If we assume it is true (premises 1-3), we can derive a contradiction.*
- <u>*But*</u>: What about premise 4?

*What is an infinite sum of finite quantities?*

- Addition is a 2-place function: takes a *pair* of numbers and outputs a single number:

| *input* | *output* |
|---|---|
| $a, b$ | $a + b$ |
| $(a + b), c$ | $(a + b) + c$ |
| $(a + b) + c), d$ | $((a + b) + c)) + d$ |
| *etc*... | |

- *So*: An *infinite* sum makes no initial sense:

$$s_1 + s_2 + s_3 + \cdots = ?$$

*Ex*
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = ?$$

- *Solution*: First form an *infinite* sequence of *finite* "partial" sums:

$$\{s_1, s_1 + s_2, (s_1 + s_2) + s_3, ((s_1 + s_2) + s_3) + s_4), \ldots\}$$

*Ex*
$$\left\{\frac{1}{2}, \frac{1}{2} + \frac{1}{4}, \left(\frac{1}{2} + \frac{1}{4}\right) + \frac{1}{8}, \left(\left(\frac{1}{2} + \frac{1}{4}\right) + \frac{1}{8}\right) + \frac{1}{16}, \ldots\right\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\right\}$$

*Next*: Define the *limit* of an infinite sequence:

*Augustin-Louis Cauchy* (1789-1857)

> **Def. (Limit of an infinite sequence)**. An infinite sequence of (increasing, positive) numbers $\{n_1, n_2, n_3, ...\}$ has a limit $L$ *if and only if* for every $\varepsilon > 0$, there exists a $\delta$ such that,
>
> $$\text{if } c \geq \delta, \quad \text{then} \quad |L - n_c| < \varepsilon$$

- *Which says*: $\{n_1, n_2, n_3, ...\}$ *has a limit L if it has a member* $n_\delta$ *afterwhich all members stay within* $\varepsilon$ *of L.*
- *Terminology*: The sequence $\{n_1, n_2, n_3, ...\}$ *converges* to $L$, if $L$ exists.

- Can now define the sum of an infinite series of numbers:

> **Def. (Sum of an infinite series)**. The sum of an infinite series $s_1 + s_2 + s_3 + \cdots$ is the limit of the sequence of partial sums $\{s_1, s_1 + s_2, (s_1 + s_2) + s_3, ...\}$, if such a limit exists.

*Example*

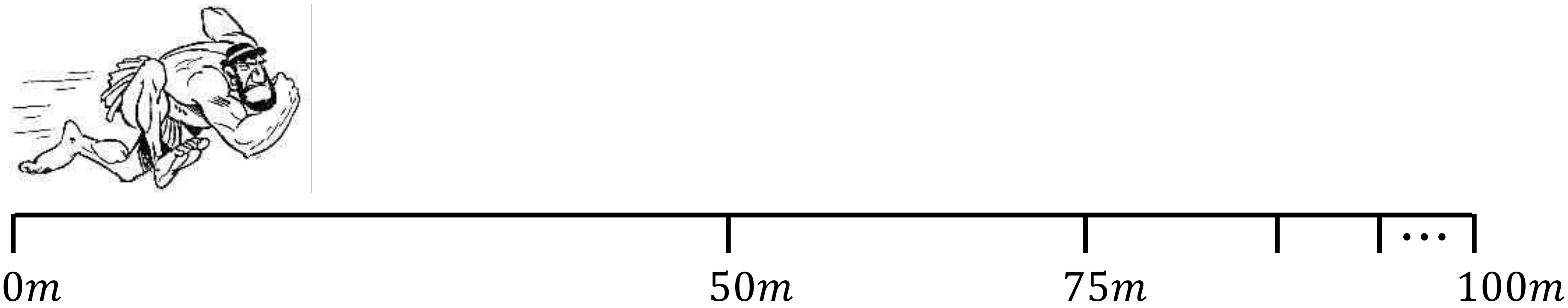
$$\left\{\frac{1}{2}, \frac{1}{2}+\frac{1}{4}, \left(\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{8}, \left(\left(\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{8}\right)+\frac{1}{16}, \ldots\right\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\right\}$$

- This infinite sequence has 1 as its limit (it *converges* to 1).
  - *Which means*: For any number $\varepsilon > 0$, no matter how small, there is a number $\delta$ such that all members of the sequence after (and including) the $\delta$th member are less than $\varepsilon$ away from 1.
  - *Which means*: For any number $\varepsilon > 0$, there is a number $\delta > 0$ such that if $c \geq \delta$, then $|1 - n_c| < \varepsilon$.

- *Thus*: According to Cauchy's definition of infinite sum,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 1$$

- *Thus*: *According to Cauchy's definition of infinite sum*, not all infinite sums of finite quantities are infinite!




- The racetrack, as a Euclidean line segment, is the sum of an infinite number of finite line segments...

$$\frac{1}{2}(100m) + \frac{1}{4}(100m) + \frac{1}{8}(100m) + \cdots = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)(100m)$$

$$= 100m$$

- ...which, according to Cauchy's definition, equals a *finite* quantity!

## 2. The Paradox of Plurality

1. All finite line segments are composed of an infinity of identical parts.

2. Points have either finite length or zero length.

-----------------------------------------------------------------

∴ The total length of a line segment is either infinite or zero.

- *Suppose*: We accept the validity of this argument.
  - *Then*: To deny the conclusion, we must deny one or more premises.
  - *But*: Both premises seem plausible.
- *So*: Can we deny validity?
  - *Suppose*: We accept Premise 1.
  - *Then*: Do the following implications hold?

(a) (*Points have finite length.*) ⇒ (*The total length of a line segment is infinite.*)

(b) (*Points have zero length.*) ⇒ (*The total length of a line segment is zero.*)

- On the surface, it *appears* as if Cauchy's definition of an infinite sum justifies both (a) and (b)!

*Suppose*: Points have finite length $\ell$.

- Consider the infinite sequence:

$$\{\ell, \ell + \ell, (\ell + \ell) + \ell, ((\ell + \ell) + \ell) + \ell, \ldots\} = \{\ell, 2\ell, 3\ell, 4\ell, \ldots\}$$

- This infinite sequence has no limit.
- *So*: According to Cauchy's definition, the following infinite sum cannot be calculated (in fact, it's infinite):

$$\ell + \ell + \ell + \ell + \cdots$$

- Which seems to justify (a)!

*Suppose*: Points have zero length.

- Consider the infinite sequence:

$$\{0, 0 + 0, (0 + 0) + 0, ((0 + 0) + 0) + 0, \ldots\} = \{0, 0, 0, 0, \ldots\}$$

- This infinite sequence has 0 as a limit.
- *So*: According to Cauchy's definition:

$$0 + 0 + 0 + 0 + \cdots = 0$$

- Which seems to justify (b)!

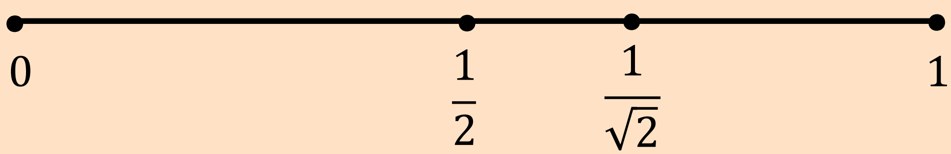
- *But*: Can Cauchy's definition of an infinite sum be applied to sums of terms corresponding to points in a line segment?


*Georg Cantor*
(1845-1918)

*Claim 1*: The points in a line segment can be mapped 1-1 to the real numbers between 0 and 1.

*Proof sketch*:

- By definition, the set of points in a line segment is *dense* (between any two there is another) and has *no gaps*.
- The set of real numbers between 0 and 1 is also dense and has no gaps.
- *So*: Can assign the endpoints of the line segment the real numbers 0 and 1; and assign each point inbetween a real number between 0 and 1:



*Claim 2*: There are more real numbers between 0 and 1 than there are natural numbers.

*Proof*:

1. Pair natural numbers with decimal expansions of real numbers between 0 and 1:

| | | | |
|---|---|---|---|
| 0 | $0\,.\,3\;3\;3\;3\cdots$ | $= 1/3$ | *all reals between* 0 *and* 1 *can be given an infinite decimal expansion* |
| 1 | $0\,.\,1\;4\;1\;5\cdots$ | $= \pi - 3$ | |
| 2 | $0\,.\,4\;1\;4\;2\cdots$ | $= \sqrt{2} - 1$ | |
| 3 | $0\,.\,5\;0\;0\;0\cdots$ | $= 1/2$ | |
| $\vdots$ | $\vdots$ | | |

2. Construct a real number between 0 and 1 that is not in the table:
   - (a) Start with first digit in decimal expansion of first real: Write 3 if it's 4, or 4 otherwise.
   - (b) Continue with second digit in expansion of second real, *etc*.

     Diagonal element: 0.3440...

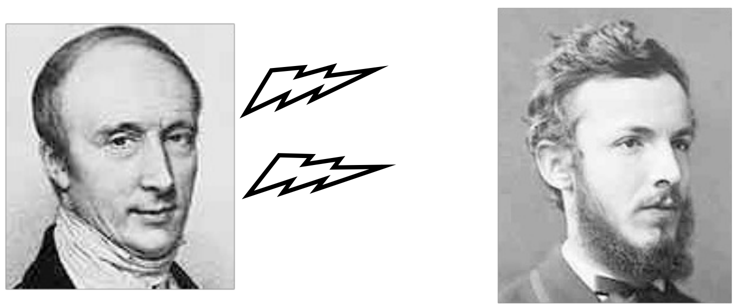
     New real number: 0.4334...

3. New real number is not in the table (it differs from the *i*th real number in the table in its *i*th digit after the decimal); but *all* natural numbers *are* in the table!

*So*: There are more points in a line segment than there are natural numbers!

- *Note*: The terms in the sum of an infinite series $s_1 + s_2 + s_3 + \cdots$ are in 1-1 correspondence with the natural numbers.
- *Thus*: There are more points in a line segment than there are terms in the sum of an infinite series!

- *Which means*: Cauchy's definition of an infinite sum *cannot* be applied to sums of terms corresponding to the points in a line segment.



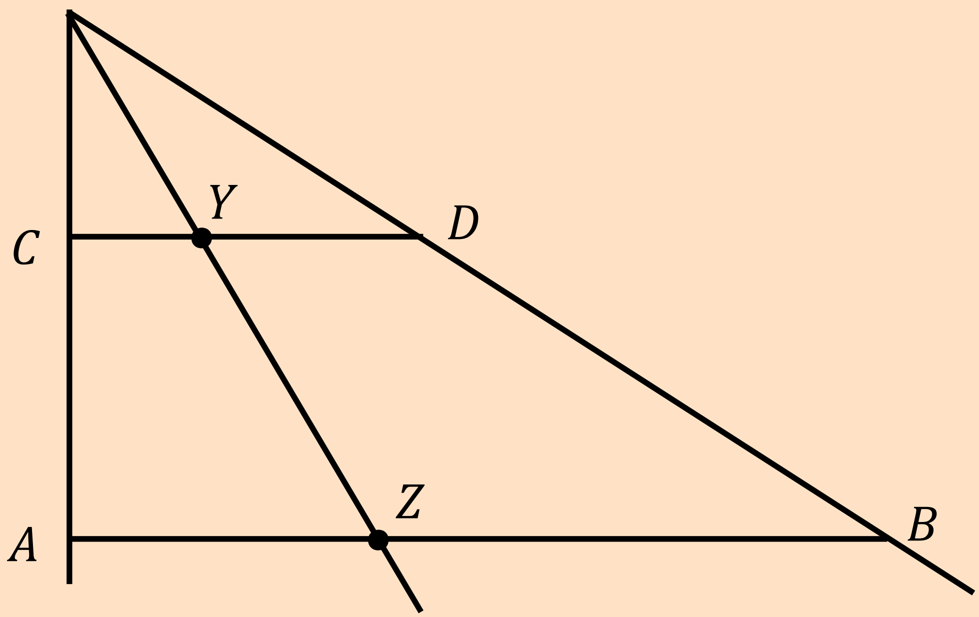
- The set $\mathbb{N}$ of natural numbers is said to be *countably infinite*.
- The set $\mathbb{R}$ of real numbers (and thus the set of points in a line segment) is said to be *uncountably infinite*.

*Can Cauchy's definition of an infinite sum be extended to treat uncountably infinite sums of lengths associated with the points in a line segment?*

*Claim 3*: All finite line segments have the same number of points.

*Proof sketch*:

1. Consider line segments *CD* and *AB*:




2. Any point *Y* on *CD* has one and only one point *Z* corresponding to it on *AB*, and *vis versa*.
3. Thus, the points on *CD* and *AB* are in 1-1 correspondence with each other.

- *Upshot*: *If* it's obvious that *CD* and *AB* have different lengths, then the length of a line segment cannot depend on the number of points in it.

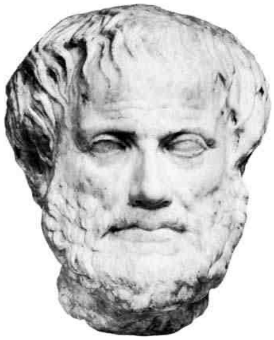
- *Modern view*:
  - The length of a line segment depends on the *metrical* properties of its points.
  - These are additional properties imposed on a set of bare points that do not depend the size of the set.

- *So*: It is *not* the case that

  (a) (*Points have finite length.*) ⇒ (*The total length of a line segment is infinite.*)

  (b) (*Points have zero length.*) ⇒ (*The total length of a line segment is zero.*)

## 3. The Arrow Paradox



"The third is... that the flying arrow is at rest, which result follows from the assumption that time is composed of moments... he says that if everything when it occupies an equal space is at rest, and if that which is in locomotion is always in a now, the flying arrow is therefore motionless."

1. Instants have no parts
2. If an arrow moves during an instant, then that instant has earlier and later parts.

---

∴ The arrow doesn't move during any instant.

- *Motivation for premise #1*: Represent an instant of time as a Euclidean point.
- *Motivation for premise #2*: If arrow moved during an instant, then it would be located at different positions at the beginning and the end of the instant.

C1. The arrow doesn't move during an instant.

3. If the arrow doesn't move during any instant, then it doesn't move at all.

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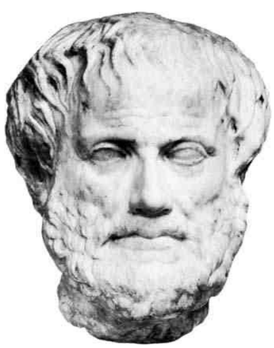
∴ The arrow doesn't move at all.

- *What about premise #3*: If the arrow doesn't move during an instant, can it be said to move at all?
- *Aristotle's Response*: Reject concept of an instant of time modeled by a Euclidean point.



"...time is not composed of indivisible nows any more than any other magnitude is composed of indivisibles".

*Huggett's Response*:

*"At-At" theory of motion*:

"To move" means "to be at a continuous series of locations over a continuous interval of instants".

- Distance traveled is a continuous function of time; specifies a location $x(t)$ for every instant $t$.

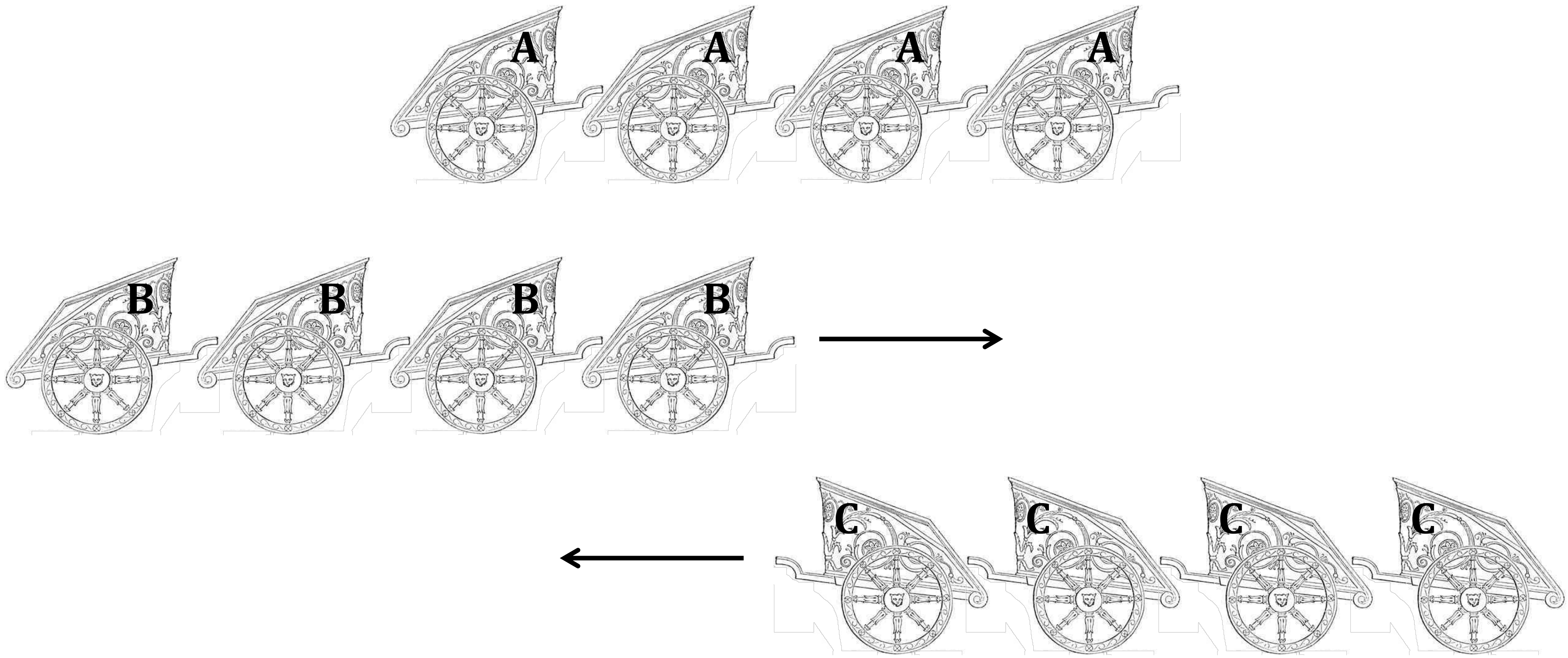
- *At-At theory*: No essence to motion-at-an-instant.
  - From a snap-shot of an arrow at an instant $t$, there is no essential way to determine if the arrow is in motion or at rest.
  - To determine "at-at" motion, we need to refer to a *range* of instants $\Delta t$ after (or before) $t$.

*Note*: An instantaneous velocity $v(t)$ can be assigned to each instant $t$ *via*:

$$v(t) \equiv \frac{dx(t)}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

- *So*: An arrow can be said to be moving at the instant $t_0$, just when $v(t_0) \neq 0$.
- *Which means*: The arrow is at an appropriate series of points at the series of subsequent times $t + \Delta t$.

## 4. Fragment 12: Chariots

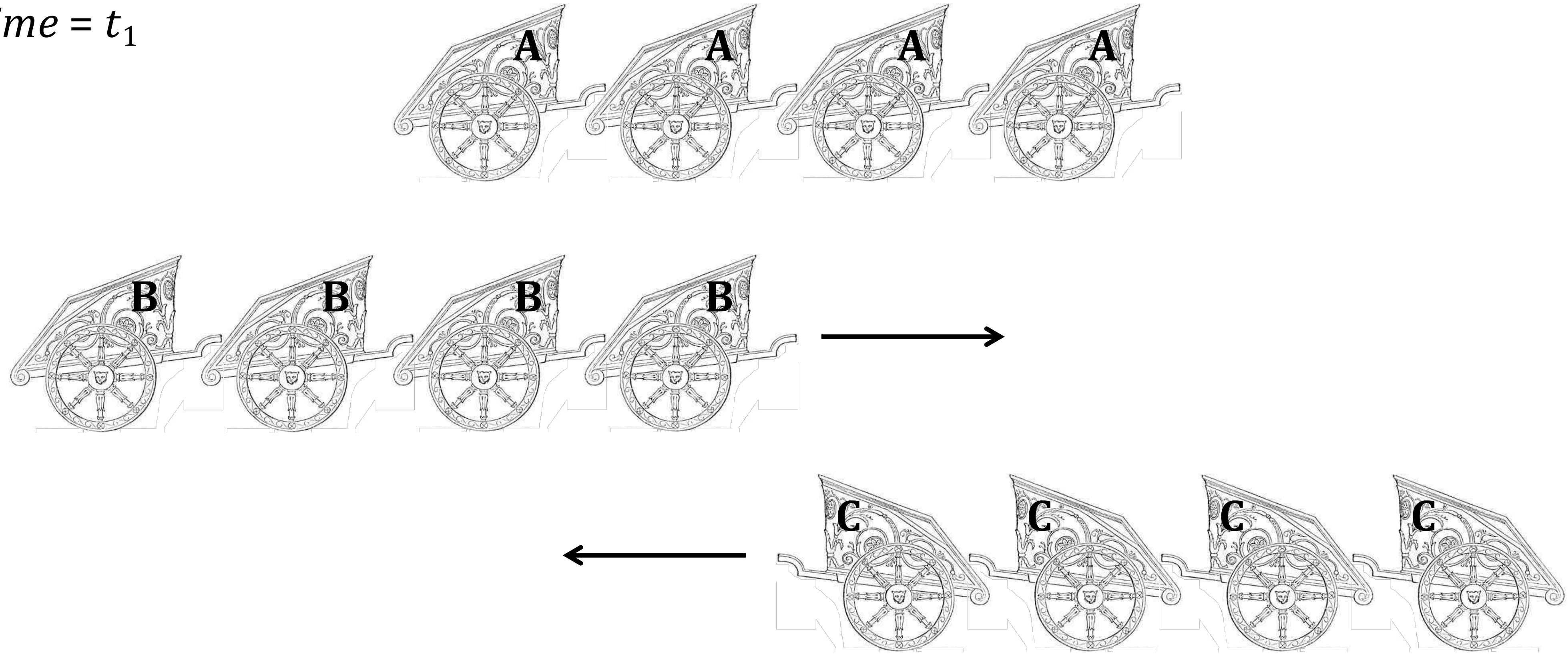



*Assumptions:*

1. Each chariot occupies 1 unit of space.
2. **B**-chariots and **C**-chariots are moving at speed $= \dfrac{\text{1 unit of space}}{\text{1 unit of time}}$
3. Space and time units have no parts.

## 4. Fragment 12: Chariots

*time* = $t_1$

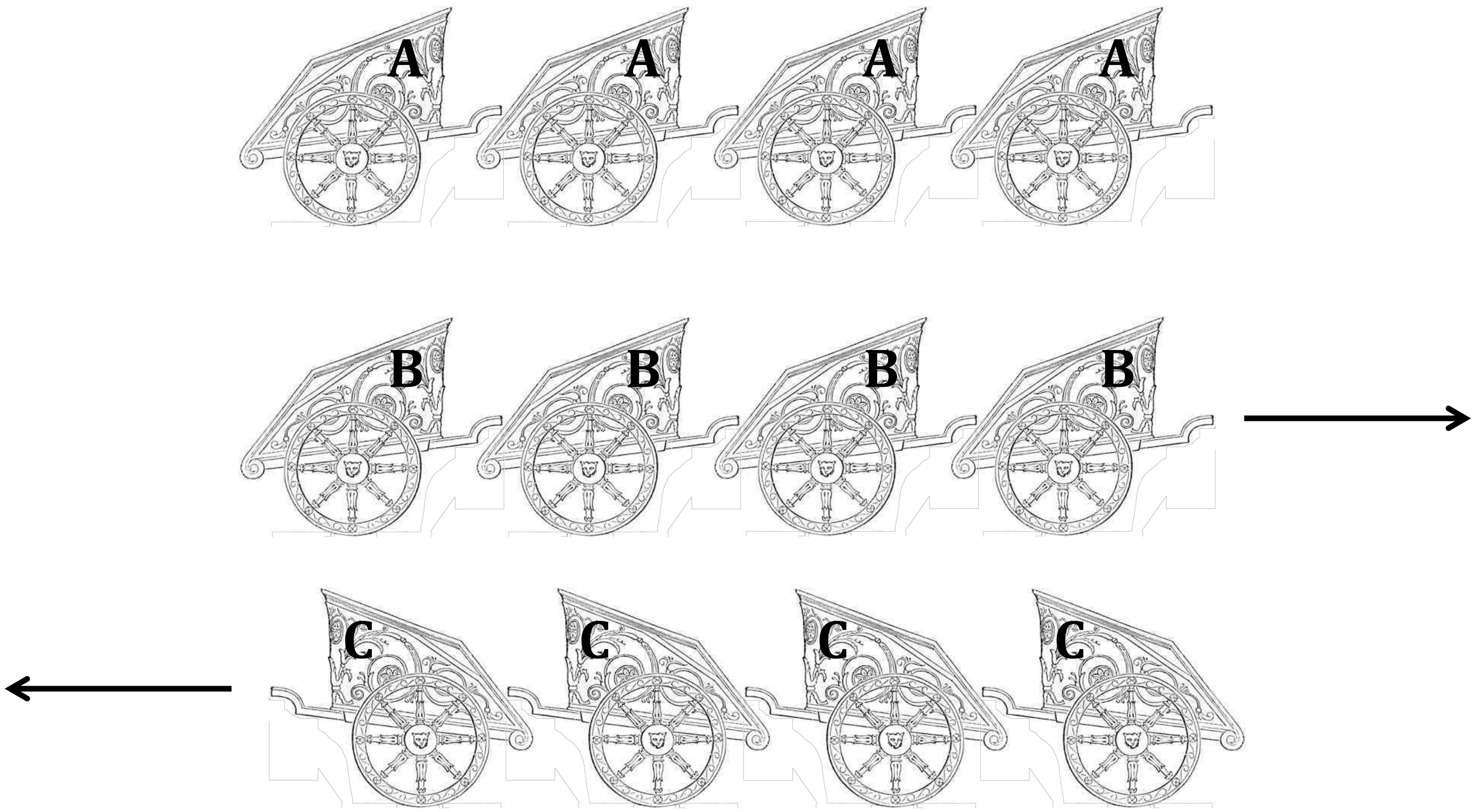



<u>*Now Suppose:*</u>

- At time $t_1$, first **B**-chariot is alligned with second **A**-chariot; and first **C**-chariot is alligned with third **A**-chariot.

# 4. Fragment 12: Chariots

*time* = $t_2$

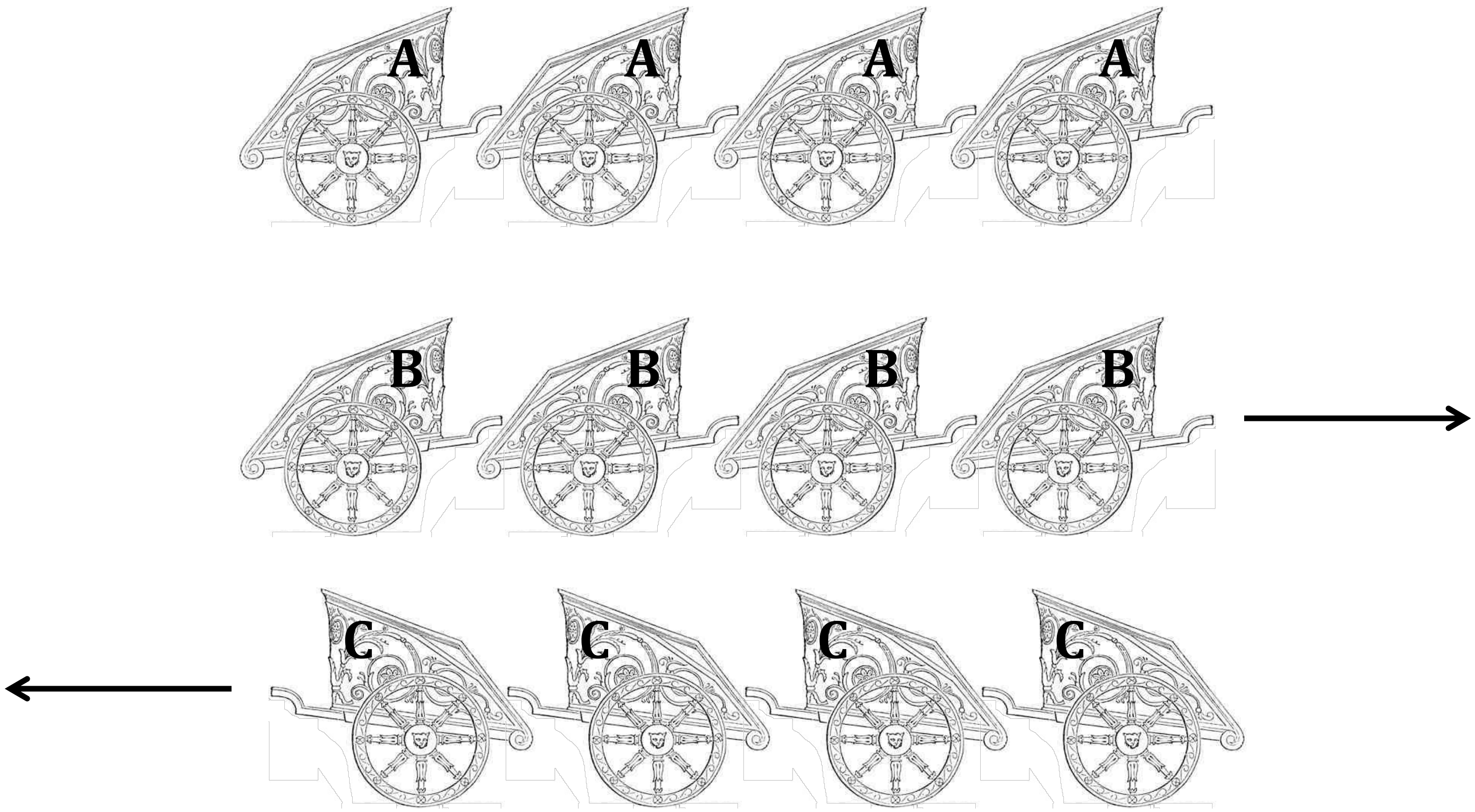



<u>*Now Suppose:*</u>

- At time $t_1$, first **B**-chariot is alligned with second **A**-chariot; and first **C**-chariot is alligned with third **A**-chariot.
- At time $t_2$, **B**-chariots and **C**-chariots are all alligned with **A**-chariots.
- *How many time units are between $t_1$ and $t_2$?*

# 4. Fragment 12: Chariots

*time* = $t_2$




(a) Two time units:

- First **B**-chariot moves by two **A**-chariots.
- First **C**-chariot moves by two **A**-chariots.

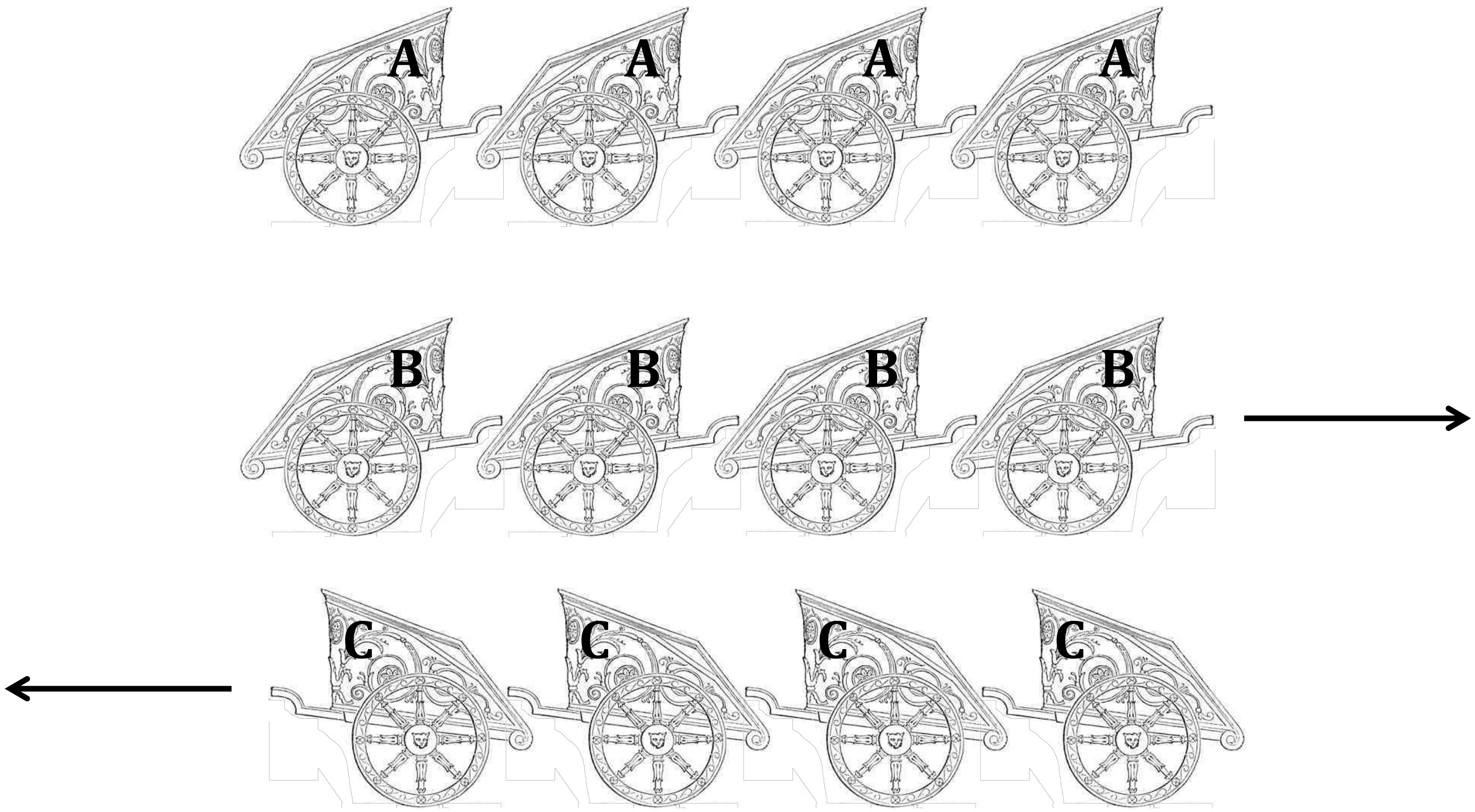
(b) Four time units:

- First **B**-chariot moves by four **C**-chariots.
- First **C**-chariot moves by four **B**-chariots.

*Contradiction!*

# 4. Fragment 12: Chariots

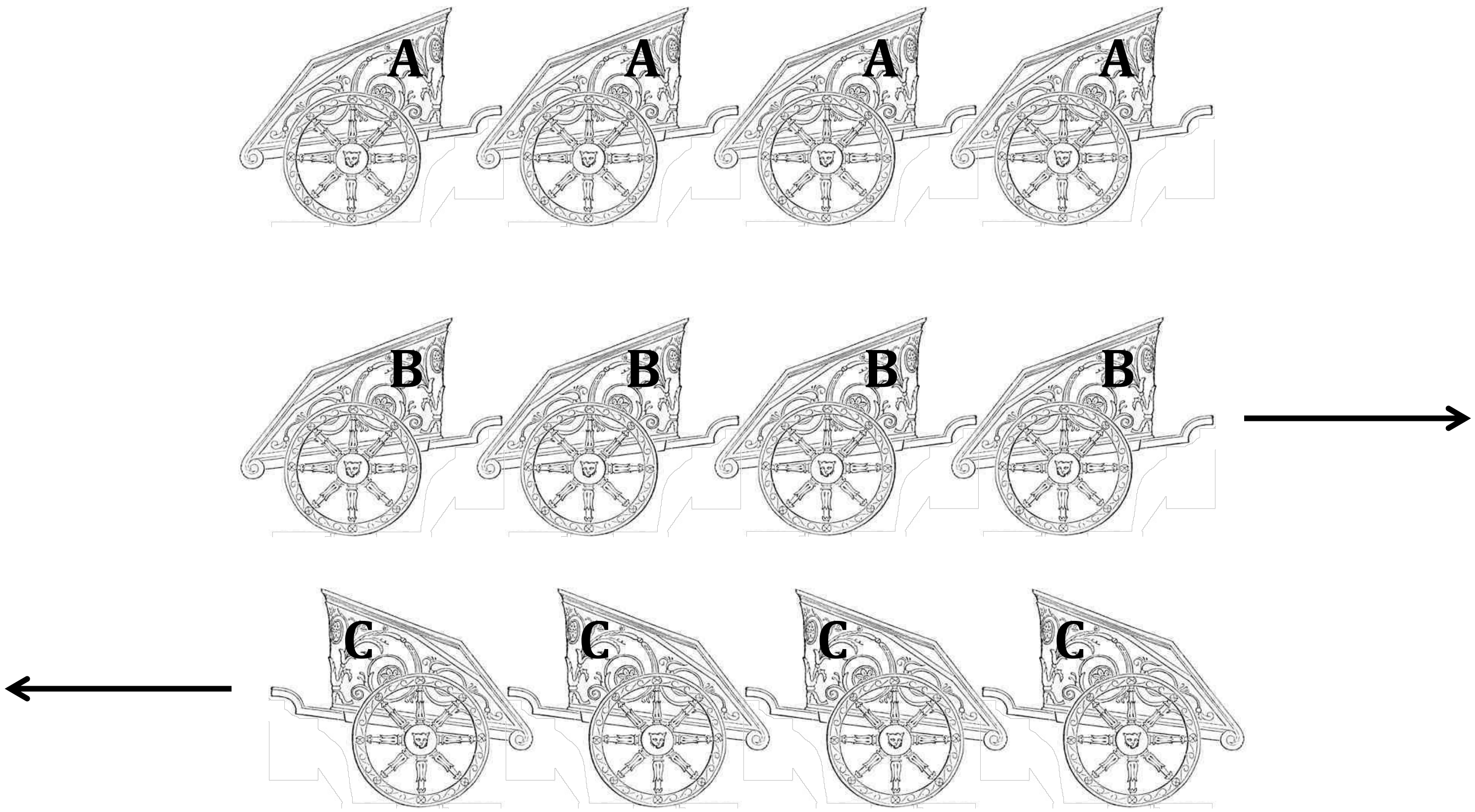
*time* = $t_2$




- *<u>So</u>*: If we accept Assumptions 1 and 3, we must reject Assumption 2: The **B-** and **C-**chariots cannot be said to be in motion.

## 4. Fragment 12: Chariots

*time* = $t_2$




- *But*: Assumption 2 claims the **B**- and **C**-chariots are moving at 1 s.u./t.u.
- *And*: This is true only with respect to the **A**-chariots.
  - The **B**-chariots are moving at 2 s.u./t.u. with respect to the **C**-chariots.
  - The **C**-chariots are moving at 2 s.u./t.u. with respect to the **B**-chariots.
- *Thus*: Conclusion (b) is false:
  - The **B**-chariots require only 2 time units to move past 4 **C**-chariots, and *vice-versa*.