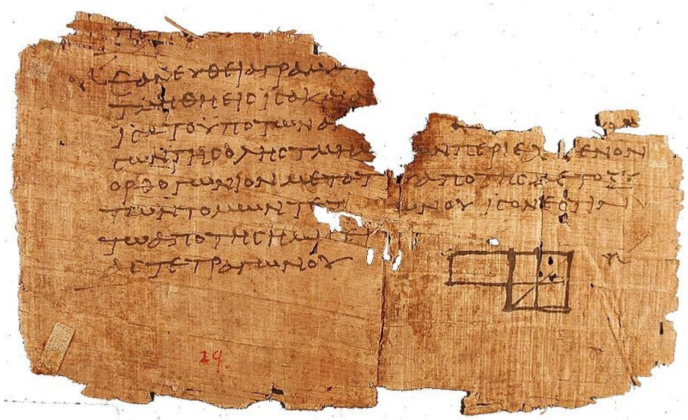
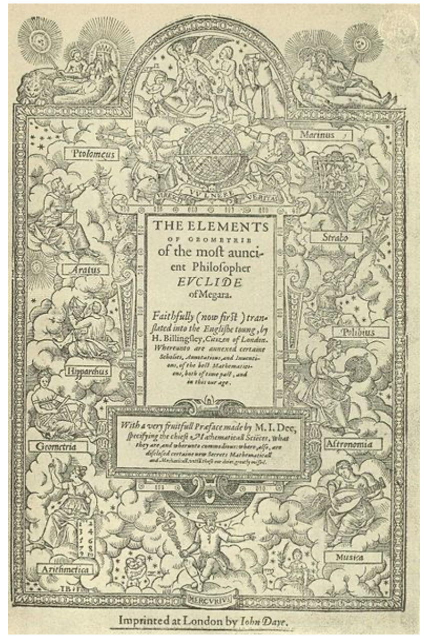
03. Euclid's *Elements* (~300 B.C.)

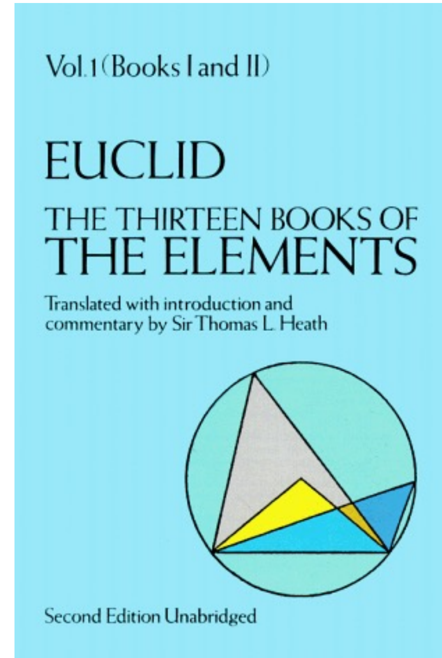
1. 5 Postulates
2. Basic Concepts
3. Euclidean Geometry as a Theory of Space



~100 A.D.
Earliest existing copy



1570 A.D.
First English translation



1956
Dover Edition

- *<u>Contents</u>*:

  I. Definitions
  II. Postulates
  III. Common Notions

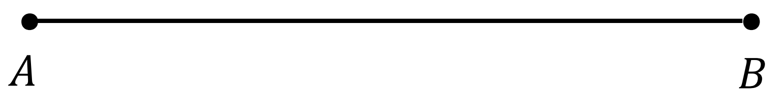
  } *basic assumptions*

  IV. Propositions ⇒ *more complex claims*

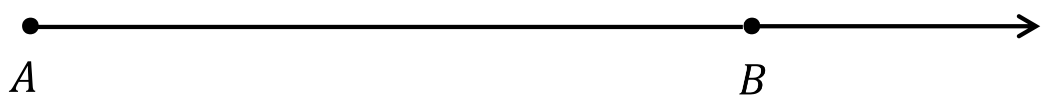
- *<u>Euclid's Accomplishment</u>*: Showed that all geometric claims then known follow from 5 postulates.

1. Euclid's 5 Postulates

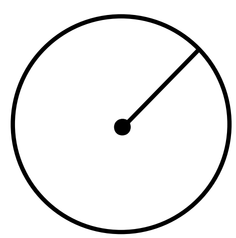
(i) To draw a straight line from any point to any point.



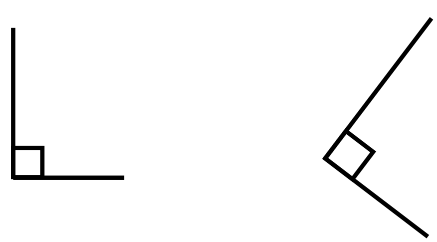
(ii) To produce a finite straight line continuously in a straight line.



(iii) To describe a circle with any center and distance.

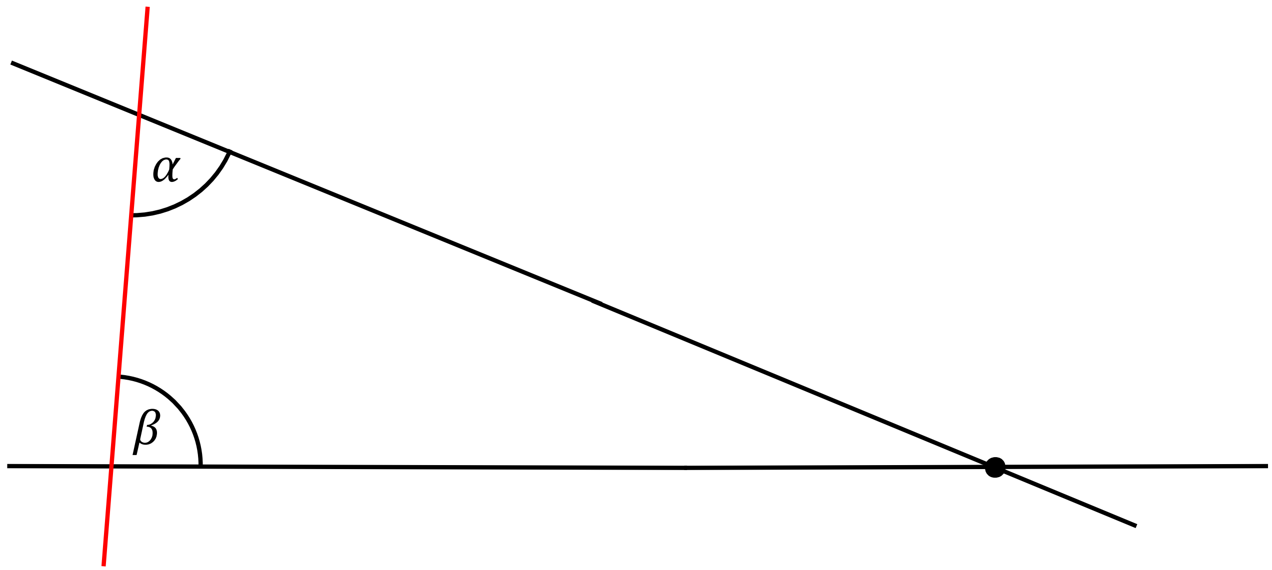


(iv) That all *right angles* are equal to one another.



Def. 10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right...

(v) That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.



$\alpha + \beta < 180^{o}$

2. Basic concepts

(i) "point" = "that which has no part"

- *Modern gloss*: no magnitude, no dimension.

(ii) "line" = "breadthless length"

- *Modern gloss*: *only* length; *i.e.*, 1-dimensional.

Two modern characteristics of a line.

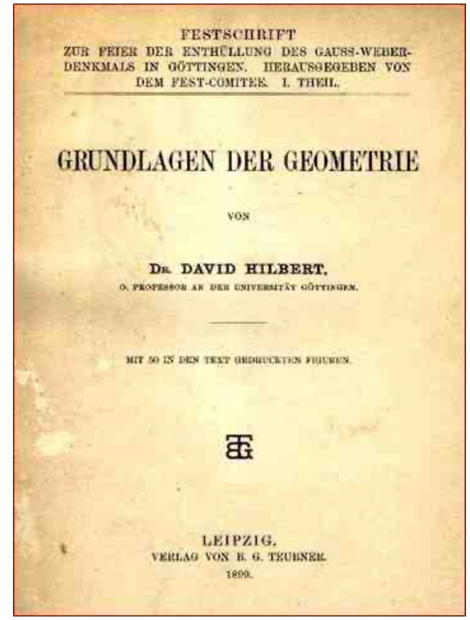
(a) A line is a *dense collection of points* with *no gaps*.

- *Dense collection of points* = between any two points is another.
- Denseness does *not* imply "no gaps":
  - The collection of rational numbers (ratios of natural numbers) is dense but contains gaps (the irrational numbers).
  - The collection of real numbers (rationals and irrationals) is dense and contains *no* gaps.
- A *dense* collection of points with *no gaps* defines a *continuum*.

(b) A line is a *straight* curve.

- A straight curve is a curve such that the distance between any of its points is the shortest.

3. Euclidean Geometry as a Theory of Space



- <u>*Recall*</u>: Two questions to ask of a theory *T*:

(i) *Is T consistent?* (If yes, then there are possible worlds in which *T* is true.)

(ii) *Does T accurately describe the actual world?*

- Euclidean geometry *is* consistent (Hilbert 1899).
- <u>*But*</u>: Does it acurately describe the actual world?
  - Consider its propositions as *predictions* about the properties of space
  - Because Euclidean geometry is consistent, each proposition is the conclusion of a *valid-deductive* argument, ultimately of the form:

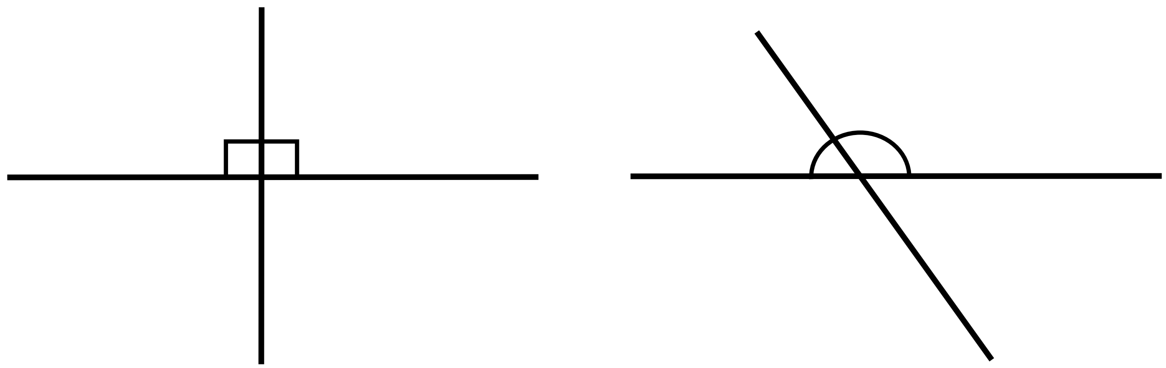
    1. Postulates 1-5.

    $\therefore$ Proposition x.

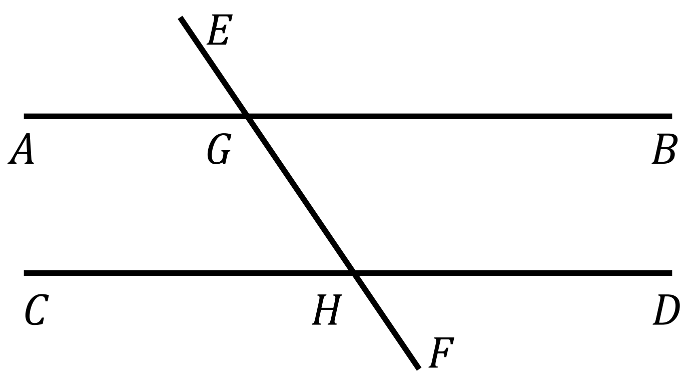
- <u>*So*</u>: If Proposition x is false in the actual world, then one or more of the premises must be false in the actual world.

Examples of propositions

Proposition 13. If a straight line stands on a straight line, then it makes either two right angles, or angles whose sum equals two right angles.



Proposition 29. A straight line falling on parallel straight lines makes the alternate angles equal, the exterior angle equal to the interior and opposite angle, the interior angles on the same side equal to two right angles.



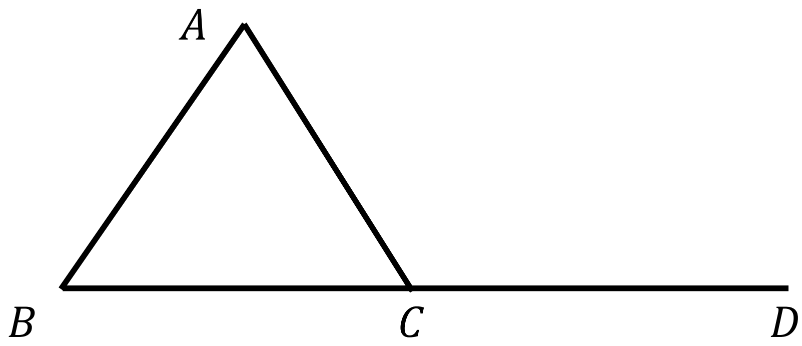
Claims:

(a) $\measuredangle AGH = \measuredangle GHD$

(b) $\measuredangle EGB = \measuredangle GHD$

(c) $\measuredangle BGH + \measuredangle GHD = \textit{two right angles}$

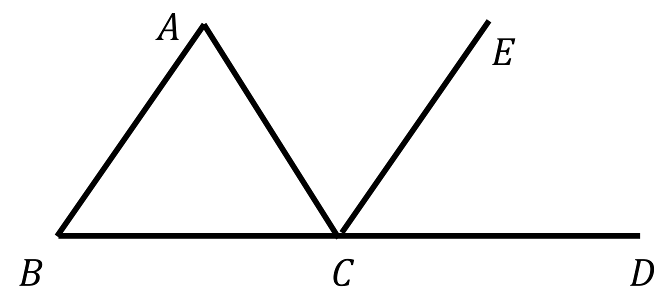
Proposition 32. In any triangle, if one of the sides is extended, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.



Claims:

(a) $\measuredangle ACD = \measuredangle CAB + \measuredangle ABC$

(b) $\measuredangle CAB + \measuredangle ABC + \measuredangle BCA = \textit{two right angles}$



Proof:

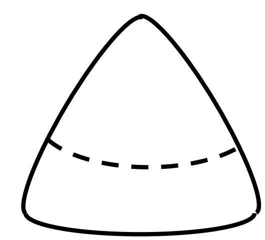
1. Draw CE parallel to BA. (Prop. 31)
2. $\measuredangle CAB = \measuredangle ACE$ (Prop. 29: *alternate angles are equal*.)
3. $\measuredangle ECD = \measuredangle ABC$ (Prop. 29: *exterior angle = interior opposite angle*)
4. $\measuredangle ACD = \measuredangle ACE + \measuredangle ECD$
   $= \measuredangle CAB + \measuredangle ABC$ (Common Notion 2: *If equals are added to equals then the resulting wholes are equal.*)
5. $\measuredangle ACD + \measuredangle BCA = \measuredangle CAB + \measuredangle ABC + \measuredangle BCA$ (Common Notion 2)
6. $\measuredangle ACD + \measuredangle BCA = \textit{two right angles}$ (Prop. 13)
7. $\measuredangle CAB + \measuredangle ABC + \measuredangle BCA = \textit{two right angles}$ (Common Notion 1: *Things equal to the same thing are equal to each other*.)

Now: Is Prop. 32 true in the actual world?

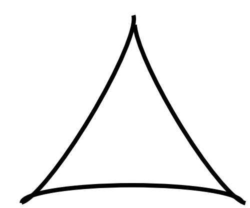
- *Suppose*: We construct a massive triangle between three mountain peaks in Bavaria, and measure its angles using reflected light rays.
- Consider the following *valid-deductive* argument:

> 1. Light rays travel along straight lines.
> 2. Euclidean geometry (*i.e.*, Postulates 1-5) is true in the actual world.
>
> --
>
> ∴ The sum of the angles of the Bavarian triangle = two right angles.

- *Suppose*: The conclusion is false.
- *Then*: One or more premises must be false... but which ones?
- Can uphold premise 2 by denying premise 1:
    - If the sum of the angles $> 180°$, then perhaps light rays "bulge outwards" between mountain peaks.



    - If the sum of the angles $< 180°$, then perhaps light rays "bulge inwards" between mountain peaks.



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2. Euclidean geometry (*i.e.*, Postulates 1-5) is true in the actual world.

∴ The sum of the angles of the Bavarian triangle = two right angles.

- *Suppose*: The conclusion is false.
- *Then*: One or more premises must be false... but which ones?
- Can uphold premise 1 by denying premise 2:
  - If the sum of the angles $> 180°$, then perhaps *spherical* geometry is true in the actual world.
  - If the sum of the angles $< 180°$, then perhaps *hyperbolic* geometry is true in the actual world.