07. Quantum Mechanics Basics.

*Stern-Gerlach Experiment* (Stern & Gerlach 1922)

*Suggests*: Electrons possess 2-valued "spin" properties. (Goudsmit & Uhlenbeck 1925)

- With respect to a given direction (axis), an electron can possess either the value "spin-up" or the value "spin-down".
- There are as many of these spin properties as there are possible axes!
- *For simplicity*: Call two such spin properties "Color" (with values "white" and "black") and "Hardness" (with values "hard" and "soft").
Experimental Result #1: There is no correlation between Color and Hardness.
**Experimental Result #2:**
Hardness measurements "disrupt" Color measurements, and *vice-versa.*

- Can we build a Hardness measuring box that doesn't "disrupt" Color values?
  - *All evidence suggests "No"!*  
- Can we determine which electrons get their Color values "disrupted" by a Hardness measurement?
  - *All evidence suggests "No"!*  
- **Thus:** All evidence suggests Hardness and Color cannot be simultaneously measured.
**Experimental Result #3**: The "2-Path" Experiment.

- Feed white electrons into the device and measure their Color as they exit.
- From previous experiments, we should expect 50% white and 50% black...
- *But*: Experimentally, 100% are white!
Experimental Result #3: The "2-Path" Experiment.

- Now insert a barrier along the $s$ path.
- 50% less electrons register at the Exit.
- And: Experimentally, of these 50% are white and 50% are black.
Experimental Result #3: The "2-Path" Experiment.

What path does an individual electron take without the barrier present?

- Not $h$. The Color statistics of hard electrons is 50/50.
- Not $s$. The Color statistics of soft electrons is 50/50.
- Not both. Place detectors along the paths and only one will will register.
- Not neither. Block both paths and no electrons register at Exit.
Experimental Result #3: The "2-Path" Experiment.

What path does an individual electron take without the barrier present?

- Not $h$.
- Not $s$.
- Not both. \[\text{Suggests that white electrons have no determinate value of Hardness.}\]
- Not neither.
How to Describe Physical Phenomena: 5 Basic Notions

(a) **Physical system.**

| Classical example: baseball | Quantum example: electron |

(b) **Property of a physical system.** Quantifiable characteristic of a physical system.

<table>
<thead>
<tr>
<th>Classical examples:</th>
<th>Quantum examples:</th>
</tr>
</thead>
<tbody>
<tr>
<td>- momentum</td>
<td>- Hardness (spin along a given direction)</td>
</tr>
<tr>
<td>- position</td>
<td>- Color (spin along another direction)</td>
</tr>
<tr>
<td>- energy</td>
<td>- momentum</td>
</tr>
<tr>
<td></td>
<td>- position</td>
</tr>
<tr>
<td></td>
<td>- energy</td>
</tr>
</tbody>
</table>

(c) **State of a physical system.** Description of system at an instant in time in terms of its properties.

<table>
<thead>
<tr>
<th>Classical example</th>
<th>Quantum example</th>
</tr>
</thead>
<tbody>
<tr>
<td>- baseball moving at 95mph, 5 ft from batter.</td>
<td>- white electron entering a Hardness box.</td>
</tr>
</tbody>
</table>

(d) **State space.** The collection of all possible states of a system.

(e) **Dynamics.** A description of how the states of a system evolve in time.
Mathematical Description of Classical Physical System
(Baseball example)

(i) A state of the baseball: Specified by momentum (3 numbers $p_1, p_2, p_3$) and position (3 numbers $q_1, q_2, q_3$). (Baseball has 6 "degrees of freedom".)

(ii) The state space of the baseball: Represented by a 6-dim set of points (phase space):

\[ \text{Each point has 6 numbers associated with it and represents a state of the baseball.} \]

(iii) Properties of the baseball: Represented by functions on the phase space. In-principle always well-defined for any point in phase space.

**Ex:** baseball's energy = $E(p_i, q_i) = (p_1^2 + p_2^2 + p_3^2)/2m$

(iv) Dynamics of the baseball: Provided by Newton's equations of motion (in their Hamiltonian form).
Will this mathematical description work for electrons?

- **No**: Experiments suggest the "spin" properties of Hardness and Color are not always well-defined.
- **So**: We can't represent them mathematically as functions on a set of points.

<table>
<thead>
<tr>
<th>physical concept</th>
<th>mathematical representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>state:</td>
<td>Classical mechanics</td>
</tr>
<tr>
<td></td>
<td>point</td>
</tr>
<tr>
<td>state space:</td>
<td>set of points (phase space)</td>
</tr>
<tr>
<td>property:</td>
<td>function on points</td>
</tr>
<tr>
<td></td>
<td><strong>Quantum mechanics</strong></td>
</tr>
<tr>
<td></td>
<td>vector</td>
</tr>
<tr>
<td></td>
<td>vector space</td>
</tr>
<tr>
<td></td>
<td>operator on vectors</td>
</tr>
</tbody>
</table>
2-State Quantum Systems

- Restrict attention to quantum properties with only two values (like Hardness and Color).
- Associated state vectors are 2-dimensional:

1. States as vectors

\[ |Q\rangle = a|0\rangle + b|1\rangle \]

Require state vector \( |Q\rangle \) to have unit length:

\[ |a|^2 + |b|^2 = 1 \]

Set of all vectors decomposable in basis \( \{ |0\rangle, |1\rangle \} \) forms a vector space \( \mathcal{H} \).

1. States as vectors
2. Properties as operators
3. Schrodinger dynamics
4. Projection postulate
5. Entangled states
Why this is supposed to help

- **Recall**: White electrons appear to have no determinate value of Hardness.
- Let's represent the values of Color and Hardness as basis vectors.
- Let's suppose the Hardness basis \{\ket{\text{hard}}, \ket{\text{soft}}\} is rotated by 45° with respect to the Color basis \{\ket{\text{white}}, \ket{\text{black}}\}:

\begin{align*}
\text{Then: } \ket{\text{white}} &= \sqrt{\frac{1}{2}} \ket{\text{hard}} + \sqrt{\frac{1}{2}} \ket{\text{soft}}
\end{align*}

- An electron in a white state... ... is in a "superposition" of hard and soft states.

- Let's assume:

"Eigenvalue-eigenvector Rule"

A quantum system possesses the value of a property if and only if it is in a state associated with that value.

- **Upshot**: Since an electron in the state \ket{\text{white}} cannot be in either of the states \ket{\text{hard}}, \ket{\text{soft}}, it cannot be said to possess values of Hardness.
• **Recall**: Experimental Result #1: There is no correlation between Hardness measurements and Color measurements.
  
  - If the Hardness of a batch of white electrons is measured, 50% will be soft and 50% will be hard.

• Let's assume:

  "**Born Rule**:"
  
The probability that a quantum system in a state \( |Q\rangle \) possesses the value \( b \) of a property \( B \) is given by the square of the expansion coefficient of the basis state \( |b\rangle \) in the expansion of \( |Q\rangle \) in the basis corresponding to all values of the property.

• **So**: The probability that a white electron has the value hard when measured for Hardness is 1/2!

\[
|white\rangle = \sqrt{\frac{1}{2}} |\text{hard}\rangle + \sqrt{\frac{1}{2}} |\text{soft}\rangle
\]

*An electron in a white state...*  

... has a probability of 1/2 of being hard upon measurement for Hardness.
2. Properties as operators

**Def. 1.** A *linear operator* $O$ is a map that assigns to any vector $|A\rangle$, another vector $O|A\rangle$, such that $O(n|A\rangle + m|B\rangle) = n(O|A\rangle) + m(O|B\rangle)$, where $n, m$ are numbers.

- **Matrix representations**

  $|Q\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$  
  
  2-dim vector as $2 \times 1$ matrix

  $O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$  

  Operator on 2-dim vectors as $2 \times 2$ matrix

  $O|Q\rangle = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} O_{11}a + O_{12}b \\ O_{21}a + O_{22}b \end{pmatrix}$  

  Matrix multiplication encodes action of $O$ on $|Q\rangle$
Def. 2. An eigenvector of an operator $O$ is a vector $|\lambda\rangle$ that does not change its direction when $O$ acts on it: $O|\lambda\rangle = \lambda|\lambda\rangle$, for some number $\lambda$.

Def. 3. An eigenvalue $\lambda$ of an operator $O$ is the number that results when $O$ acts on one of its eigenvectors.

This allows the following correspondences
- Let an operator $O$ represent a property.
- Let its eigenvectors $|\lambda\rangle$ represent the value states ("eigenstates") associated with the property.
- Let its eigenvalues $\lambda$ represent the (numerical) values of the property.

- The Eigenvalue-Eigenvector Rule can now be stated as:

"Eigenvalue-eigenvector Rule"
A quantum system possesses the value $\lambda$ of a property represented by an operator $O$ if and only if it is in an eigenstate $|\lambda\rangle$ of $O$ with eigenvalue $\lambda$. 
3. The Schrödinger Dynamics

*States evolve in time via the Schrödinger equation*

Plug an initial state $|\psi(t_1)\rangle$ into the Schrödinger equation, and it produces a unique final state $|\psi(t_2)\rangle$.

\[
|\psi(t_1)\rangle \xrightarrow{\text{Schrödinger evolution}} |\psi(t_2)\rangle
\]

- The Schrödinger equation can be encoded in an operator $S$ (which is a function of the Hamiltonian operator $H$ that encodes the energy of the system).

\[
e^{iHt_2/\hbar}|A\rangle \equiv S|A\rangle = |A'\rangle
\]

**Important property:** $S$ is a linear operator.

\[
S(\alpha|A\rangle + \beta|B\rangle) = \alpha S|A\rangle + \beta S|B\rangle, \quad \text{where} \ \alpha, \beta \text{are numbers.}
\]
4. The Projection Postulate

**Projection Postulate (2-state systems)**

When a measurement of a property represented by an operator $B$ is made on a system in the state $|Q\rangle = a|\lambda_1\rangle + b|\lambda_2\rangle$ expanded in the eigenvector basis of $B$, and the result is the value $\lambda_1$, then $|Q\rangle$ collapses to the state $|\lambda_1\rangle$, $|Q\rangle \xrightarrow{\text{collapse}} |\lambda_1\rangle$.

**Example:** Suppose we measure a white electron for Hardness.
- The pre-measurement state is given by:
  $$|\text{white}\rangle = \sqrt{\frac{1}{2}}|\text{hard}\rangle + \sqrt{\frac{1}{2}}|\text{soft}\rangle$$
- **Suppose:** The outcome of the measurement is the value hard.
- **Then:** The post-measurement state is given by $|\text{hard}\rangle$.

**Motivations:**
- Guarantees that if we obtain the value $\lambda_1$ once, then we should get the same value $\lambda_1$ on a second measurement (provided the system is not interfered with).
- Guarantees that measurements have unique outcomes.

---

John von Neumann
(1903-1957)
5. Entangled states

- Consider state spaces $\mathcal{H}_1$, $\mathcal{H}_2$ for two quantum 2-state systems (electrons, say). State space for combined system is represented by $\mathcal{H}_1 \otimes \mathcal{H}_2$.

- Suppose: $\{|0\>_1, |1\>_1\}$ is a basis for $\mathcal{H}_1$ and $\{|0\>_2, |1\>_2\}$ is a basis for $\mathcal{H}_2$.

- Then: $\{|0\>_1|0\>_2, |0\>_1|1\>_2, |1\>_1|0\>_2, |1\>_1|1\>_2\}$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

- Any 2-particle state $|Q\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be expanded in this basis:

$$|Q\rangle = a|0\>_1|0\>_2 + b|0\>_1|1\>_2 + c|1\>_1|0\>_2 + d|1\>_1|1\>_2$$

An entangled state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a vector that cannot be written as a product of two terms, one from $\mathcal{H}_1$ and the other from $\mathcal{H}_2$. 
According to the Eigenvalue-Eigenvector Rule:

- **Entangled:** $|\Psi^+\rangle = \sqrt{\frac{1}{2}} \left\{ |0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2 \right\}$

- **Nonentangled (Separable):**

  $|A\rangle = \sqrt{\frac{1}{4}} \left\{ |0\rangle_1 |0\rangle_2 + |0\rangle_1 |1\rangle_2 + |1\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2 \right\} = \sqrt{\frac{1}{4}} \left\{ |0\rangle_1 + |1\rangle_1 \right\} \left\{ |0\rangle_2 + |1\rangle_2 \right\}$

  $|B\rangle = \sqrt{\frac{1}{2}} \left\{ |0\rangle_1 |0\rangle_2 + |1\rangle_1 |0\rangle_2 \right\} = \sqrt{\frac{1}{2}} \left\{ |0\rangle_1 + |1\rangle_1 \right\} |0\rangle_2$

  $|C\rangle = |0\rangle_1 |0\rangle_2$

- Suppose $|0\rangle$ and $|1\rangle$ are eigenstates of Hardness (i.e., $|0\rangle = |\text{hard}\rangle$ and $|1\rangle = |\text{soft}\rangle$).

According to the Eigenvalue-Eigenvector Rule:

- In states $|\Psi^+\rangle$ and $|A\rangle$, both electrons have no determinate Hardness value.
- In state $|B\rangle$, electron_1 has no determinate Hardness value, but electron_2 does (i.e., hard).
- In state $|C\rangle$, both electrons have determinate Hardness values.
According to the Projection Postulate:

- Suppose $|0\rangle$ and $|1\rangle$ are eigenstates of Hardness (i.e., $|0\rangle = |\text{hard}\rangle$ and $|1\rangle = |\text{soft}\rangle$).

- **Entangled:** $|\Psi^+\rangle = \sqrt{\frac{1}{2}} \left\{|0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2 \right\}$

- **Nonentangled (Separable):**
  
  $|A\rangle = \sqrt{\frac{1}{4}} \left\{|0\rangle_1 |0\rangle_2 + |0\rangle_1 |1\rangle_2 + |1\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2 \right\} = \sqrt{\frac{1}{4}} \left\{|0\rangle_1 + |1\rangle_1 \right\} \left\{|0\rangle_2 + |1\rangle_2 \right\}$

  $|B\rangle = \sqrt{\frac{1}{2}} \left\{|0\rangle_1 |0\rangle_2 + |1\rangle_1 |0\rangle_2 \right\} = \sqrt{\frac{1}{2}} \left\{|0\rangle_1 + |1\rangle_1 \right\} |0\rangle_2$

  $|C\rangle = |0\rangle_1 |0\rangle_2$

- In the entangled state $|\Psi^+\rangle$, when a measurement is performed on electron$_1$, its state collapses (to either $|0\rangle_1$ or $|1\rangle_1$), and this instantaneously affects the state of electron$_2$!

- In any of the separable states, a measurement performed on electron$_1$ will not affect the state of electron$_2$. 