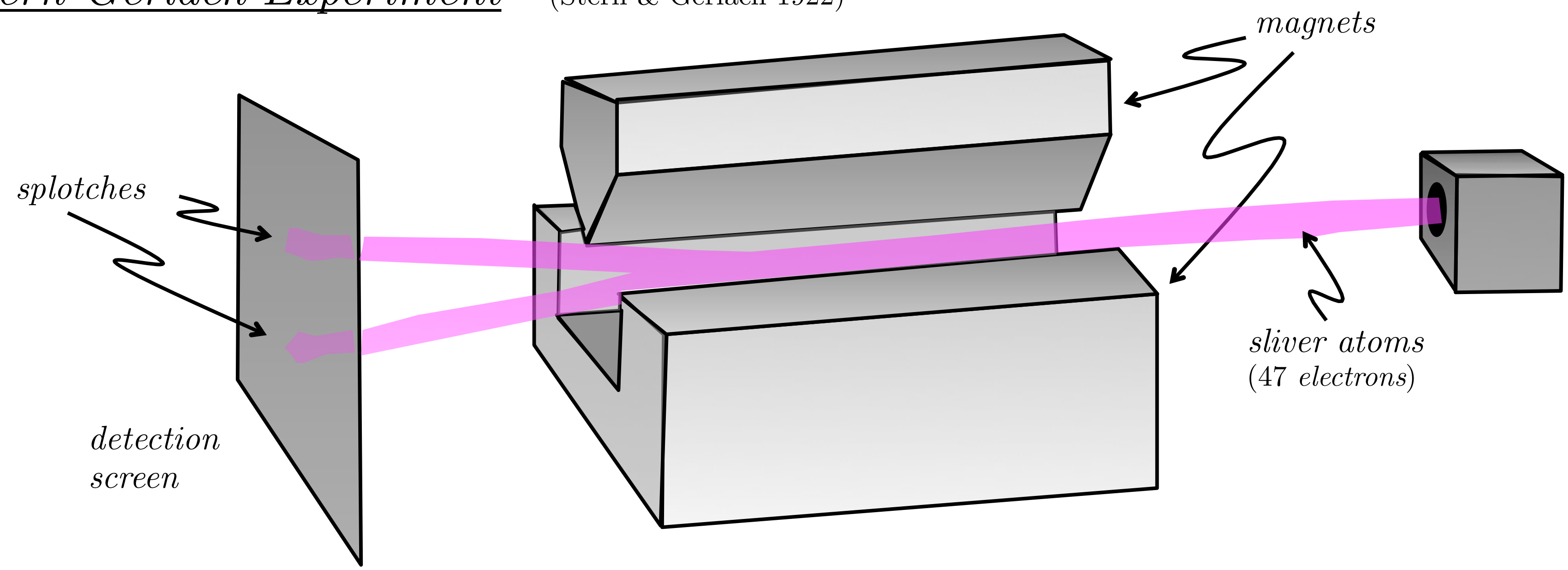
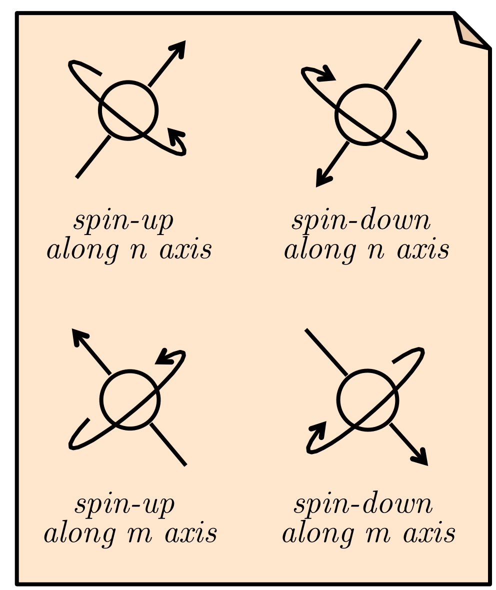
07. Quantum Mechanics Basics.

Stern-Gerlach Experiment (Stern & Gerlach 1922)

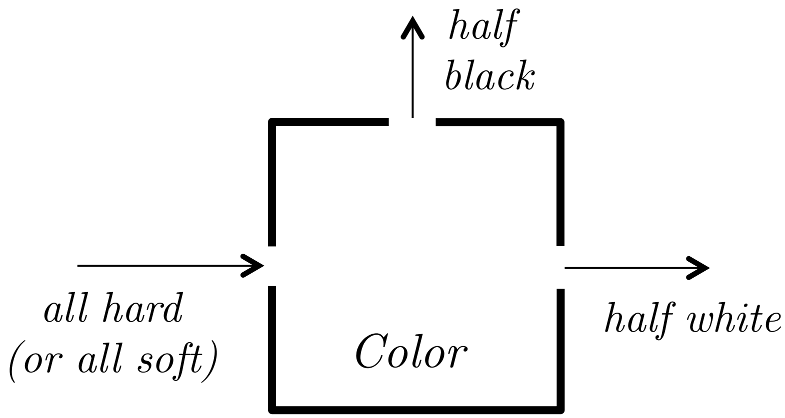


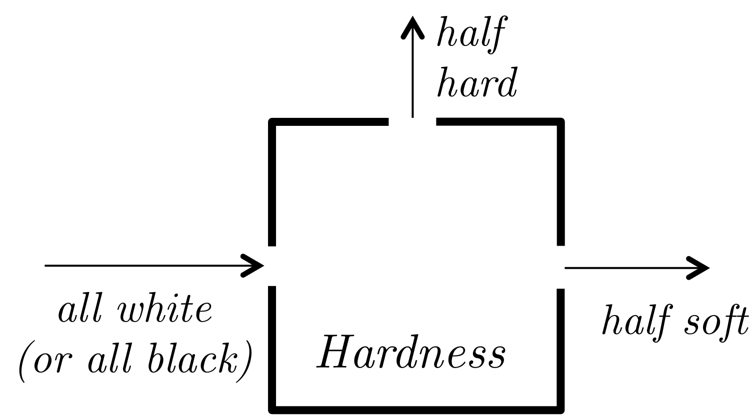
Suggests: Electrons possess 2-valued "spin" properties. (Goudsmit & Uhlenbeck 1925)

- With respect to a given direction (axis), an electron can possess either the value "spin-up" or the value "spin-down".
- There are as many of these spin properties as there are possible axes!
- *For simplicity*: Call two such spin properties "Color" (with values "white" and "black") and "Hardness" (with values "hard" and "soft").



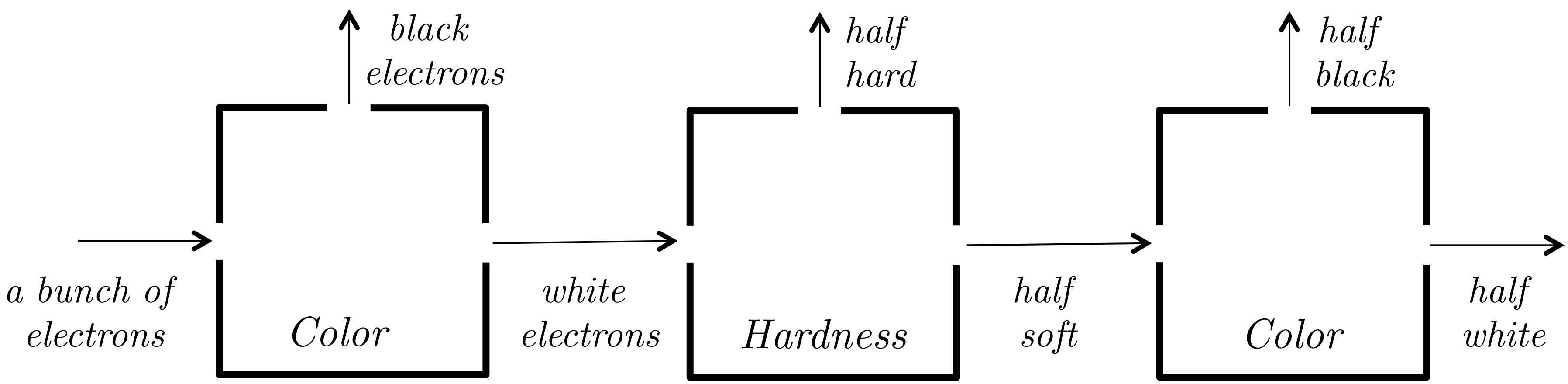
Experimental Result #1: There is no correlation between Color and Hardness.





<u>*Experimental Result #2*</u>:

Hardness measurements "disrupt" Color measurements, and *vice-versa*.



- Can we build a Hardness measuring box that doesn't "disrupt" Color values?

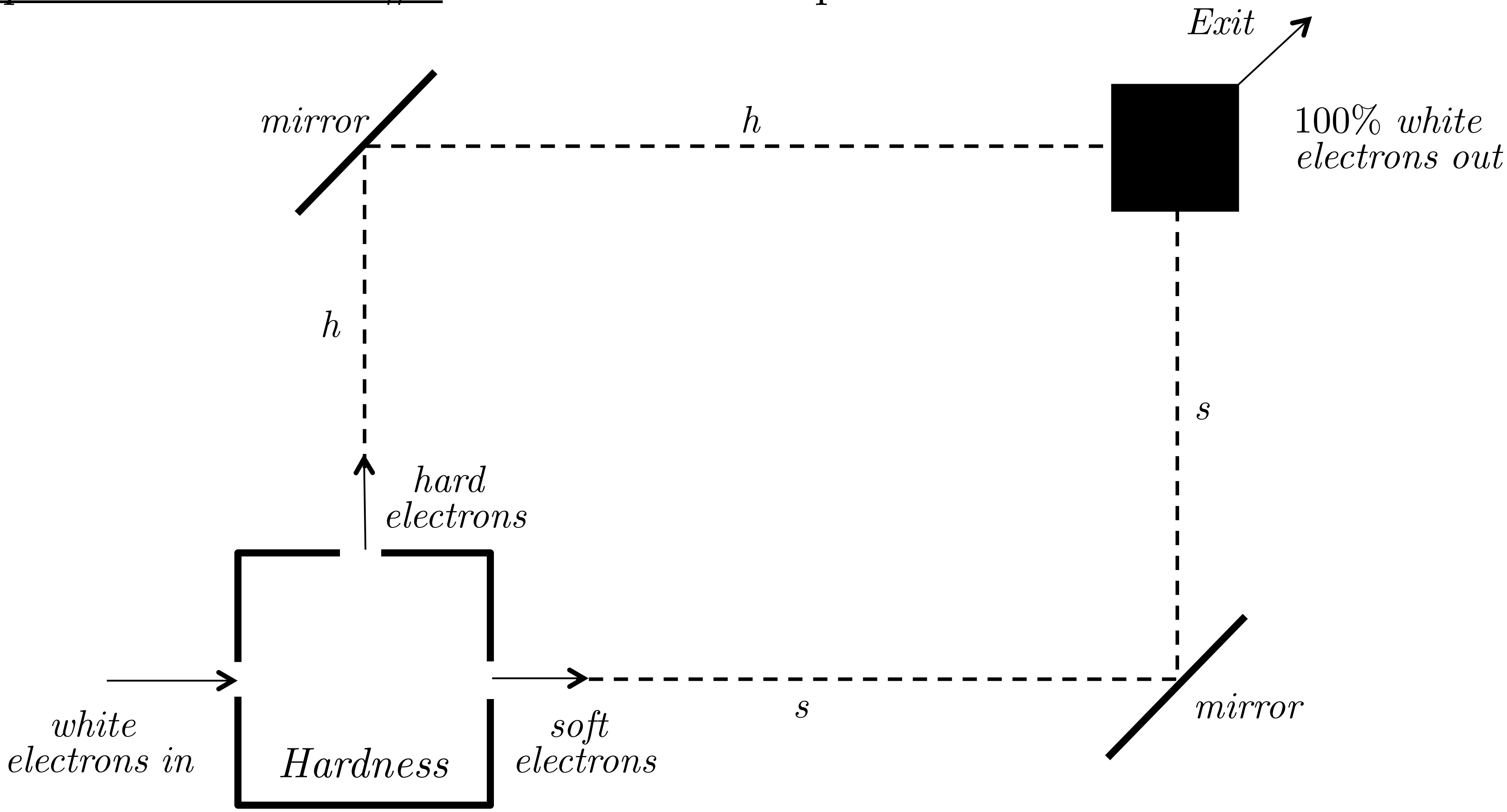
  - *All evidence suggests "No"!*

- Can we determine which electrons get their Color values "disrupted" by a Hardness measurement?

  - *All evidence suggests "No"!*

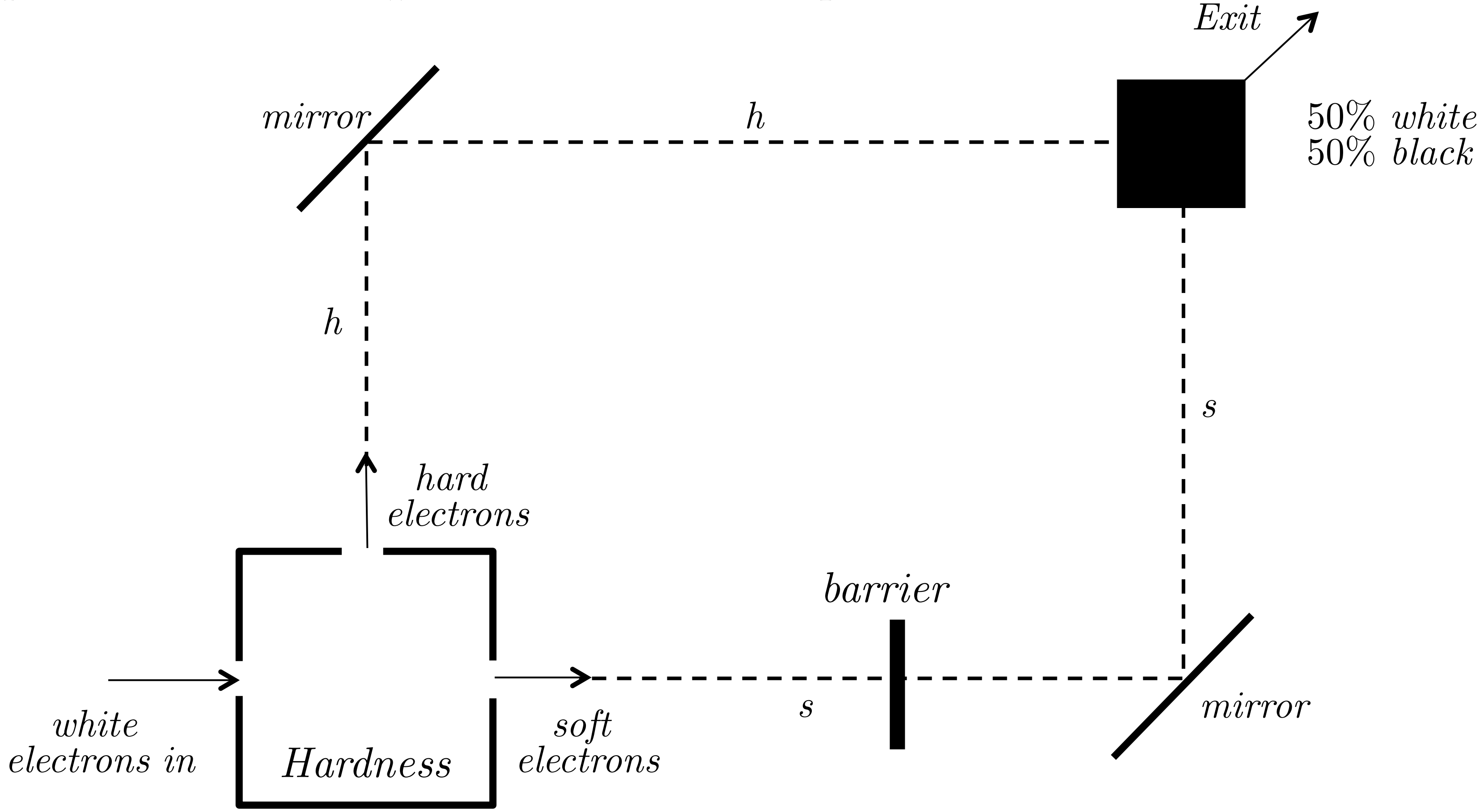
- <u>*Thus*</u>: All evidence suggests Hardness and Color cannot be simultaneously measured.

Experimental Result #3: The "2-Path" Experiment.



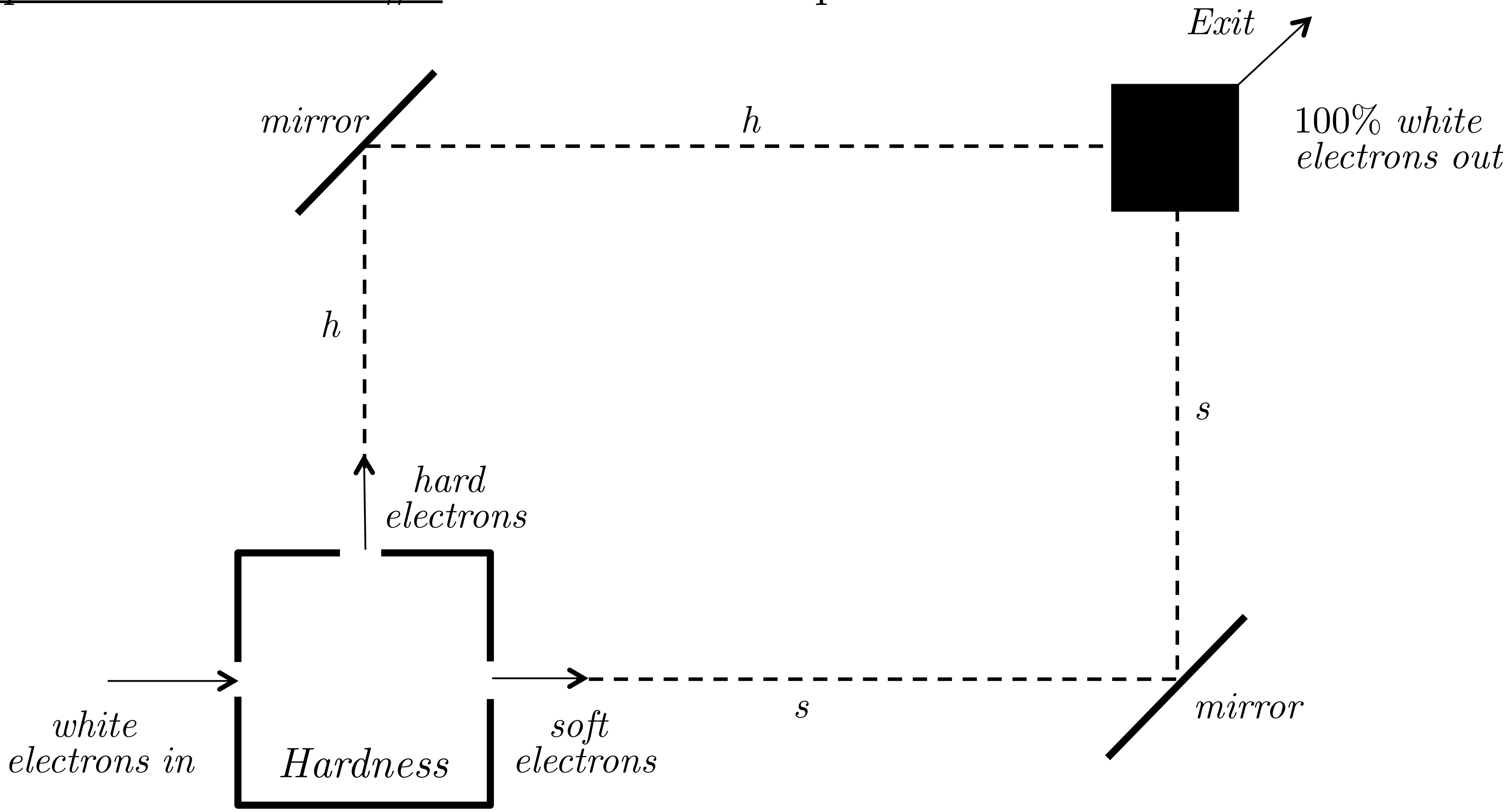
- Feed white electrons into the device and measure their Color as they exit.
- From previous experiments, we should expect 50% white and 50% black...
- *But*: Experimentally, 100% are white!

Experimental Result #3: The "2-Path" Experiment.



- Now insert a barrier along the *s* path.
- 50% less electrons register at the Exit.
- *And*: Experimentally, of these 50% are white and 50% are black.

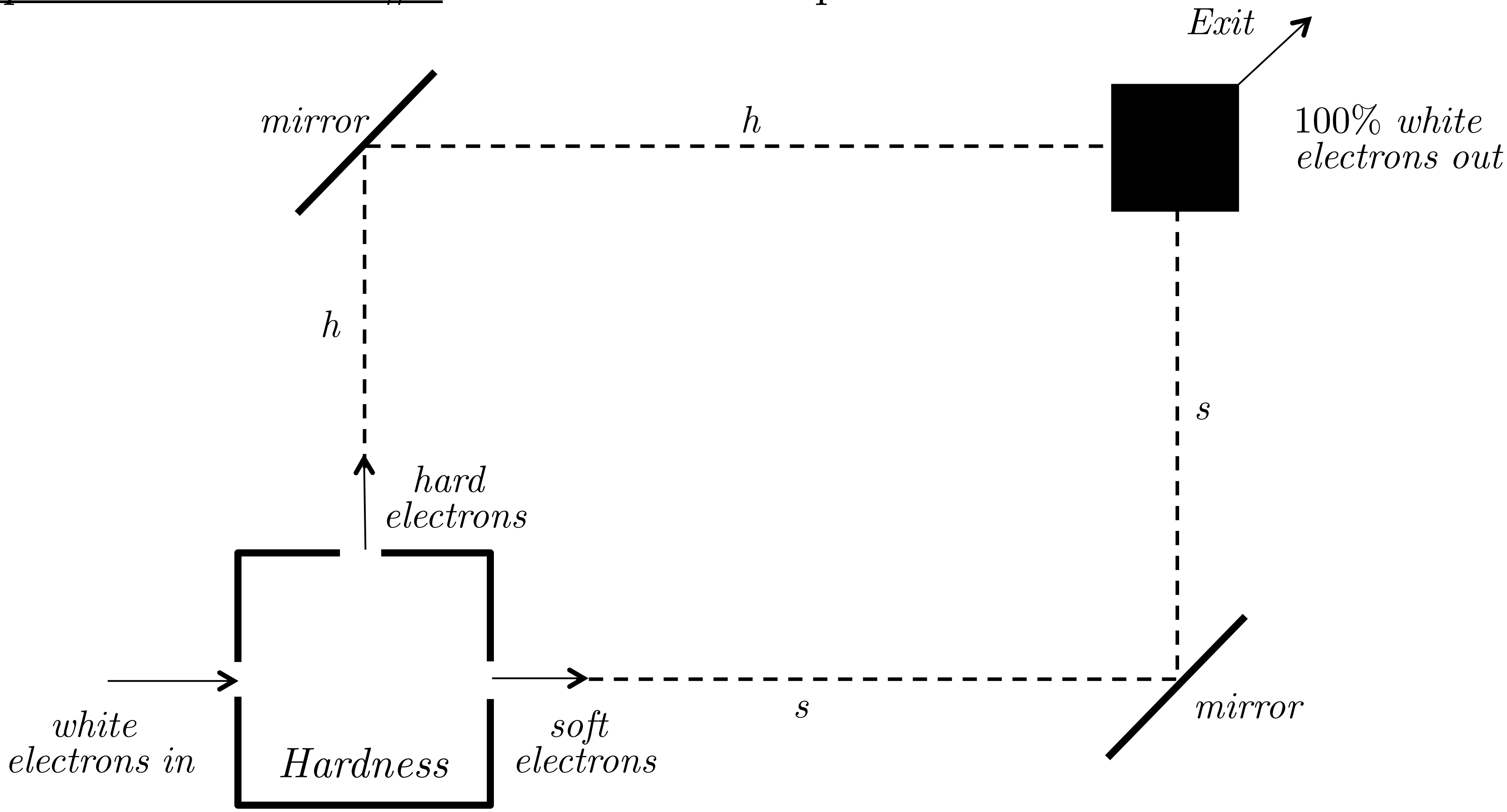
Experimental Result #3: The "2-Path" Experiment.



What path does an individual electron take without the barrier present?

- Not *h*. The Color statistics of hard electrons is 50/50.
- Not *s*. The Color statistics of soft electrons is 50/50.
- Not *both*. Place detectors along the paths and only one will register.
- Not *neither*. Block both paths and no electrons register at Exit.

Experimental Result #3: The "2-Path" Experiment.



What path does an individual electron take without the barrier present?

- Not *h*.
- Not *s*.
- Not *both*.
- Not *neither*.

Suggests that white electrons have no determinate value of Hardness.

How to Describe Physical Phenomena: 5 Basic Notions

(a) *Physical system*.

> *Classical example*: baseball   *Quantum example*: electron

(b) *Property of a physical system*. Quantifiable characteristic of a physical system.

Classical examples:	*Quantum examples:*
- momentum	- Hardness (spin along a given direction)
- position	- Color (spin along another direction)
- energy	- momentum
	- position
	- energy

(c) *State of a physical system*. Description of system at an instant in time in terms of its properties.

Classical example	*Quantum example*
- baseball moving at 95mph, 5 ft from batter.	- white electron entering a Hardness box.

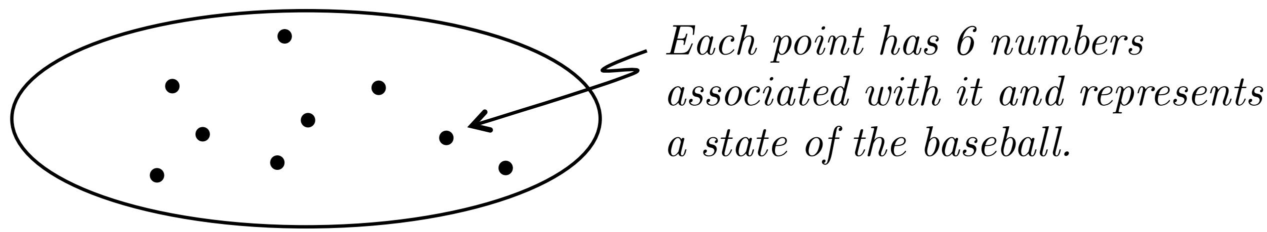
(d) *State space*. The collection of all possible states of a system.

(e) *Dynamics*. A description of how the states of a system evolve in time.

Mathematical Description of Classical Physical System
(Baseball example)

(i) A *state* of the baseball: Specified by *momentum* (3 numbers p_1, p_2, p_3) and *position* (3 numbers q_1, q_2, q_3). (Baseball has 6 "degrees of freedom".)

(ii) The *state space* of the baseball: Represented by a 6-dim *set of points* (*phase space*):



(iii) *Properties* of the baseball: Represented by *functions* on the phase space. *In-principle always well-defined for any point in phase space.*

Ex: baseball's *energy* $= E(p_i, q_i) = (p_1{}^2 + p_2{}^2 + p_3{}^2)/2m$

(iv) *Dynamics* of the baseball: Provided by Newton's equations of motion (in their Hamiltonian form).

Will this mathematical description work for electrons?

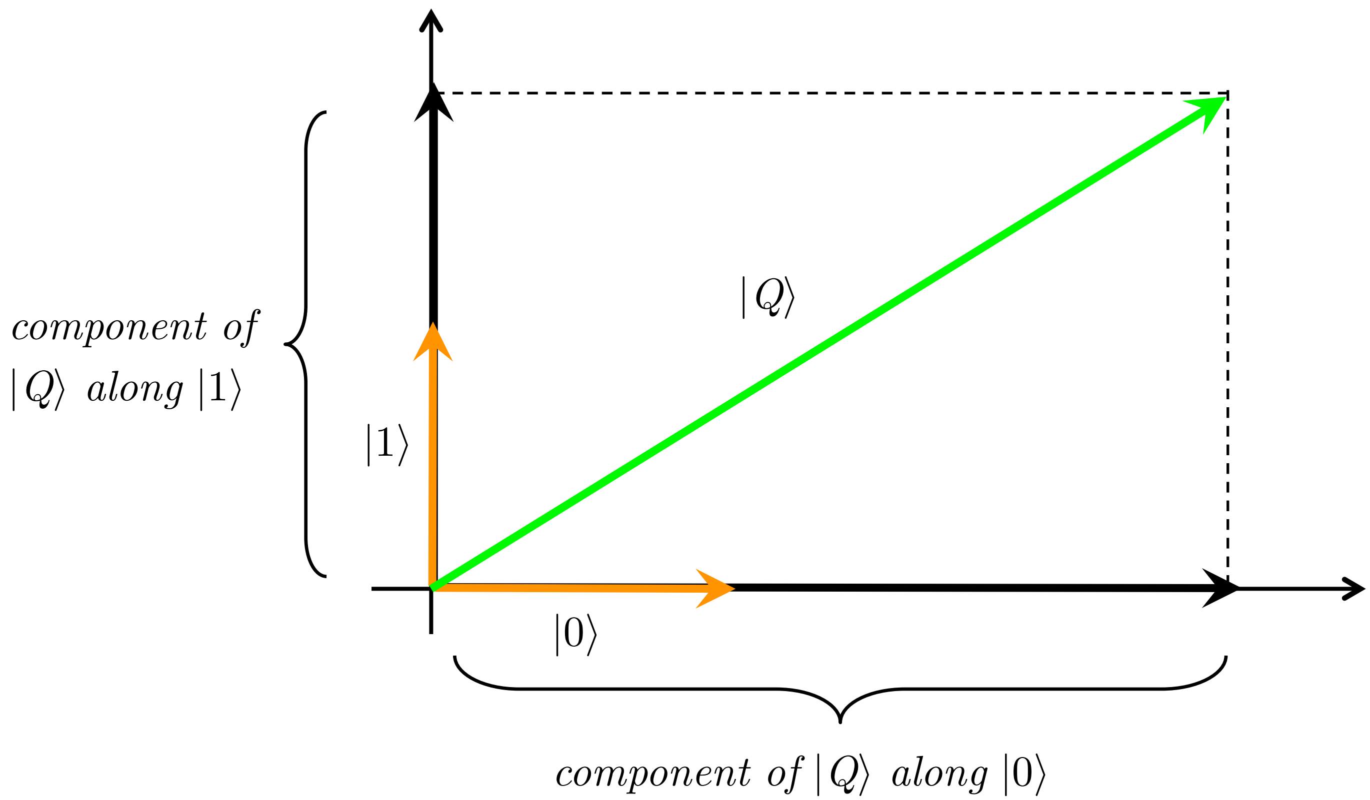
- *No*: Experiments suggest the "spin" properties of Hardness and Color are not always well-defined.
- *So*: We can't represent them mathematically as functions on a set of points.

physical concept	*mathematical representation*	
	Classical mechanics	*Quantum mechanics*
state:	point	vector
state space:	set of points (phase space)	vector space
property:	function on points	operator on vectors

2-State Quantum Systems

- Restrict attention to quantum properties with only two values (like *Hardness* and *Color*).
- Associated state vectors are 2-dimensional:

1. States as vectors



$$|Q\rangle = a|0\rangle + b|1\rangle$$

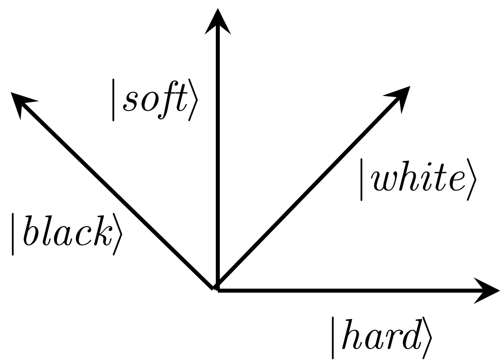
Require state vector $|Q\rangle$ *to have unit length:*

$$|a|^2 + |b|^2 = 1$$

- Set of all vectors decomposible in basis $\{|0\rangle, |1\rangle\}$ forms a *vector space* $\mathcal{H}$.

Why this is supposed to help

- *Recall*: White electrons appear to have no determinate value of Hardness.
- Let's represent the *values* of Color and Hardness as basis vectors.
- Let's suppose the Hardness basis $\{|hard\rangle, |soft\rangle\}$ is rotated by 45° with respect to the Color basis $\{|white\rangle, |black\rangle\}$:



Then: $|white\rangle = \sqrt{\tfrac{1}{2}}|hard\rangle \; + \; \sqrt{\tfrac{1}{2}}|soft\rangle$

An electron in a white state...

... is in a "superposition" of hard and soft states.

- Let's assume:

> *"Eigenvalue-eigenvector Rule"*
> A quantum system possesses the value of a property *if and only if* it is in a state associated with that value.

- *Upshot*: Since an electron in the state $|white\rangle$ cannot be in either of the states $|hard\rangle$, $|soft\rangle$, it cannot be said to possess values of Hardness.

- *Recall*: Experimental Result #1: There is no correlation between Hardness measurements and Color measurements.
  - If the Hardness of a batch of white electrons is measured, 50% will be soft and 50% will be hard.

- Let's assume:

> *"Born Rule"*:
> The probability that a quantum system in a state $|Q\rangle$ possesses the value b of a property B is given by the square of the expansion coefficient of the basis state $|b\rangle$ in the expansion of $|Q\rangle$ in the basis corresponding to all values of the property.

Max Born
1882-1970

- *So*: The probability that a *white* electron has the value *hard* when measured for Hardness is 1/2!

$$|white\rangle = \sqrt{\tfrac{1}{2}}|hard\rangle + \sqrt{\tfrac{1}{2}}|soft\rangle$$

An electron in a white state...

... has a probability of 1/2 of being hard upon measurement for Hardness.

2. Properties as operators

Def. 1. A *linear operator* O is a map that assigns to any vector $|A\rangle$, another vector $O|A\rangle$, such that $O(n|A\rangle + m|B\rangle) = n(O|A\rangle) + m(O|B\rangle)$, where n, m are numbers.

- <u>*Matrix representations*</u>

$$|Q\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

2-dim vector as 2×1 *matrix*

$$O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$$

Operator on 2-dim vectors as 2×2 *matrix*

$$O|Q\rangle = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} O_{11}a + O_{12}b \\ O_{21}a + O_{22}b \end{pmatrix}$$

Matrix multiplication encodes action of O *on* $|Q\rangle$

Def. 2. An *eigenvector* of an operator O is a vector $|\lambda\rangle$ that does not change its direction when O acts on it: $O|\lambda\rangle = \lambda|\lambda\rangle$, for some number λ.

Def. 3. An *eigenvalue* λ of an operator O is the number that results when O acts on one of its eigenvectors.

This allows the following correspondences

- Let an operator O represent a *property*.
- Let its eigenvectors $|\lambda\rangle$ represent the *value states* ("eigenstates") associated with the property.
- Let its eigenvalues λ represent the (numerical) *values* of the property.

• The Eigenvalue-Eigenvector Rule can now be stated as:

"Eigenvalue-eigenvector Rule"
A quantum system possesses the value λ of a property represented by an operator O *if and only if* it is in an eigenstate $|\lambda\rangle$ of O with eigenvalue λ.

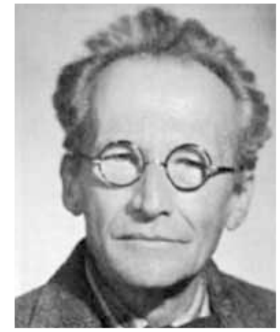
3. The Schrödinger Dynamics

<u>*States evolve in time via the Schrödinger equation*</u>

Plug an initial state $|\psi(t_1)\rangle$ into the Schrödinger equation, and it produces a unique final state $|\psi(t_2)\rangle$.

$$|\psi(t_1)\rangle \xrightarrow[\text{evolution}]{\text{Schrödinger}} |\psi(t_2)\rangle$$

state at time t_1 — *state at later time* t_2



Erwin Schrödinger
(1887-1961)

- The Schrödinger equation can be encoded in an operator S (which is a function of the Hamiltonian operator H that encodes the energy of the system).

$$e^{iHt_2/\hbar}|A\rangle \equiv S|A\rangle = |A'\rangle$$

state at time t_1 — *state at later time* t_2

<u>*Important property*</u>: S is a *linear operator.*

$$S(\alpha|A\rangle + \beta|B\rangle) = \alpha S|A\rangle + \beta S|B\rangle, \qquad \text{where } \alpha, \beta \text{ are numbers.}$$

4. The Projection Postulate

Projection Postulate (2-state systems)

When a measurement of a property represented by an operator B is made on a system in the state $|Q\rangle = a|\lambda_1\rangle + b|\lambda_2\rangle$ expanded in the eigenvector basis of B, and the result is the value λ_1, then $|Q\rangle$ *collapses* to the state $|\lambda_1\rangle$, $|Q\rangle \xrightarrow[\text{collapse}]{} |\lambda_1\rangle$.

John von Neumann
(1903-1957)

Example: Suppose we measure a *white* electron for Hardness.

- The pre-measurement state is given by:

$$|white\rangle = \sqrt{\tfrac{1}{2}}|hard\rangle + \sqrt{\tfrac{1}{2}}|soft\rangle$$

- *Suppose*: The outcome of the measurement is the value *hard*.
- *Then*: The post-measurement state is given by $|hard\rangle$.

Motivations:

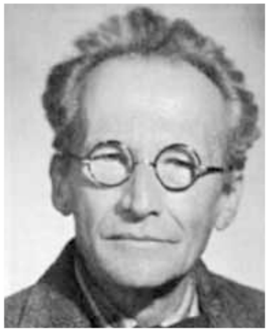
- Guarantees that if we obtain the value λ_1 once, then we should get the same value λ_1 on a second measurement (provided the system is not interferred with).
- Guarantees that measurements have unique outcomes.

5. Entangled states

- Consider state spaces $\mathcal{H}_1$, $\mathcal{H}_2$ for *two* quantum 2-state systems (electrons, say). State space for combined system is represented by $\mathcal{H}_1 \otimes \mathcal{H}_2$.
- *Suppose*: $\{|0\rangle_1, |1\rangle_1\}$ is a basis for $\mathcal{H}_1$ and $\{|0\rangle_2, |1\rangle_2\}$ is a basis for $\mathcal{H}_2$.
- *Then*: $\{|0\rangle_1|0\rangle_2, |0\rangle_1|1\rangle_2, |1\rangle_1|0\rangle_2, |1\rangle_1|1\rangle_2\}$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.
- Any 2-particle state $|Q\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be expanded in this basis:

$$|Q\rangle = a|0\rangle_1|0\rangle_2 + b|0\rangle_1|1\rangle_2 + c|1\rangle_1|0\rangle_2 + d|1\rangle_1|1\rangle_2$$

An *entangled state* in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a vector that cannot be written as a product of two terms, one from $\mathcal{H}_1$ and the other from $\mathcal{H}_2$.



Examples:

- *Entangled:* $|\Psi^+\rangle = \sqrt{\tfrac{1}{2}}\left\{|0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2\right\}$
- *Nonentangled (Separable):*

$$|A\rangle = \sqrt{\tfrac{1}{4}}\left\{|0\rangle_1|0\rangle_2 + |0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\right\} = \sqrt{\tfrac{1}{4}}\left\{|0\rangle_1 + |1\rangle_1\right\}\left\{|0\rangle_2 + |1\rangle_2\right\}$$

$$|B\rangle = \sqrt{\tfrac{1}{2}}\left\{|0\rangle_1|0\rangle_2 + |1\rangle_1|0\rangle_2\right\} = \sqrt{\tfrac{1}{2}}\left\{|0\rangle_1 + |1\rangle_1\right\}|0\rangle_2$$

$$|C\rangle = |0\rangle_1|0\rangle_2$$

- Suppose $|0\rangle$ and $|1\rangle$ are eigenstates of Hardness (*i.e.*, $|0\rangle = |hard\rangle$ and $|1\rangle = |soft\rangle$).

According to the Eigenvalue-Eigenvector Rule:

- In states $|\Psi^+\rangle$ and $|A\rangle$, both electrons have no determinate Hardness value.
- In state $|B\rangle$, electron$_1$ has no determinate Hardness value, but electron$_2$ *does* (*i.e.*, hard).
- In state $|C\rangle$, both electrons have determinate Hardness values.

Examples:

- *Entangled:* $|\Psi^+\rangle = \sqrt{\tfrac{1}{2}}\left\{|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\right\}$
- *Nonentangled (Separable):*

$$|A\rangle = \sqrt{\tfrac{1}{4}}\left\{|0\rangle_1|0\rangle_2 + |0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\right\} = \sqrt{\tfrac{1}{4}}\left\{|0\rangle_1 + |1\rangle_1\right\}\left\{|0\rangle_2 + |1\rangle_2\right\}$$

$$|B\rangle = \sqrt{\tfrac{1}{2}}\left\{|0\rangle_1|0\rangle_2 + |1\rangle_1|0\rangle_2\right\} = \sqrt{\tfrac{1}{2}}\left\{|0\rangle_1 + |1\rangle_1\right\}|0\rangle_2$$

$$|C\rangle = |0\rangle_1|0\rangle_2$$

- Suppose $|0\rangle$ and $|1\rangle$ are eigenstates of Hardness (*i.e.*, $|0\rangle = |hard\rangle$ and $|1\rangle = |soft\rangle$).

According to the Projection Postulate:

- In the entangled state $|\Psi^+\rangle$, when a measurement is performed on electron$_1$, its state collapses (to either $|0\rangle_1$ or $|1\rangle_1$), *and this instantaneously affects the state of electron$_2$!*
- In any of the separable states, a measurement performed on electron$_1$ will not affect the state of electron$_2$.