06. Malament-Hogarth Spacetimes and Non-Turing Computability

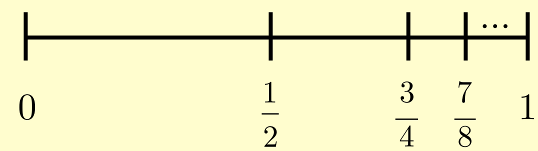
1. Supertasks

- *Recall*: The decision problem for 1st-order arithmetic is *Turing unsolvable*.
- *Which means*: No TM that halts after a finite number of steps can determine if a given statement in arithmetic is a theorem.
- What if we allow the TM to perform an *infinite* number of steps?
- *Initial question*: Should we allow such an infinity TM a finite or an infinite amount of time to do this?

Option A: An infinity TM that performs an infinite number of steps in a *finite* amount of time.

Goldbach's Conjecture: Every even integer greater than 2 can be expressed as the sum of two primes.

- Begin at $t = 0$.
- At $t = 1/2$, check 4.
- At $t = 3/4$, check 6.
- At $t = 7/8$, check 8.
- *etc*....
- At $t = 1$, all even integers greater than 2 will have been checked!



- *Advantage*: Puny finite humans can access the infinity TM's output in a finite amount of time.
- *But*: Is such a "supertask" conceptually possible?

Thomson's Lamp:

- At $t = 0$, switch lamp on.
- At $t = 1/2$, switch lamp off.
- At $t = 3/4$, switch lamp on.
- *etc....*
- At $t = 1$, is lamp on or off?

- *Thomson (1954)*: The lamp's operation requires:

  (i) For any time t, $0 < t < 1$, if the lamp is *off* at t, then there's another time t', $t < t' \leq 1$, such that the lamp is *on* at t'.

  (ii) For any time t, $0 < t < 1$, if the lamp is *on* at t, then there's another time t', $t < t' \leq 1$, such that the lamp is *off* at t'.

- (i) implies that the lamp is on at $t = 1$.
- (ii) implies that the lamp is off at $t = 1$.

} *Contradiction! So the lamp is not possible.*

Thomson's Lamp:

- At $t = 0$, switch lamp on.
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- *etc*....
- At $t = 1$, is lamp on or off?

- *Thomson (1954)*: The lamp's operation requires:
  (i) For any time t, $0 < t < 1$, if the lamp is *off* at t, then there's another time t', $t < t' \leq 1$, such that the lamp is *on* at t'.
  (ii) For any time t, $0 < t < 1$, if the lamp is *on* at t, then there's another time t', $t < t' \leq 1$, such that the lamp is *off* at t'.

Benacerraf's (1962) response:
- The limit of a sequence is not a member of the sequence.
- *So*: Any properties of a sequence cannot necessarily be attributed to its limit.
- *Thus*: Since 1 is not a member of (0, 1/2, 3/4, 7/8, ...), any properties of the latter cannot be attributed to the former.
- *In particular*: The "$\leq$" in (i) and (ii) is unjustified.

Option B: An infinity TM that performs an infinite number of steps in an *infinite* amount of time.

> *Goldbach's Conjecture*: Every even integer greater than 2 can be expressed as the sum of two primes.
>
> Use an infinity TM to check:
>
> - At $t = 1$, check 4; at $t = 2$, check 6; *etc.*
> - At $t = \infty$, all integers greater than 2 will have been checked!

- *But*: How can puny humans access the data from such an infinity TM?
- *Answer*: In a Malament-Hogarth spacetime!

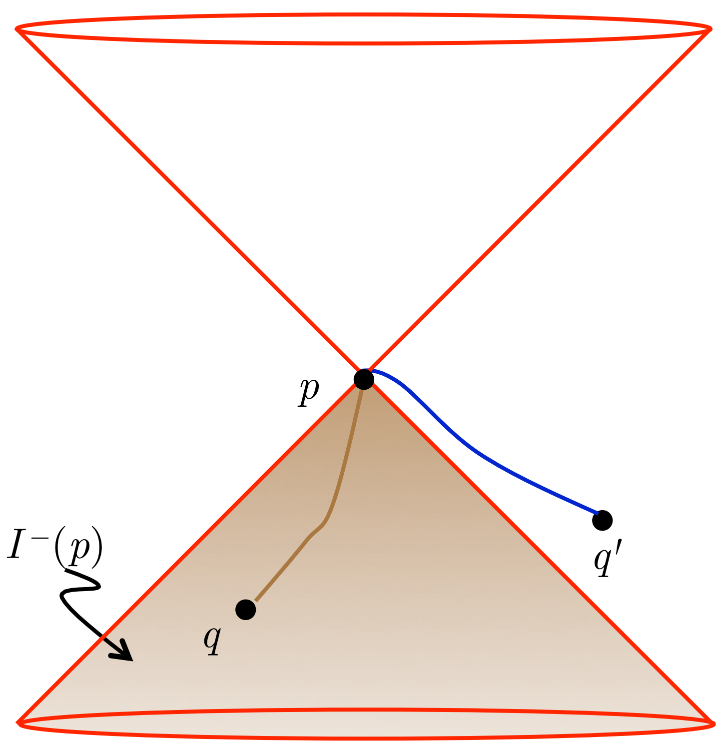
2. Malament-Hogarth Spacetimes

- *Idea*: We want the infinity TM's worldline to be infinitely long, *and* to be *accessible* (*via* a causal signal, say) from the programmer's worldline.

> *A Malament-Hogarth spacetime* = a spacetime with a worldline of *infinite proper length* that lies in the *past* of some point.

Let's try to make this more precise...

- *Recall:* In a relativistic spacetime, there is a lightcone at every point p.



Def. 1. The *chronological past* $I^-(p)$ *of* p consists of all points q that can be connected to p by a timelike worldline.

$q \in I^-(p)$.
$q' \notin I^-(p)$.

So: All points in $I^-(p)$ are accessible from p.

Def. 2. The *proper length* of a timelike worldline γ is the sum of all infinitesimal intervals ds along it: (*proper length of* γ) $= \int_\gamma ds$.

Def. 3. A *timelike half-worldline* is a timelike worldline with a past endpoint.

- *Idea*: We want the infinity TM's worldline γ to have a point, call it q, in its past that coincides with a point of the programmer's worldline. (q is the event at which the programmer programs the infinity TM.)

A Malament-Hogarth spacetime = a 4-dim collection of points with a metric $g_{\mu\nu}$, a timelike half worldline γ, and a point p such that

(MH1) the proper length of γ is infinite: $\int_\gamma ds = \infty$.

(MH2) γ lies completely in the chronological past of p: $\gamma \subset I^-(p)$.

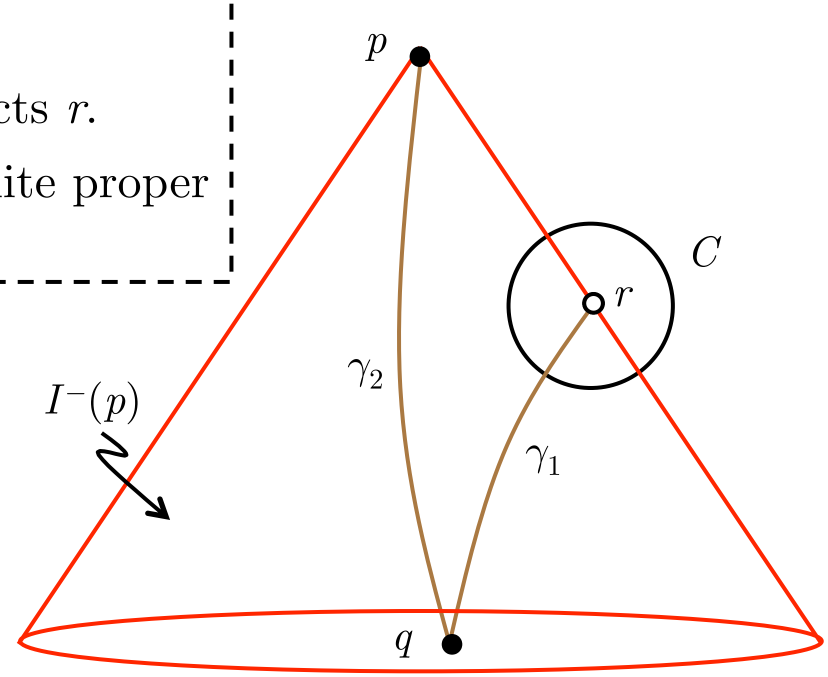
Example: Toy M-H spacetime.

1. Start with Minkowski spacetime.
2. Add scalar field Ω that is the identity outside of a compact region C, and goes to ∞ as the point $r \in C$ is approached.
3. Remove r.

Claim: Resulting spacetime with metric $\Omega^2 \eta_{\mu\nu}$ is M-H.

Proof (by construction):

- Let γ_1 be a timelike worldline with future endpoint r and that passes through point q.
- Let p be a point whose past lightcone intersects r.
- Then γ_1 is a timelike half worldline with infinite proper length that lies in the chronological past of p.



- γ_1 is the TM worldline.
- γ_2 is the programmer worldline.
- At p, programmer has access to output of TM after it has performed ∞ steps.

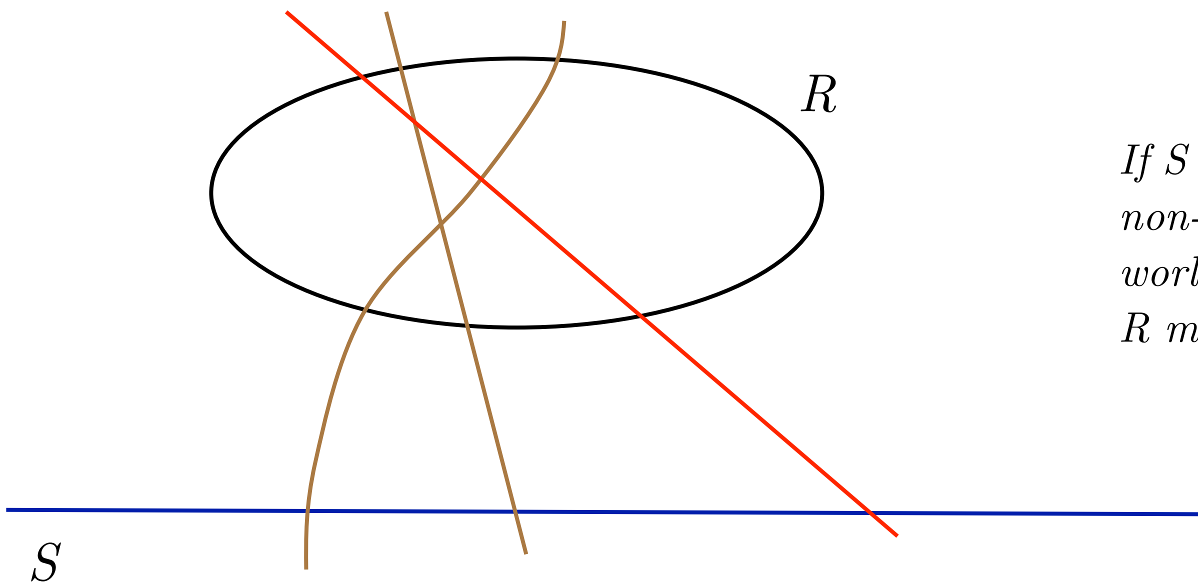
Are M-H spacetimes physically possible?

(1) M-H spacetimes are not globally hyperbolic.

Def. 4. A *globally hyperbolic spacetime* is a spacetime that admits a *Cauchy surface*.

A *Cauchy surface* is a *spacelike* surface S such that every *non-spacelike* worldline without endpoints intersects S exactly once.

- *Why is this important?* Cauchy surfaces serve as initial data surfaces and thus provide a basis for *predictability* in relativistic spacetimes.



If S is Cauchy then all non-spacelike (causal) worldlines interacting in R must register on S.

Claim: *M-H spacetimes are not globally hyperbolic.*

Proof 1: Suppose there is an M-H spacetime that is globally hyperbolic.

- Select a Cauchy surface S that contains the M-H point p.
- Extend the M-H worldline γ maximally into the past; call it γ'.
- Since γ' now has no endpoints and is timelike, it must intersect S.
- Since γ' is in $I^-(p)$, all points on it can be connected to p *via* timelike worldlines.
- In particular, the point p' where γ' intersects S can be connected to p *via* a timelike worldline.
- But but both p and p' are in S, and S is spacelike.

Proof 2: Suppose there is an M-H spacetime that is globally hyperbolic.

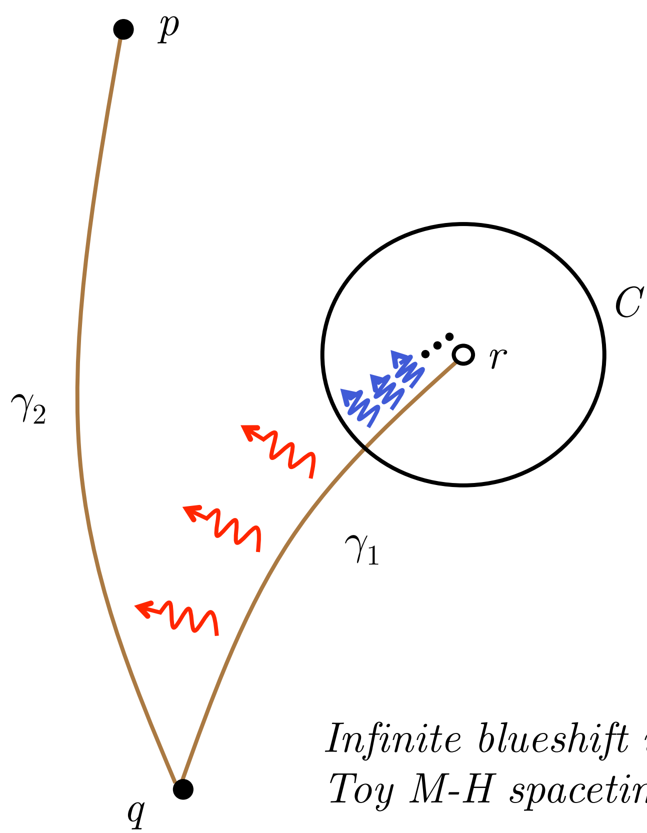
- *Lemma*: If a spacetime is globally hyperbolic and if $p \in J^+(q)$ (the *causal future* of q), for any p, q, then there is a non-spacelike worldline from q to p whose length is maximal.
- Let p be a M-H point and γ a M-H worldline with endpoint q.
- Let γ' be the maximal worldline from q to p guaranteed by the *Lemma*.
- But a longer route from q to p can always be taken by going far enough along γ before heading towards p!

Response:

- Is global hyperbolicity necessary for a spacetime to be physically possible?
- Some solutions to the Einstein equations are M-H spacetimes, and thus not globally hyperbolic:
  - *anti-de Sitter spacetime*
  - *Reissner-Nordstrom spacetime (charged black hole)*
  - *Kerr spacetime (charged, rotating black hole)*
- *Open question*: Is the spacetime that describes our universe globally hyperbolic?

(2) M-H spacetimes may entail infinite blueshifts.

- TM must send an infinite number of signals to the programmer.
- Programmer must receive these signals in a finite amount of her proper time.
- *So*: She must receive them in ever *decreasing* intervals = ever *increasing* frequencies.
- *So*: She perceives these frequencies to increase without bound.



Infinite blueshift in Toy M-H spacetime

More precisely:
The frequency ω_1 of a signal sent by γ_1 is related to the frequency ω_2 of a signal received by γ_2 by:

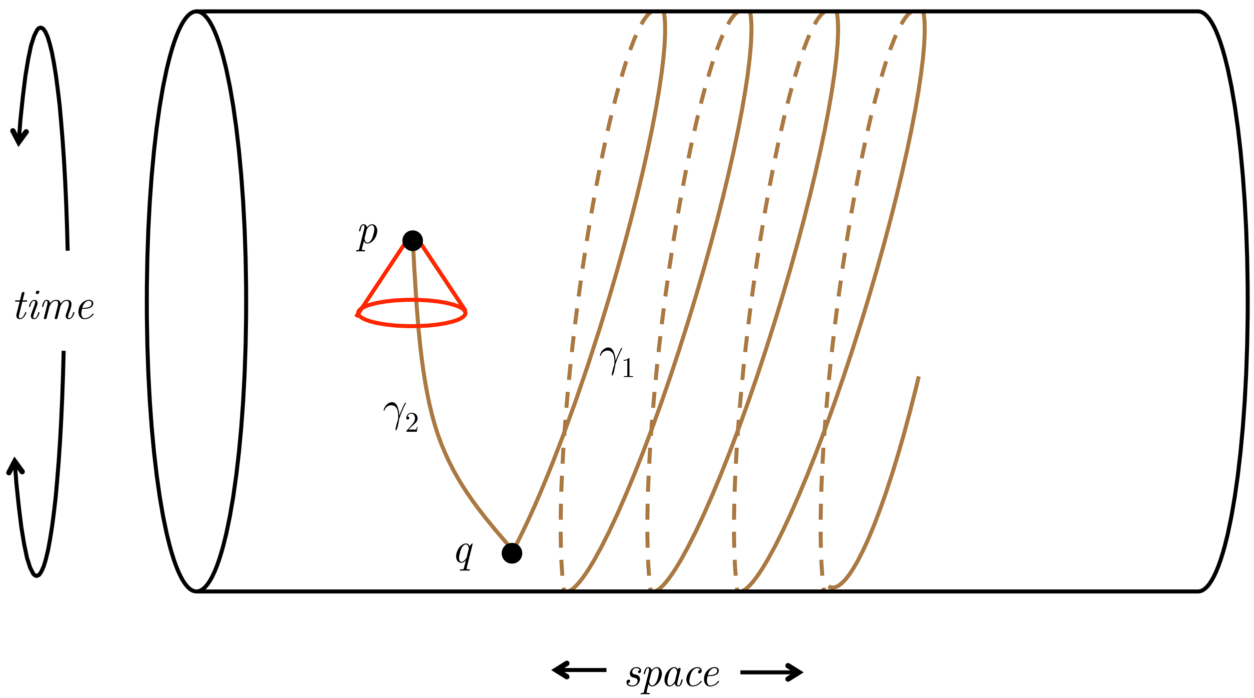
$$\int_{\gamma_1} \omega_1 \, ds = \int_{\gamma_2} \omega_2 \, ds$$

Since $\int_{\gamma_1} ds = \infty$ *and* $\int_{\gamma_2} ds < \infty$

it must be the case that $\int_{\gamma_2} \omega_2 \, ds = \infty$.

Response: Not all M-H spacetimes face this problem.

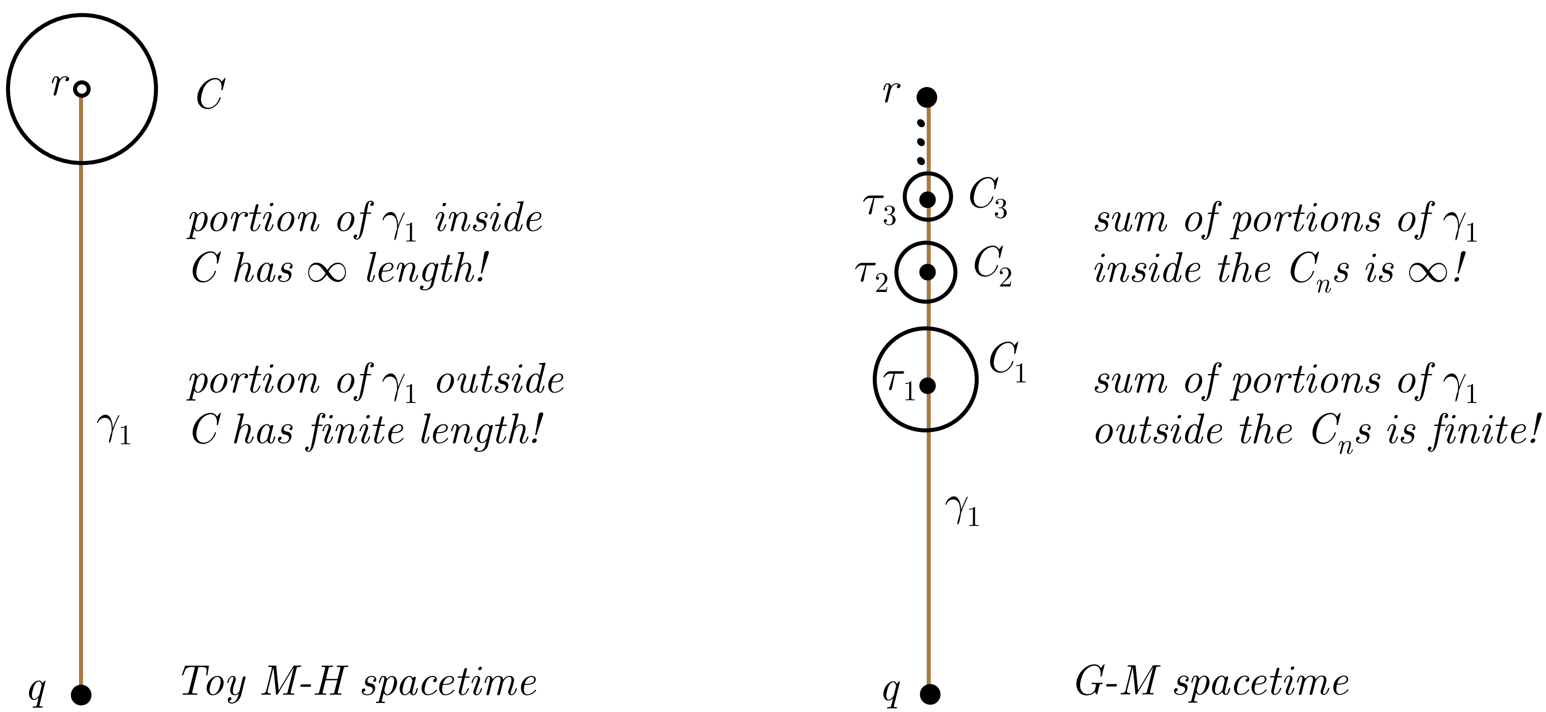
(a) Rolled-up Minkowski spacetime



- γ_1's proper length is infinite.
- Chronological past of p is the entire spacetime!
- No infinite blueshifts: signals from γ_1 to γ_2 need not increase in frequency.
- *But*: Rolled-up Minkowski spacetime has closed timelike curves (time travel!). Are such spacetimes physically realistic?

(b) Geroch-Malament (G-M) spacetime

- *Recall*: In Toy M-H spacetime γ_1 is guaranteed to have ∞ length due to the scalar field Ω in the interior of the sphere C centered at the missing point r.
- *But*: The frequencies of signals sent from γ_1 inside C blow up as r is approached.
- *Strategy of G-M spacetime*: Guarantee that γ_1 has ∞ length by constructing scalar fields for spheres centered at an infinite sequence of points τ_1, τ_2, τ_3, ... along γ_1. Signals sent from points *not* in this sequence will not blow up!



How to construct G-M spacetime.

Step 1. Let $(\tau_1, \tau_2, ...)$ be an infinite sequence of points on γ_1 given by $\tau_n = 1 - (3/4)(1/2)^n$.

Step 2. At each point τ_n, construct a sphere C_n of radius $r_n = (1/2)^{n+3}$.

Step 3. In the interior of each sphere C_n, construct a scalar field Ω_n such that

(i) Ω_n goes to 1 as r_n is approached, and reaches it maximum value as τ_n is approached; and

(ii) the length of γ_1 inside C_n is equal to 1.

Consequences:

- The total length of the portion of γ_1 in all the spheres is ∞.
- The total length of the portion of γ_1 outside the spheres is finite.
- Signals sent from the points of γ_1 where its length is finite will *not* be blueshifted.
- In particular, signals sent from the sequence of points (1/2, 3/4, 7/8, ...) will not be blueshifted.
- *But*: How physically realistic is a G-M spacetime?

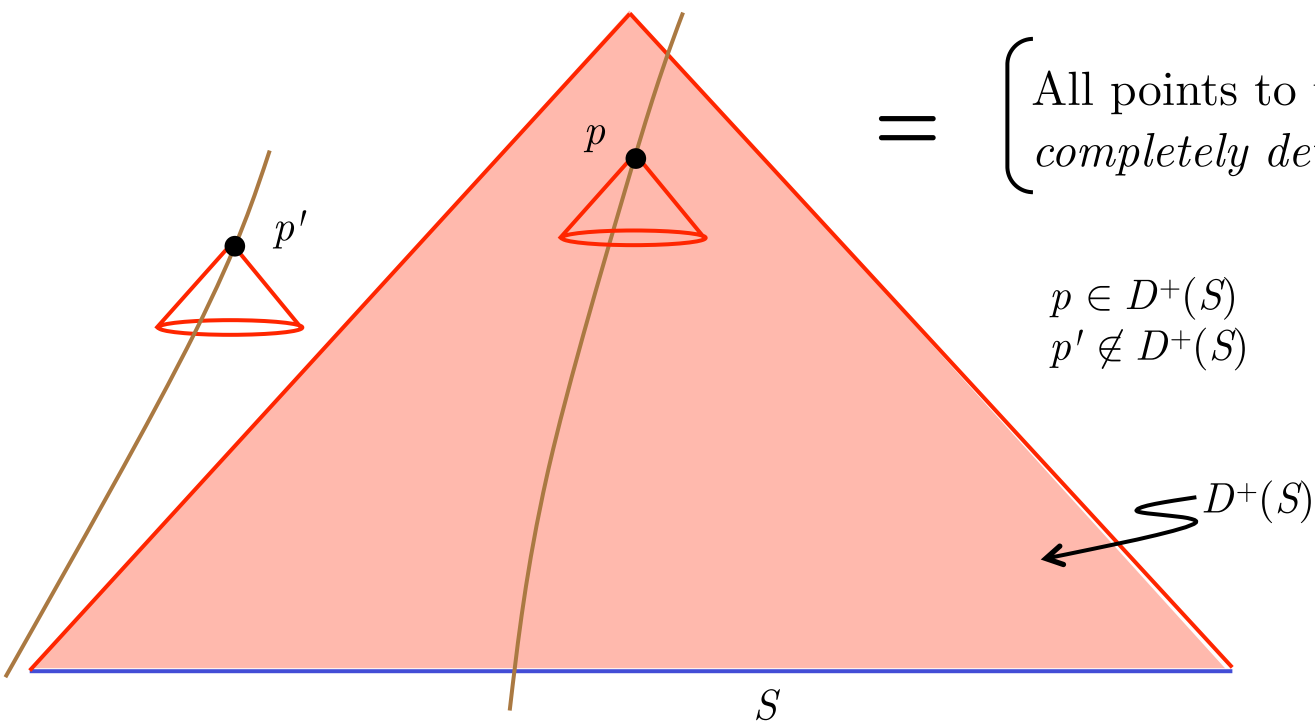
(3) M-H spacetimes may violate the Cosmic Censureship Hypothesis (CCH).

<u>CCH (Penrose 1974)</u>: "Naked" singularities do not develop in physically reasonable solutions to the Einstein equations.

- Naked singularities entail a breakdown of *determinism*.
- Let S be a spacelike surface.

(The *future domain of dependence* $D^+(S)$ of S) = (All points p such that every non-spacelike worldline through p with no future endpoint intersects S exactly once)

= (All points to the future of S that are *completely determined* by data on S.)



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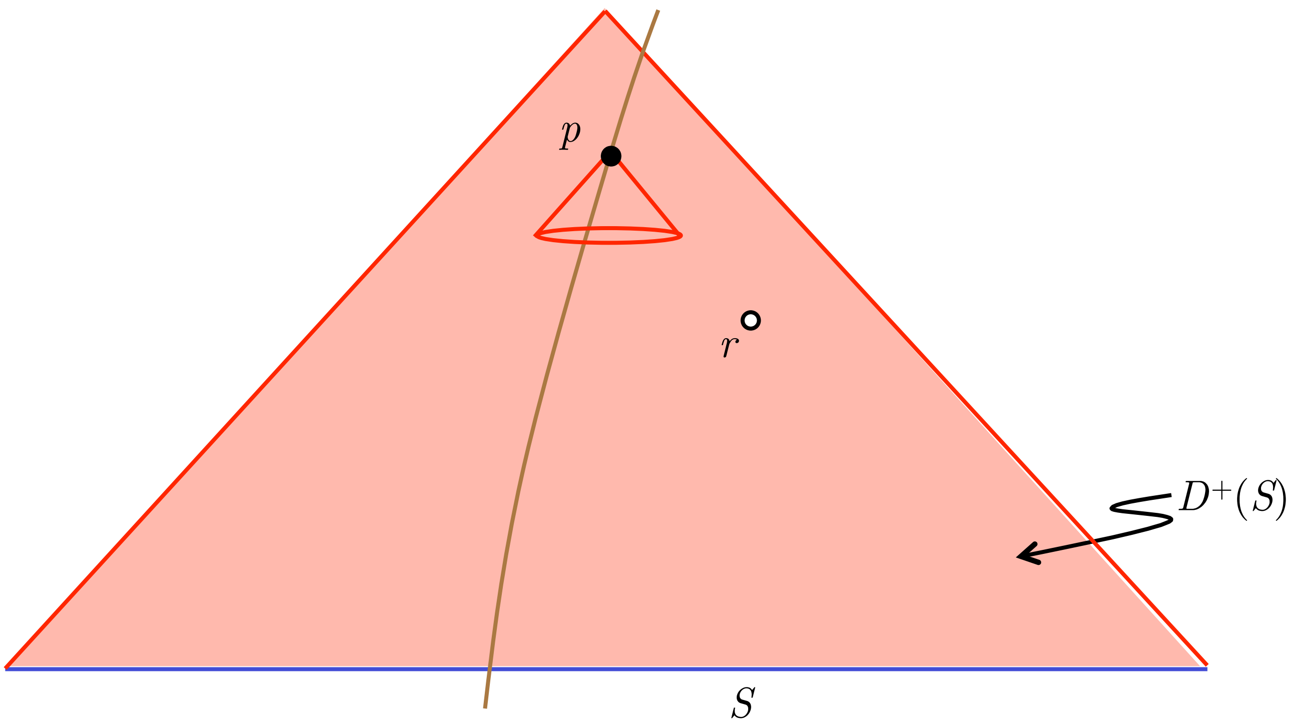

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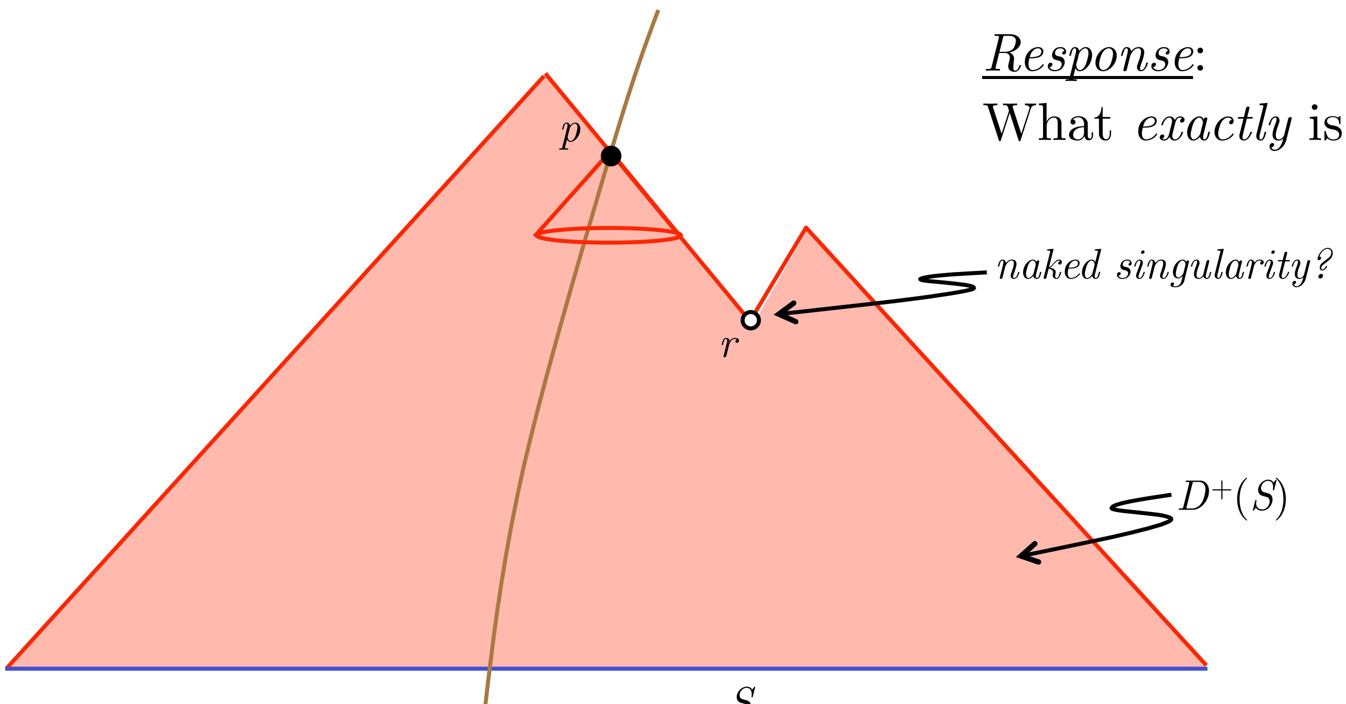
<u>CCH (Penrose 1974)</u>: "Naked" singularities do not develop in physically reasonable solutions to the Einstein equations.

- Naked singularities entail a breakdown of *determinism*.
- Remove r.
- p is no longer in $D^+(S)$!
- Causal influences that do not register on S can now affect p.



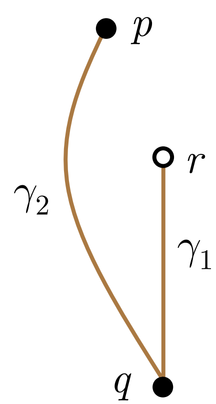
Response:
What *exactly* is a naked singularity?

3. Non-Turing Computability

- A *simple infinity TM* is a TM that is allowed to complete an infinite number of steps. Call it TM_∞.
- A TM following a M-H worldline is a TM_∞.

The Big Difference between a regular TM and a TM_∞:

- All TMs either halt or fail to halt.
- If a regular TM fails to halt, this could mean either that it may or may not halt.
- If a TM_∞ fails to halt, this *definitely* means it will never halt.



At M-H point p, programmer knows whether TM has halted or failed to halt. If the latter, this definitely means the TM will never halt.

Toy M-H spacetime

What can be solved by a TM_∞

(i) The halting problem.

- Does arbitrary TM T_t halt on input n?
- Run T_t on a TM_∞:
  - If the TM_∞ halts, T_t halts.
  - If the TM_∞ fails to halt, T_t does not halt.

(ii) The decision problem for purely existential and purely universal statements in arithmetic.

Ex: Fermat's Last Theorem.

- "For $n \geq 3$, there are no x, y, z, such that $x^n + y^n = z^n$."
- *Or*: $\forall x \forall y \forall z \forall n \neg F(x, y, z, n)$, where $F(x, y, z, n)$ means "x, y, z, n are natural numbers, and $n \geq 3$, and $x^n + y^n = z^n$".

- *Note*: Any statement (in prenex form) with two adjacent quantifiers of the same type is equivalent to a statement with a single quantifer of that type in place of these two.
- *Thus*: Any purely existential/universal statement can be rewritten as a statement with a *single* quantifier.

For purely universal statements $\forall x F(x)$:

(i) For each number x, determine if F holds.

(ii) If F holds, then go to $x + 1$.

(iii) If F does not hold, halt.

Consequence:

- If TM_∞ halts, then statement is false.
- If TM_∞ fails to halt, then statement is true.

For purely existential statements $\exists x F(x)$:

(i) For each number x, determine if F holds.

(ii) If F holds, then halt.

(iii) If F does not hold, then go to $x + 1$.

Consequence:

- If TM_∞ halts, then statement is true.
- If TM_∞ fails to halt, then statement is false.

- *But*: What about *mixed* quantifier statements?
- *Ex*: $\exists x \forall y R(x, y)$, which we can take to mean "There's an ultimate number x which stands in the relation R to all numbers."

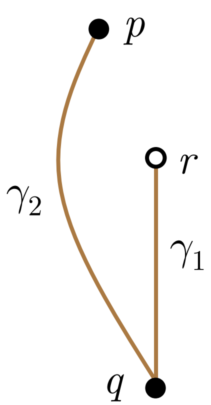
(i) For each x, check to see if there's an y for which R doesn't hold.
(ii) If so, then go to $x + 1$ and start checking y's again.

Consequence:

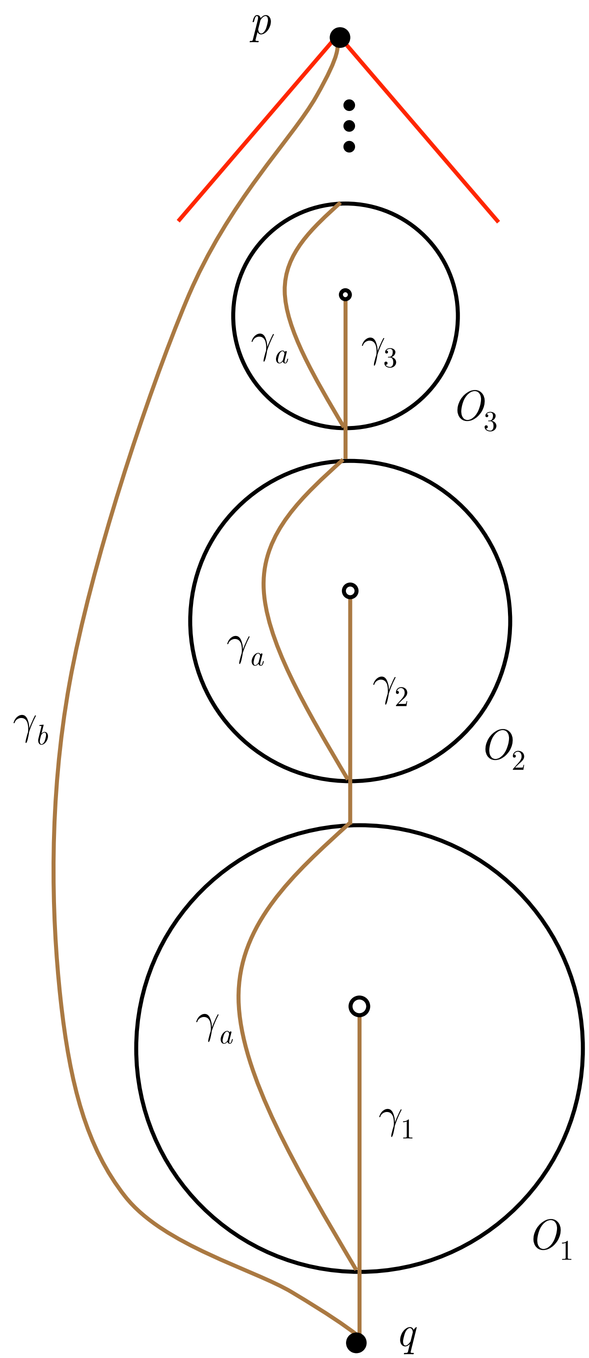
- The TM_∞ will fail to halt, but what does this mean?
  - *Either* the x currently being checked against all numbers is the ultimate (so the statement is true).
  - *Or* the TM_∞ is still checking candidates for x (so the statement may be false).
- *Thus*: The truth of the statement cannot be determined.
- *So*: A *single* TM_∞ cannot solve the decision problem for *arbitrary* statements in arithmetic.

What about using an infinite number of TM_∞'s?

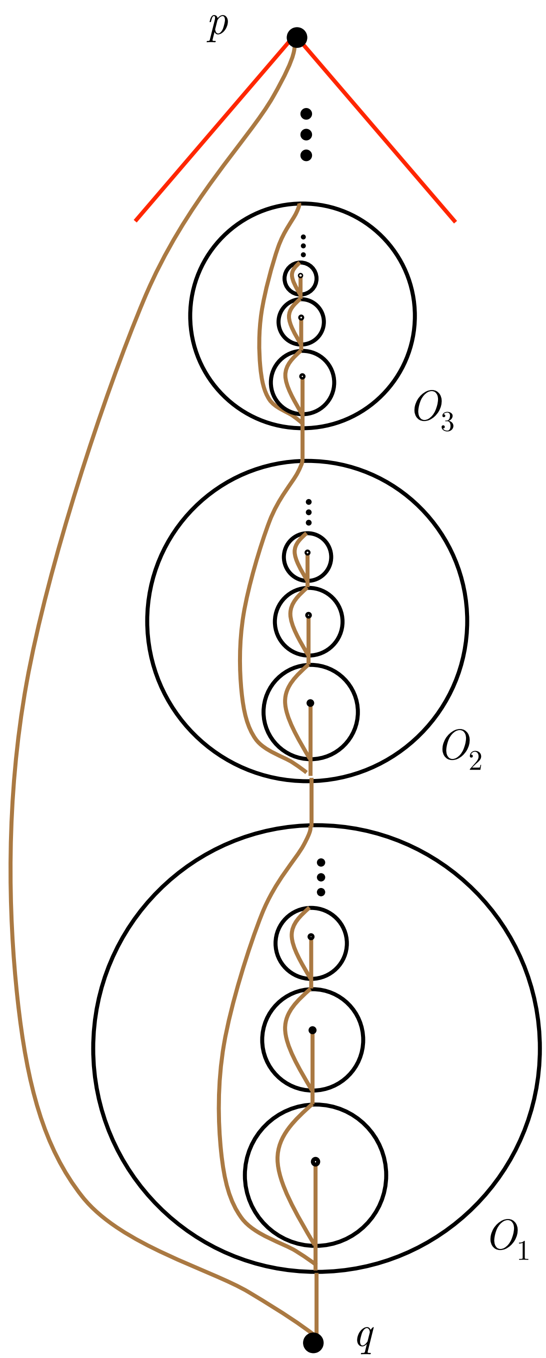
- *Idea*: To decide the truth of $\exists x \forall y R(x, y)$,
  - TM^1_∞ checks $\forall y R(1, y)$, TM^2_∞ checks $\forall y R(2, y)$, ...
  - Collect results using another TM_∞.
- Can we devise a spacetime that can support such a computation (Hogarth 1994)?



- M-H spacetime.
- γ_1 is a single TM_∞.
- γ_2 is the programmer.
- Decides *singly*-quantified statements $\exists x R(x)$, $\forall x R(x)$.



- Each region O_i is a M-H spacetime such that, for all i
  (a) $O_i \subset I^-(O_{i+1})$.
  (b) $O_i \subset I^-(p)$.
- Each γ_i is a TM_∞ that determines $\forall y R(i, y)$.
- γ_a is another TM_∞ that collects these results to determine $\exists x \forall y R(x, y)$.
- γ_b is the programmer.
- Decides *doubly*-mixed quantifier statements $\exists x \forall y R(x, y)$, $\forall x \exists y R(x, y)$.
- Call this a SAD_2 (2*nd-order arithmetical sentence deciding*) spacetime.



- Each region O_i is a SAD_2 spacetime such that, for all i
  (a) $O_i \subset I^-(O_{i+1})$.
  (b) $O_i \subset I^-(p)$.
- Decides *triply*-mixed quantifier statements $\forall x \exists y \forall z R(x, y, z)$, $\exists x \forall y \exists z R(x, y, z)$).
- Call this a SAD_3 spacetime.

A SAD$_n$ (nth-order arithmetical sentence deciding) spacetime = a 4-dim collection of points such that:

(1) If $n = 1$, the spacetime is M-H.

(2) If $n > 1$, the spacetime admits a *string of SAD$_{n-1}$ spacetimes*: a collection of non-intersecting open regions O_i, $i = 1, 2, \ldots$, such that

  (a) Each O_i is in the chronological past of the next O_{i+1}.

  (b) There is a point p such that all regions are in the chronological past of p.

  (c) Each region O_i is a SAD$_{n-1}$ spacetime.

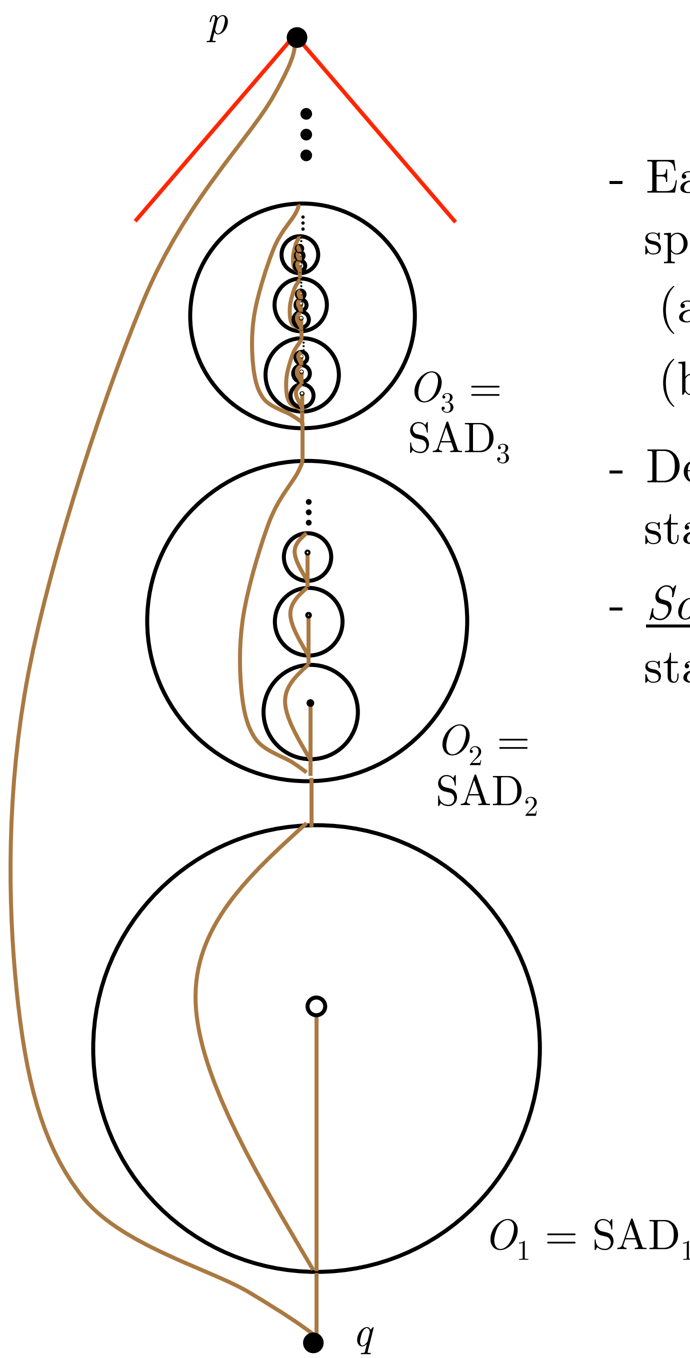
An AD (arithmetical sentence deciding) spacetime = a 4-dim collection of points that admits a string of open regions O_i, $i = 1, 2, \ldots$, such that:

(a) Each O_i is in the chronological past of the next O_{i+1}.

(b) There is a point p such that all regions are in the chronological past of p.

(c) For each $n \geq 1$, O_n is a SAD$_n$ spacetime.

An AD spacetime



- Each region O_i is a SAD$_i$ spacetime such that, for all i
  (a) $O_i \subset I^-(O_{i+1})$.
  (b) $O_i \subset I^-(p)$.
- Decides mixed quantifier statements of all orders.
- <u>*So*</u>: Decides truth of arbitrary statements in arithmetic!

Recap:

- A TM operating in a classical spacetime cannot decide (any) quantifier statements in arithmetic.
- A TM_∞ operating in a M-H spacetime can decide purely existential/ universal statements in arithmetic.
- A configuration of TM_∞'s operating in a SAD_n spacetime, $n > 1$, can decide nth-order mixed quantifier statements in arithmetic.
- A configuration of TM_∞'s operating in an AD spacetime can decide all types of quantifier statements (of arbitrary mixed order) in arithmetic.

- *Thus*: The decision problem for arithmetic is solvable in an AD spacetime.

Consequences for the Concept of Computability:

> *The Church-Turing Thesis*:
> Turing computability = effective computability.

Question 1: What is "Turing computability"?

- TM computable (computability in classical spacetimes)
- TM_∞ computable
- SAD_n computable
- AD computable

$\left.\right\}$ (three types of computability in relativistic spacetimes)

- Are these *distinct* types of computing devices?
- Or are they all simply regular TMs operating in different types of spacetimes?

Question 2: How do physical constraints affect the concept of computability?

- Do physical constraints rule in favor of TM computability?

> *Possible claim*: TM_∞-, SAD_n-, and AD-computability are only possible in highly idealized relativistic spacetimes.

- *But*: Ordinary TMs are themselves ideal concepts: they assume an infinite amount of memory!

> *The Physical Church-Turing Thesis*:
> Turing computability = an *upper bound* on effective computability.

- Should this idealized upper bound be replaced by one associated with relativistic spacetimes?
- *Practical response*: The spacetime arena in which real computers (designed by computer engineers) operate is, for all intents and purposes, a classical spacetime.
- *But*: If the issue involves formulating a *theoretical* notion of "computability"; *i.e.*, what in-principle can and cannot be computed, then practical issues hold lesser sway.