II. Classical and Relativistic Spacetimes.

<u>*Motivation:*</u> Turing solvable problems require a TM to halt after a finite number of steps with a given output.

- What if we allow TMs to perform an *infinite* number of steps?
  - <u>*Then*</u>: *Some Turing unsolvable problems may become solvable!*
  - <u>*But*</u>: *How could puny finite humans access the output of a TM that performs an infinite number of steps?*

<u>*Answer to come*</u>: Place the puny human in a spacetime that:

(a) Allows the TM to live to infinity in its rest frame; and

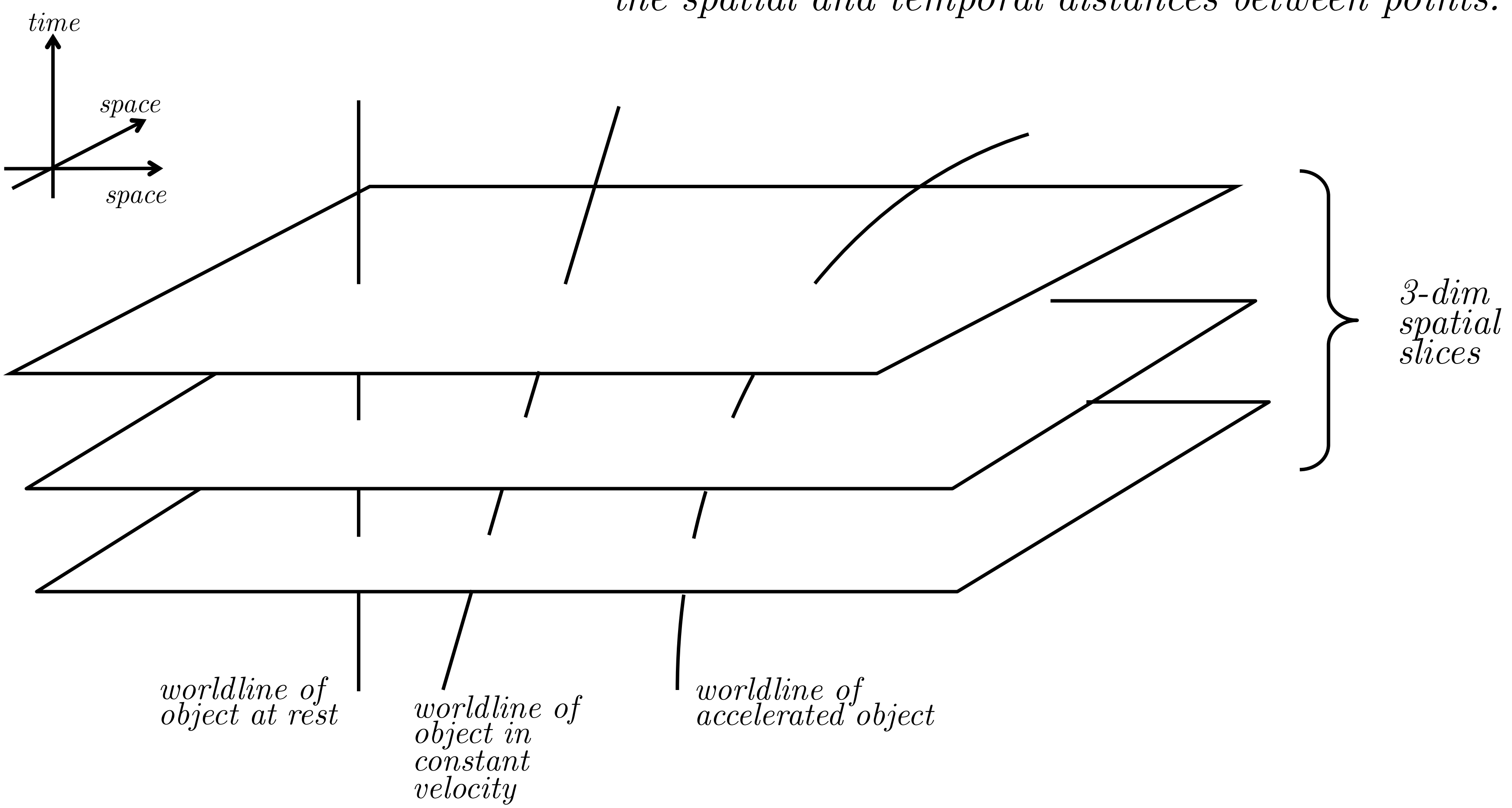
(b) Allows the human to access the TM's output in a finite amount of time.

- <u>*Mathematically*</u>: A matter of determining the appropriate curved geometry for the given spacetime.
- <u>*Physically*</u>: Are such appropriately curved spacetimes physically possible?

1. Types of Spacetimes

- A *spacetime* is a 4-dim collection of points with *additional structure*.

Typically, one or more metrics = a specification of the spatial and temporal distances between points.

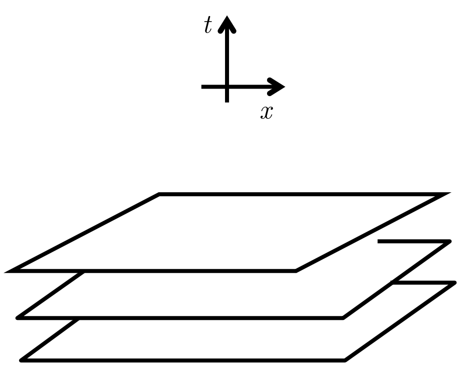


Two ways spacetimes can differ:

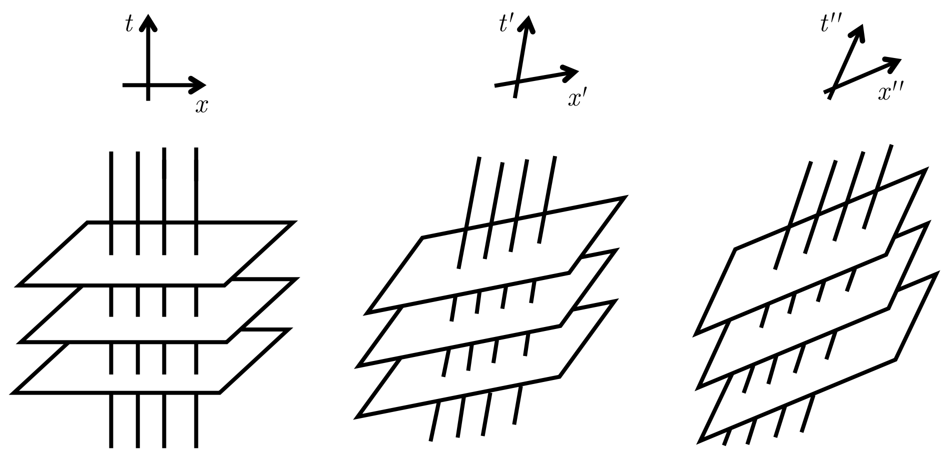
(1) Different ways of specifying distances between points yield different types of spacetimes.

- *Classical spacetimes* have *separate* spatial and temporal metrics: only one way to split time from space (spatial and temporal distances are *absolute*).

- *Relativistic spacetimes* have a *single* spatiotemporal metric, and how it gets split into spatial and temporal parts depends on one's inertial reference frame (spatial and temporal distances are *relative*).



Classical spacetime: only one way to split time from space.

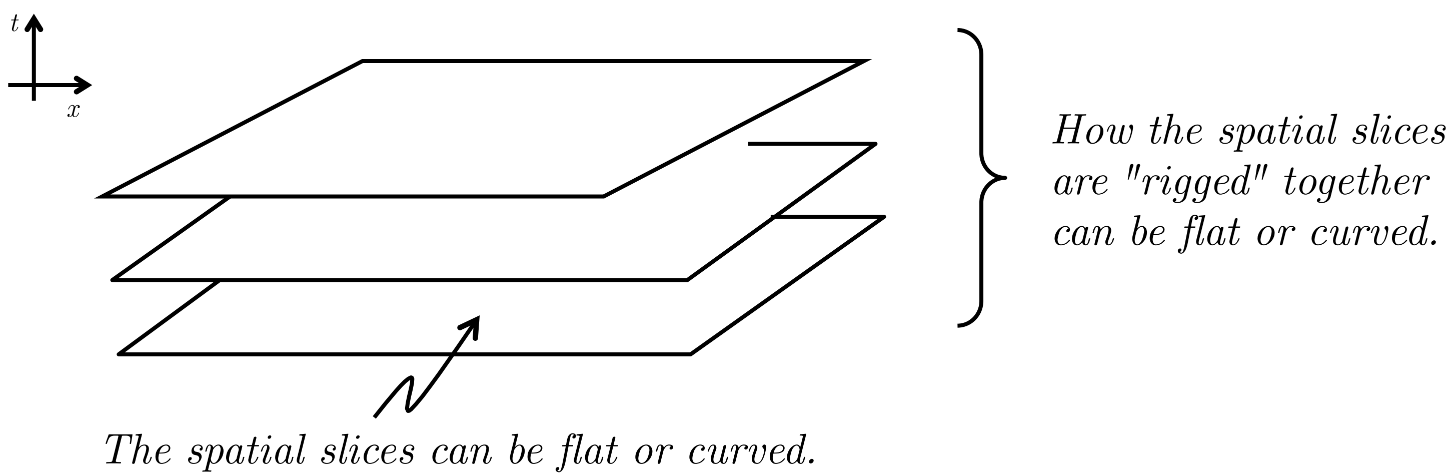


Relativistic spacetime: many ways to split time from space.

(2) Metrics can be *flat* or *curved*: how one specifies the distance between points encodes the curvature of the spacetime.

- *Classical spacetimes* can be flat or curved.
- *Relativistic spacetimes* can be flat (Minkowski spacetime) or curved (general relativistic spacetimes).

• Two ways curvature can manifest itself:



2. Classical Spacetimes

Newtonian spacetime is a 4-dim collection of points such that:

(N1) Between any two points p, q with coordinates (t,x,y,z) and $(t+\Delta t, x+\Delta x, y+\Delta y, z+\Delta z)$ there is a definite *temporal interval* $T(p,q) = \Delta t$.

(N2) Between any two points p, q with coordinates (t,x,y,z) and $(t+\Delta t, x+\Delta x, y+\Delta y, z+\Delta z)$ there is a definite *Euclidean distance*

$$R(p,q) = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

(N1) and (N2) entail:

(a) All worldlines have a definite *absolute velocity*.

> For worldline γ, and any two points p, q on γ, the *absolute velocity of* γ *with respect to* p, q can be defined by $R(p,q)/T(p,q)$.

(b) There is a privileged collection of worldlines defined by $R(p,q)/T(p,q) = 0$.

(c) All worldlines have a definite *absolute acceleration*.

> For worldline γ, and points p, q on γ, the *absolute acceleration of* γ *with respect to* p, q can be defined by $d/dt\{R(p,q)/T(p,q)\}$.

Newton's absolute space!

But absolute space and absolute velocity are unobservable!

Neo-Newtonian spacetime is a 4-dim collection of points such that:

(NN1) Between any two points p, q with coordinates (t,x,y,z) and $(t+\Delta t, x+\Delta x, y+\Delta y, z+\Delta z)$ there is a definite *temporal interval* $T(p,q) = \Delta t$.

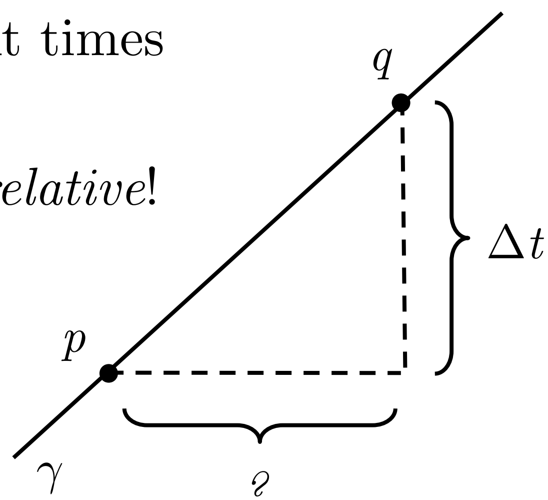
(NN2) Between any two *simultaneous* points p_t, q_t with coordinates (t,x,y,z) and $(t, x+\Delta x, y+\Delta y, z+\Delta z)$ there is a definite *Euclidean distance*,

$$R(p_t, q_t) = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

(NN3) Any worldline γ through a point p has a definite curvature $S(\gamma, p)$.

(NN2) entails:

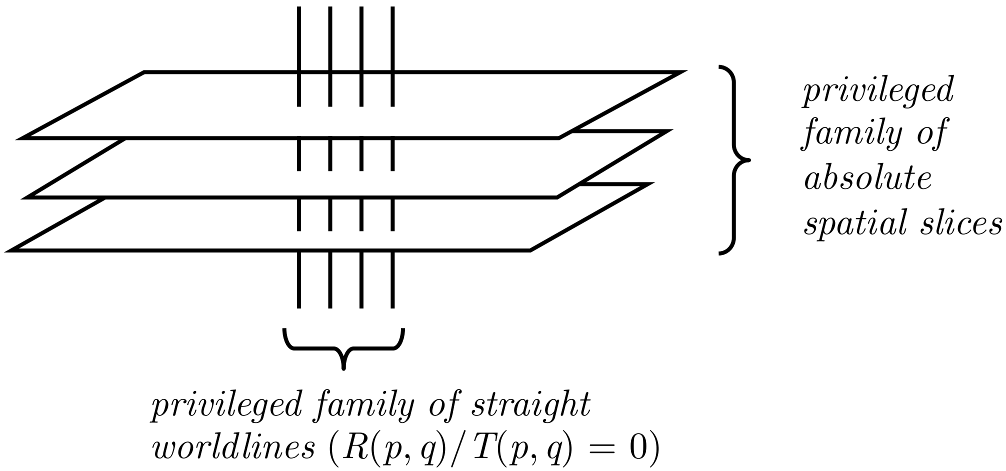
- No absolute spatial distance between points at different times on any worldline γ.
- *So*: No absolute velocity for any worldline: *velocity is relative*!
- *So*: *No single privileged inertial reference frame.*



(NN3) entails: acceleration remains absolute!

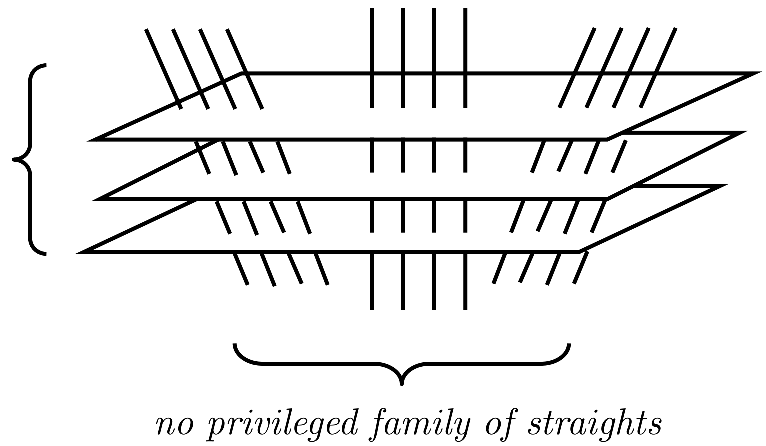
For worldline γ and point p on γ, the *absolute acceleration of γ with respect to p* is given by $S(\gamma, p)$.

Newtonian Spacetime



1. Single, privileged inertial frame.
2. Velocity is absolute.
3. Acceleration is absolute.
4. Simultaneity is absolute.

Neo-Newtonian Spacetime



1. Many inertial frames; none privileged.
2. Velocity is relative.
3. Acceleration is absolute.
4. Simultaneity is absolute.

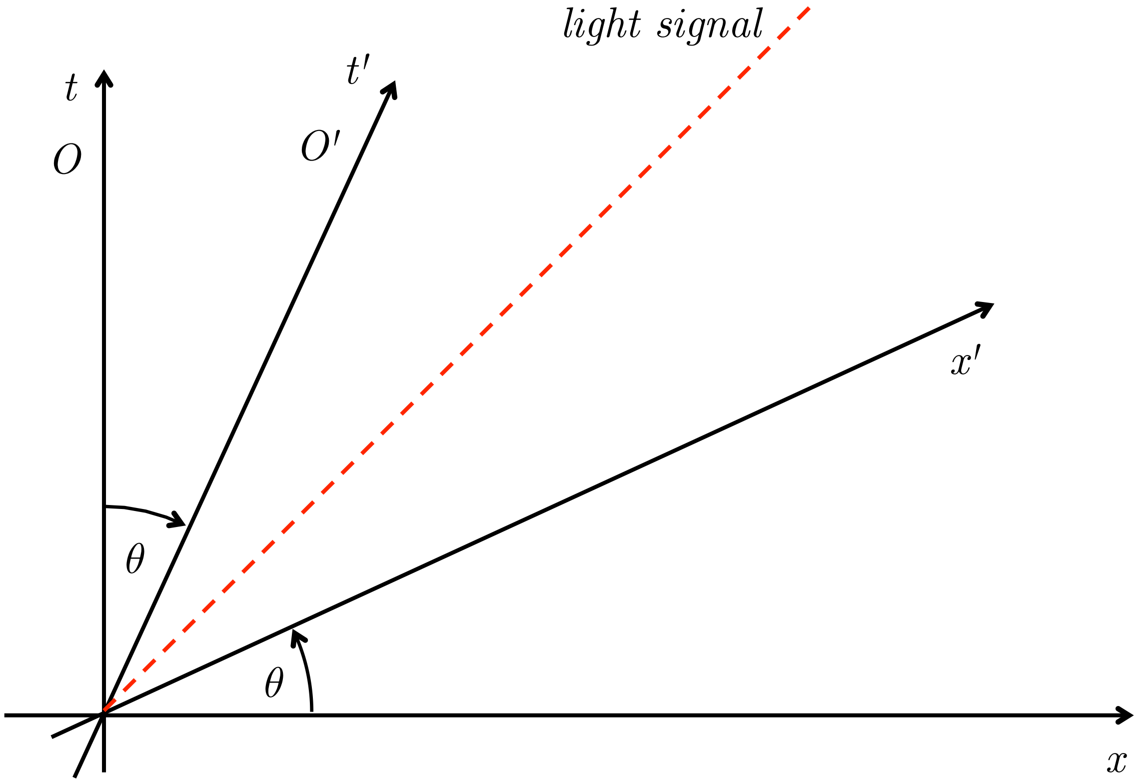
- Both Newtonian spacetime and Neo-Newtonian spacetime have absolute temporal metrics: *Everyone agrees on what time it is.*
- Relativistic spacetimes have no absolute temporal metric: *What time it is depends on your inertial reference frame.*

3. Relativistic Spacetimes

- Light Postulate of Special Relativity entails:

The speed of light c is the same in all inertial reference frames.

Albert Einstein
(1879-1955)

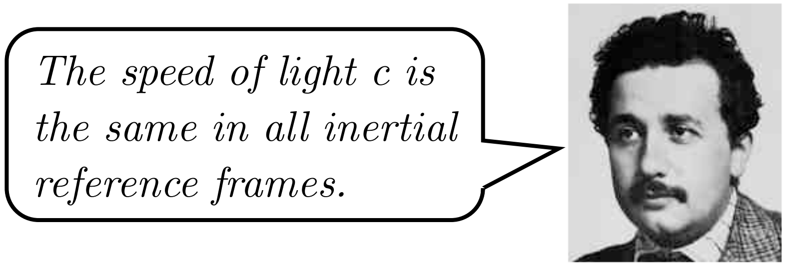


$$c = \frac{\Delta x}{\Delta t} = \frac{\Delta x'}{\Delta t'}$$

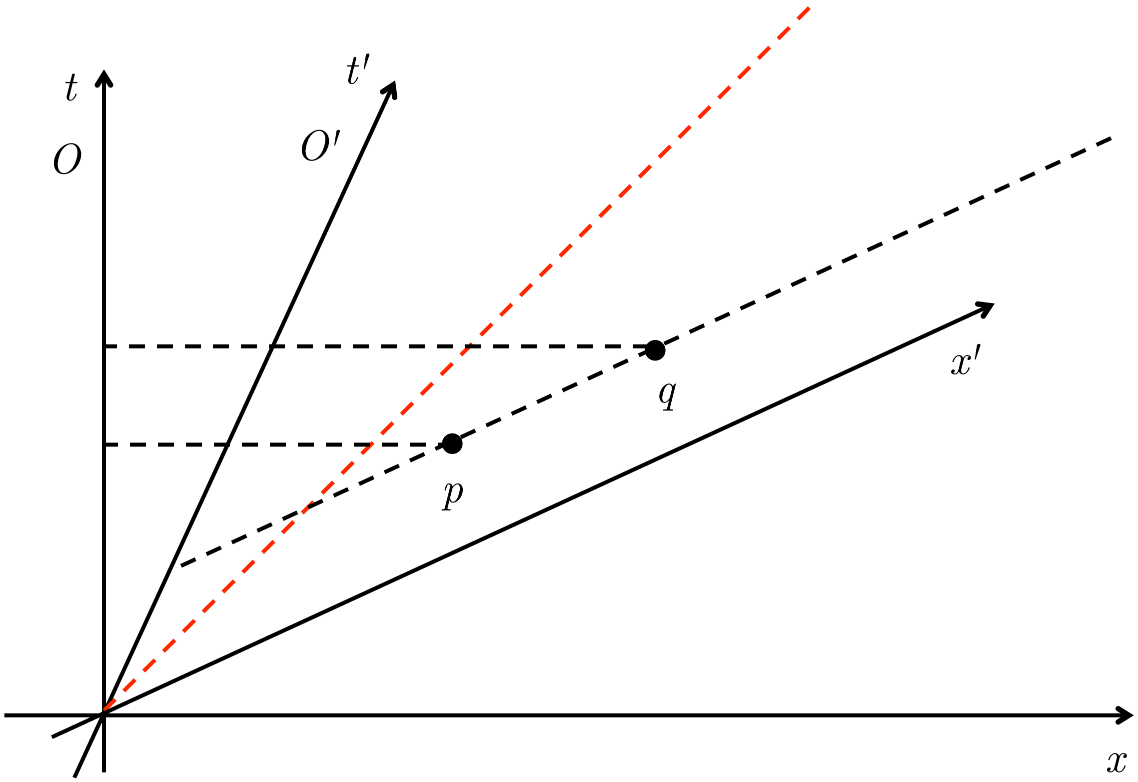
value of c for O | *value of c for O′*

- Object O' is moving at constant velocity with respect to object O.
- O and O' must measure same speed c for light signal.
- So: The x' axis must be inclined by the same amount θ from the x axis as the t' axis is inclined from the t axis.
- Thus: O and O' must disagree on spatial and temporal measurements!

3. Relativistic Spacetimes



- Light Postulate of Special Relativity entails:



- O and O' make different judgements of simultaneity (*relativity of simultaneity*).
- p and q are simultaneous according to O'.
- p happens before q according to O.

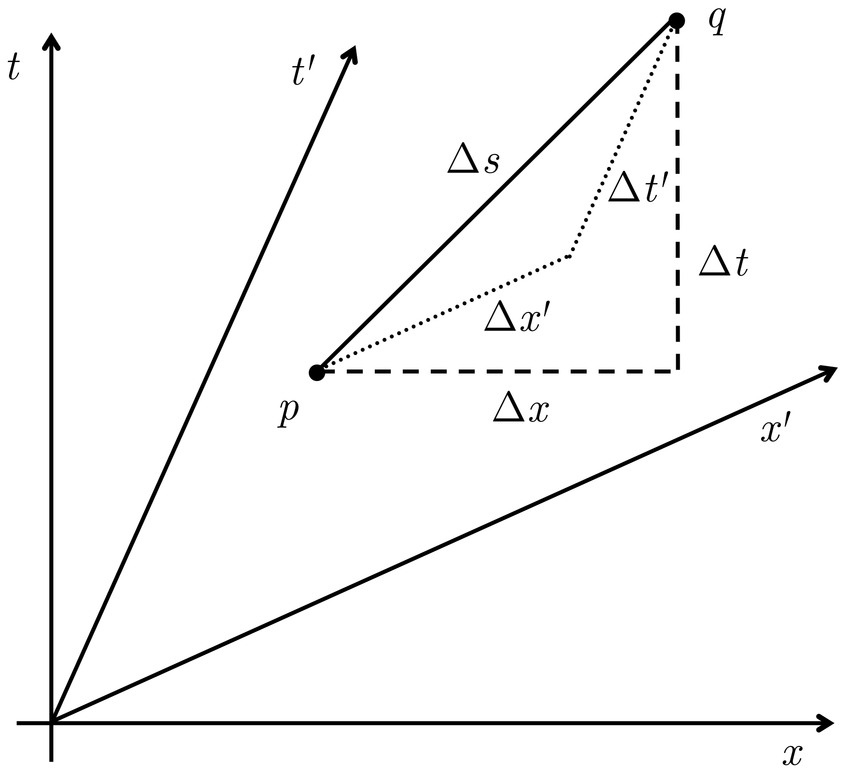
Spacetime of Special Relativity = Minkowski spacetime

Hermann Minkowski (1864-1909)

Minkowski spacetime = 4-dim collection of points such that between any two points p, q with coordinates (t, x, y, z) and $(t+\Delta t,\ x+\Delta x,\ y+\Delta y,\ z+\Delta z)$ there is a definite spacetime interval given by

$$\Delta s = \sqrt{-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}.$$

- Similar to Euclidean *spatial* interval $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$.
- *But*: Includes the time coordinate difference, too! And it's *negative*!
  - *Idea*: All inertial frames will agree on the spatiotemporal distance Δs between any points p and q.
  - But they will disagree on how Δs gets split into a temporal part and a spatial part: they will disagree on measurements of time and measurements of space.



$$\Delta s = \sqrt{-(c\Delta t)^2 + (\Delta x)^2}$$

$$= \sqrt{-(c\Delta t')^2 + (\Delta x')^2}$$

- All inertial frames agree on the *spacetime* distance between any two points p and q.
- They will disagree on the *temporal* distance between p and q (time dilation) and on the *spatial* distance (length contraction).
  - *They will disagree on how they split the spacetime distance into temporal and spatial parts.*

The Minkowski spacetime interval is encoded in the *Minkowski metric* $\eta_{\mu\nu}$

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$
$$= \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

$\Delta x^0 = c\Delta t\,,\ \Delta x^1 = \Delta x\,,$
$\Delta x^2 = \Delta y\,,\ \Delta x^3 = \Delta z\,,$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Infinitesimally: $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

Three forms of $(\Delta s)^2$

(a) *Timelike.* $(\Delta s)^2 < 0$, or: $\dfrac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{\Delta t} < c$

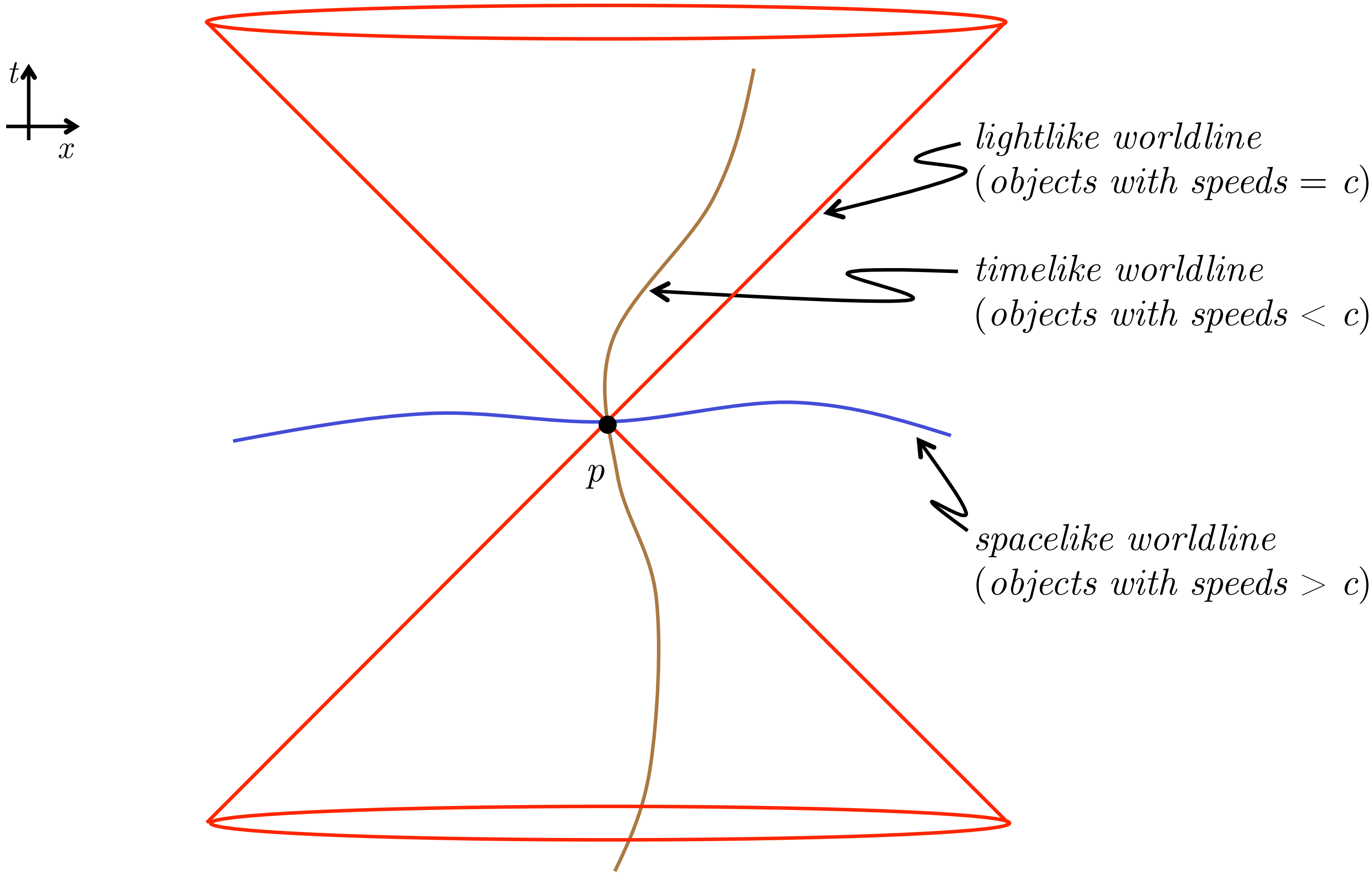
(b) *Lightlike.* $(\Delta s)^2 = 0$, or: $\dfrac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{\Delta t} = c$

(c) *Spacelike*: $(\Delta s)^2 > 0$, or: $\dfrac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{\Delta t} > c$

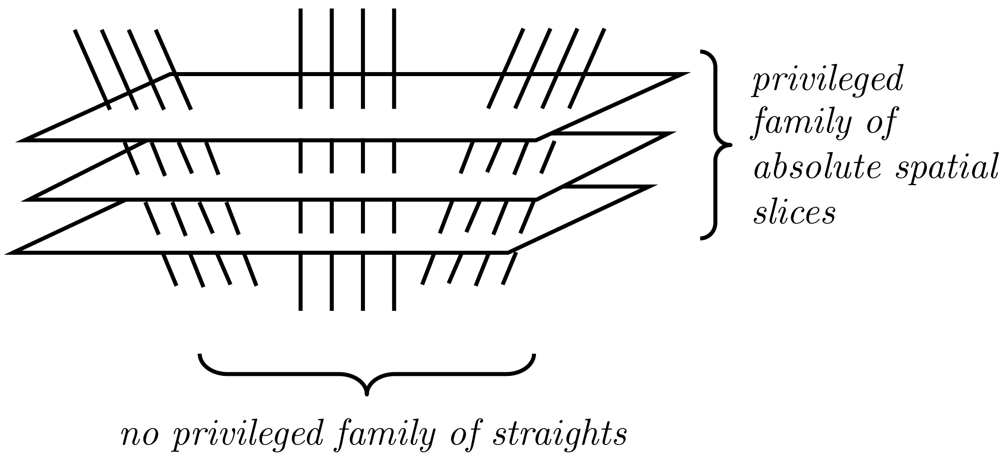
Three different types of worldline in Minkowski spacetime!

- *Absolute distinction*: All inertial frames agree on Δs, so all inertial frames agree on which worldlines are timelike, lightlike, and spacelike!

Hence: The Minkowski metric defines a *lightcone* at any point p:

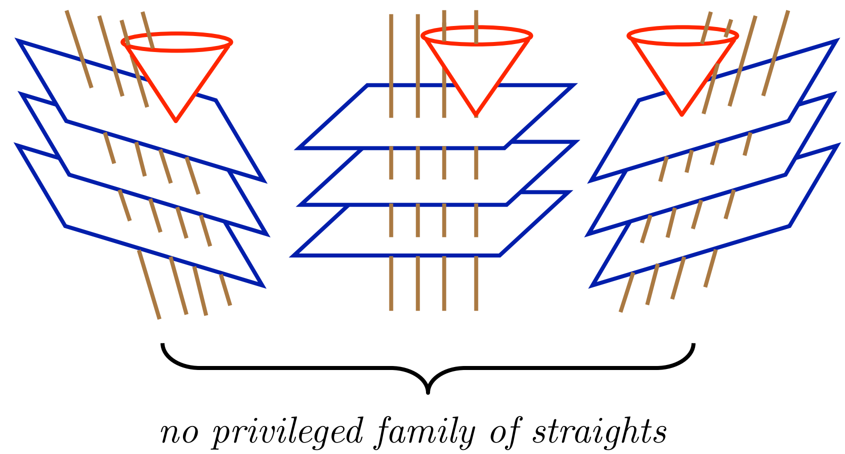


Neo-Newtonian Spacetime



1. Many inertial frames; none privileged.
2. Velocity is relative.
3. Acceleration is absolute.
4. Simultaneity is absolute.

Minkowski Spacetime



1. Many inertial frames; none privileged.
2. Velocity is relative.
3. Acceleration is absolute.
4. Simultaneity is relative.
5. Invariant light-cone structure at each point.

Problem: Special relativity does not account for the gravitational force.

- To include gravity...

Geometricize it! Make it a feature of spacetime geometry.

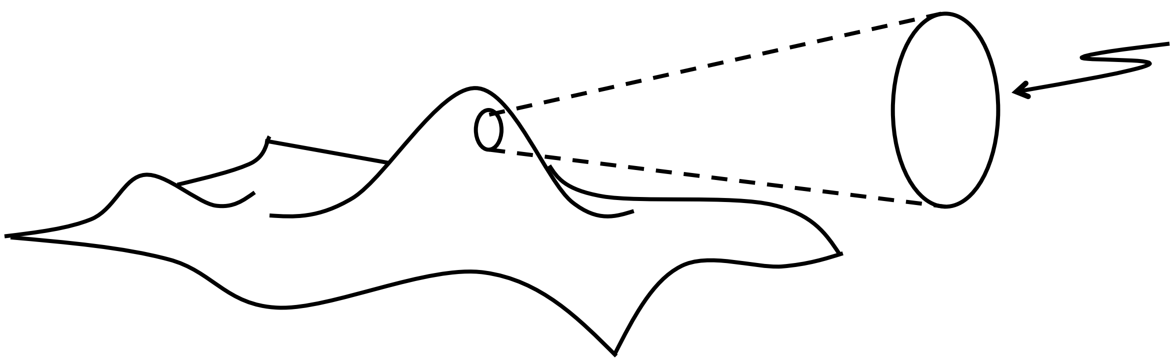
Two requirements:

(1) New theory ("general relativity") must reduce to special relativity in sufficiently flat regions of spacetime:

- Replace $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ with $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

flat Minkowski metric *non-flat metric*

- Require $g_{\mu\nu}$ to reduce to $\eta_{\mu\nu}$ in small regions of spacetime.



arbitrarily curved surface

<u>Problem</u>: Special relativity does not account for the gravitational force.

- To include gravity...

> *Geometricize it! Make it a feature of spacetime geometry.*

<u>Two requirements</u>:

(2) Curvature of spacetime must be related to matter density:

- The Einstein equations (1916):

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

Einstein tensor = encodes curvature of spacetime as a function of $g_{\mu\nu}$

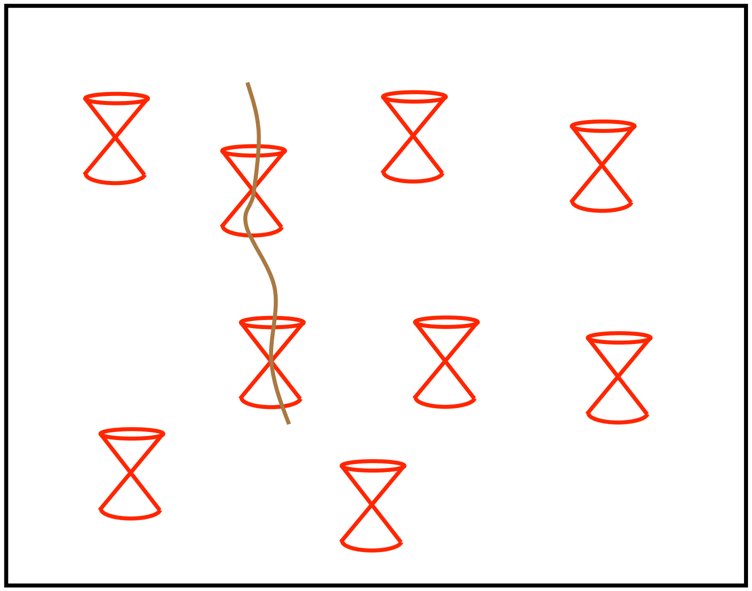
Stress-energy tensor = encodes matter density

- *<u>Consequence</u>*: The Minkowski metric is the solution for zero curvature $G_{\mu\nu} = 0$ (*i.e.*, spatiotemporal flatness).

<u>*A general relativistic spacetime*</u> = 4-dim collection of points such that between any two (infinitesimally close) points, there is a definite spacetime interval given by $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, where $g_{\mu\nu}$ is a <u>*Lorentzian*</u> metric that satisfies the Einstein equations.

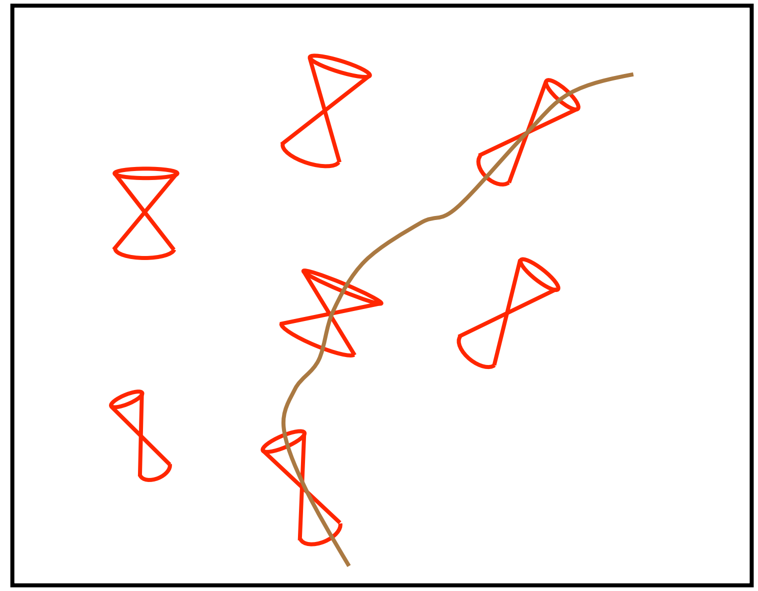
"reduces to the Minkowski metric at any point"

Minkowski Spacetime



Invariant light-cone structure at each point: light-cones all have same size and orientation.

Arbitrary General Relativistic Spacetime



Light-cone structure at each point is not invariant: light-cones may not have same size and orientation due to curvature.

- *Idea*: The light-cone structure constrains the motion of physical objects (traveling on timelike worldlines).
- *And*: In an arbitrary general relativistic spacetime, the matter density determines the metric, which determines the light-cone structure.