10. The Einstein Equations

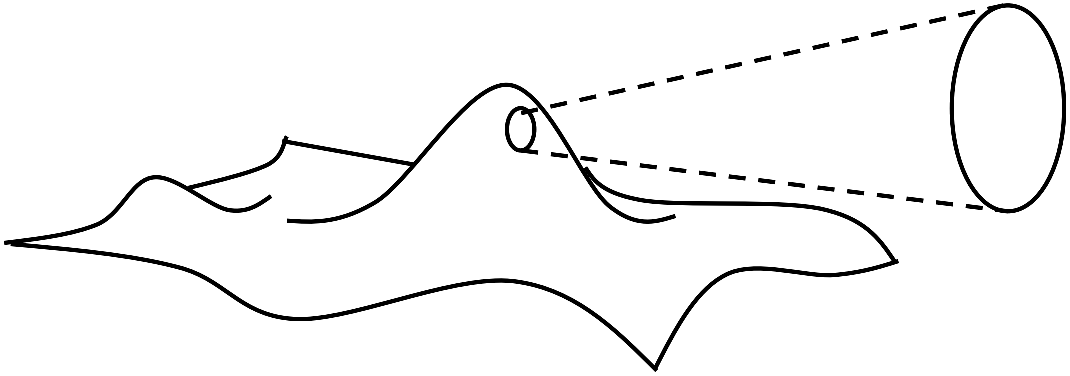
Two requirements to geometrize the gravitational force in a relativistic spacetime:

(I) *Relativistic theory*: Our theory must get everything right that special relativity gets right.

- Replace $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ with $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

  flat Minkowski metric — non-flat metric

- Require $g_{\mu\nu}$ to reduce to $\eta_{\mu\nu}$ in small regions of spacetime.



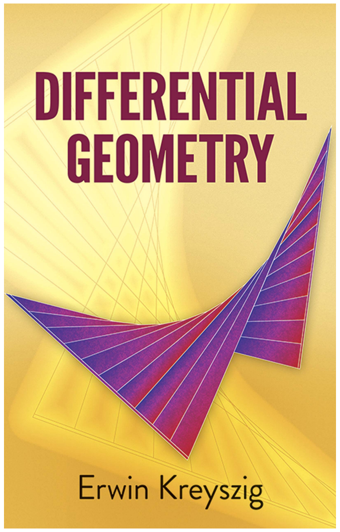
Any sufficiently small piece looks flat

arbitrarily curved surface

(II) *Geometrization*: We want to construct a spacetime in which the equation for a straight line takes the form $(d^2x/dt^2 + \nabla\Phi) = 0$, in the *Newtonian limit*, where Φ is the Newtonian gravitational potential.

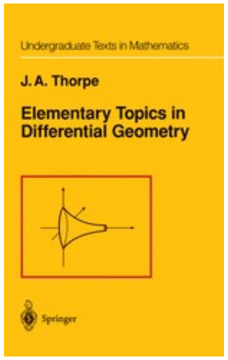
- *This means our theory will reproduce Newtonian gravity in the appropriate limit.*

To accomplish (II), Einstein had to learn differential geometry!



DIFFERENTIAL GEOMETRY

Erwin Kreyszig

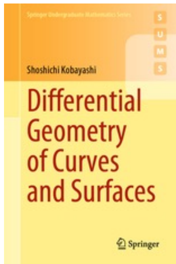


Undergraduate Texts in Mathematics

J. A. Thorpe

Elementary Topics in Differential Geometry

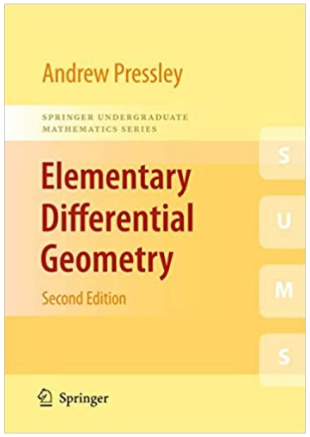
Springer



Shoshichi Kobayashi

Differential Geometry of Curves and Surfaces

Springer



Andrew Pressley

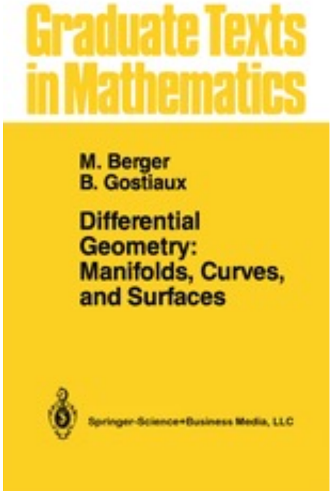
SPRINGER UNDERGRADUATE MATHEMATICS SERIES

Elementary Differential Geometry

Second Edition

Springer

SUMS



Graduate Texts in Mathematics

M. Berger
B. Gostiaux

Differential Geometry: Manifolds, Curves, and Surfaces

Springer-Science+Business Media, LLC

Two Facts from Differential Geometry

1. *Geodesic Equation*. The general equation for a straight line (a *geodesic*) in a curved space is given by,

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{dt}\frac{dx^\sigma}{dt} = 0$$

(a) The "correction factors" $\Gamma^\mu_{\nu\sigma}$ are (64!) functions that encode the curvature.

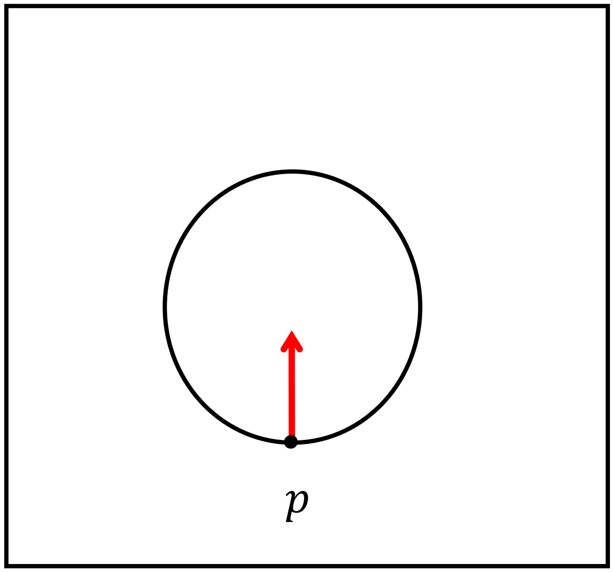
(b) They depend explicitly on the metric $g_{\mu\nu}$.

(c) $\Gamma^\mu_{\nu\sigma} = 0$ is a *sufficient*, but not *necessary*, condition for flatness.

2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*

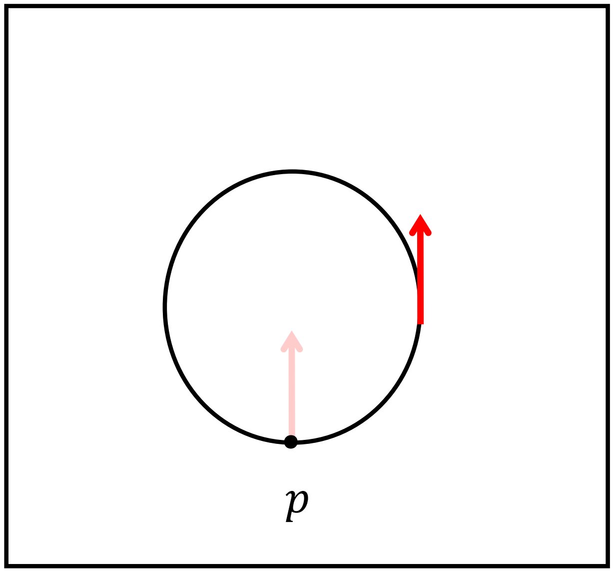


- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*

2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*

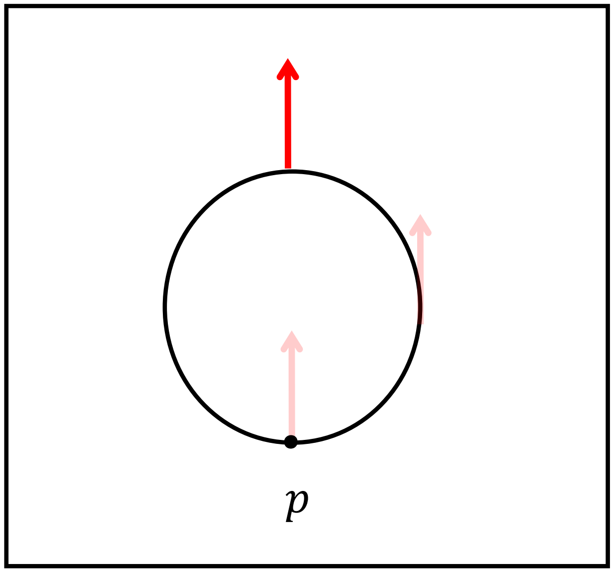


- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*

2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*

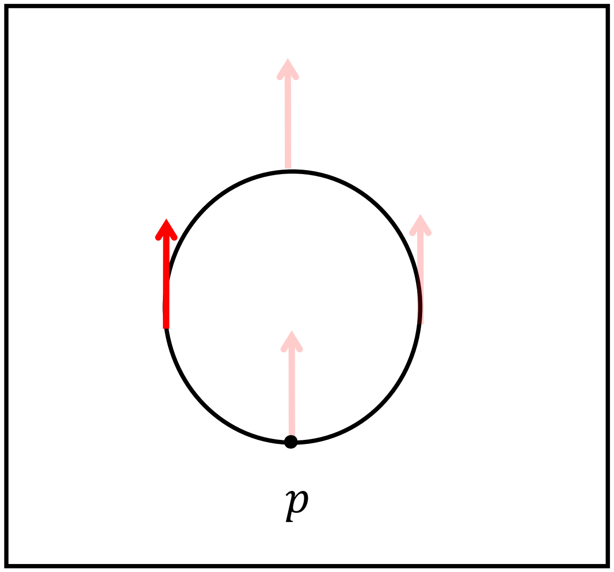


- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*

2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*

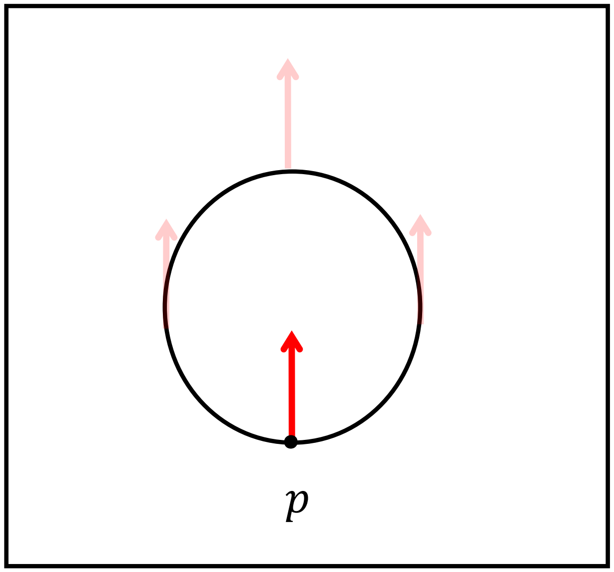


- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*

2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*

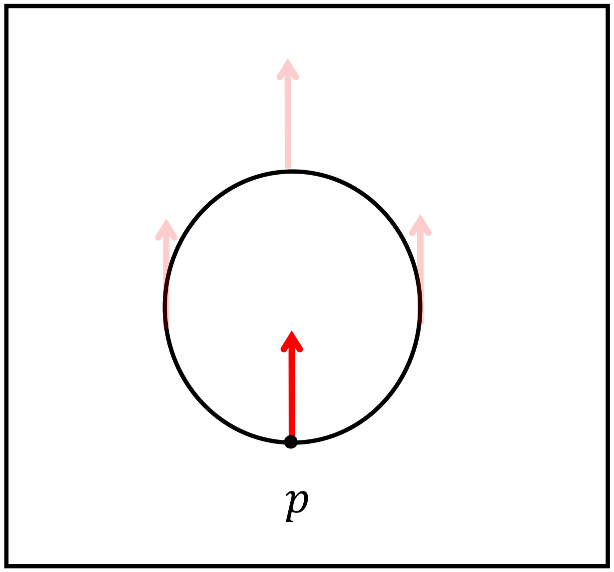


- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*
- *It ends back pointing in same direction.*

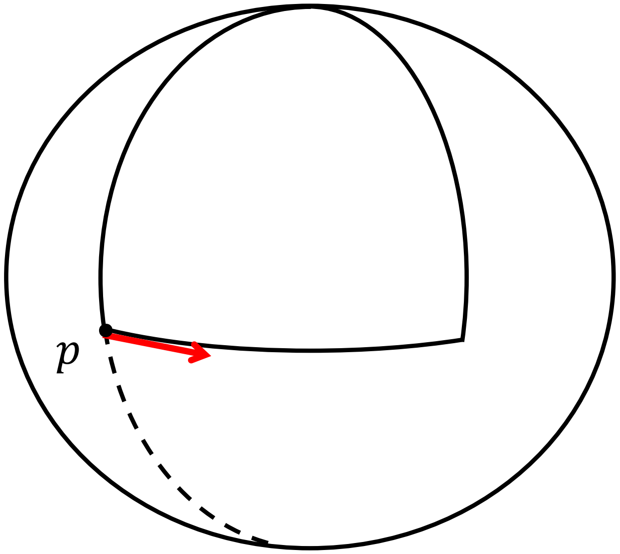
2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*



- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*
- *It ends back pointing in same direction.*

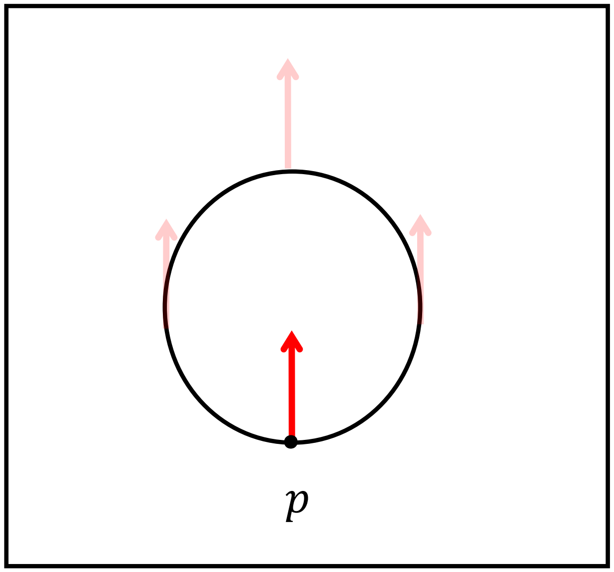


- *In a curved space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*

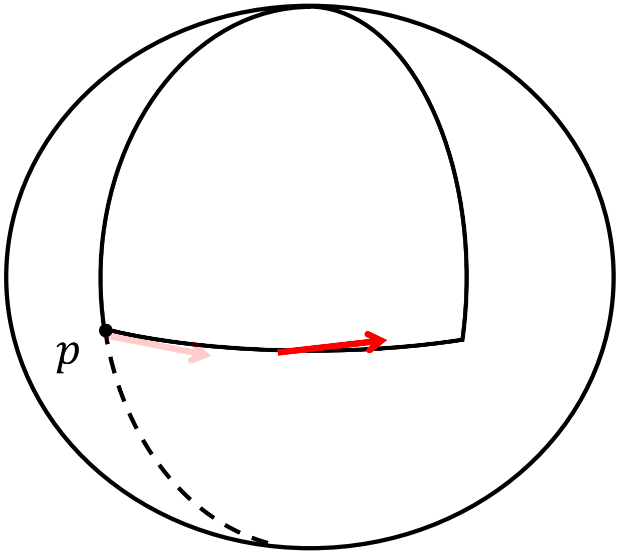
2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*



- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*
- *It ends back pointing in same direction.*

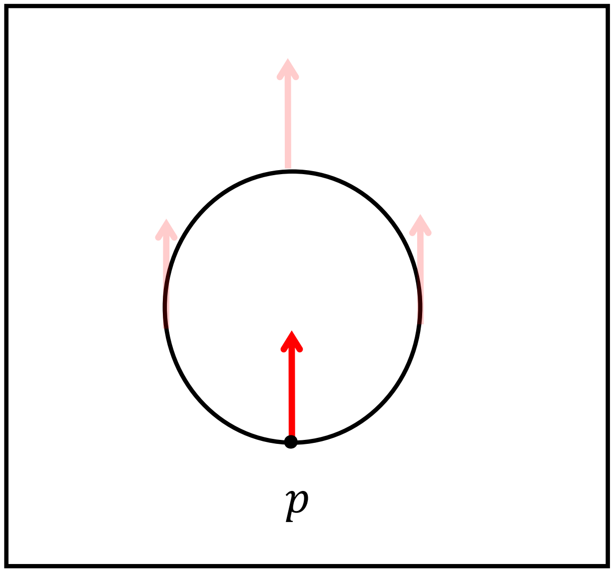


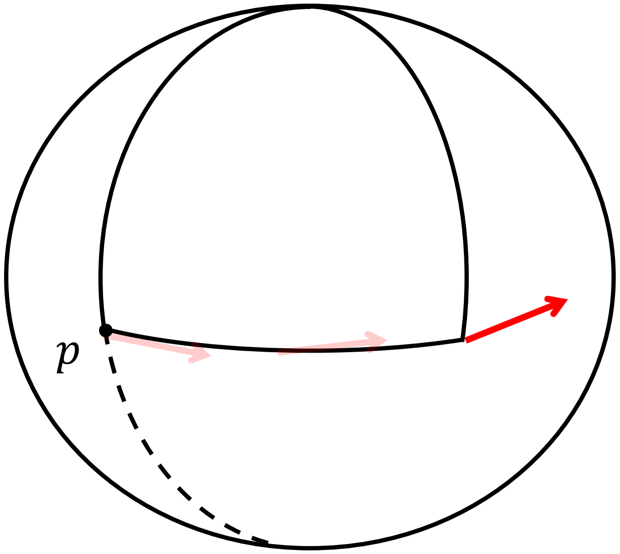
- *In a curved space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*

2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*





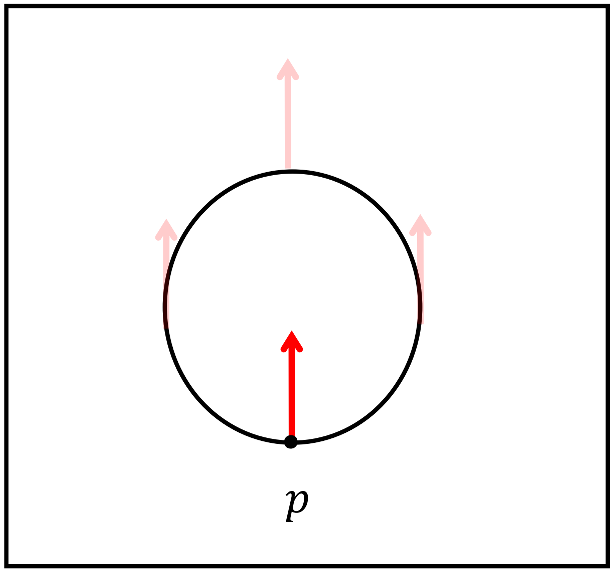
- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*
- *It ends back pointing in same direction.*

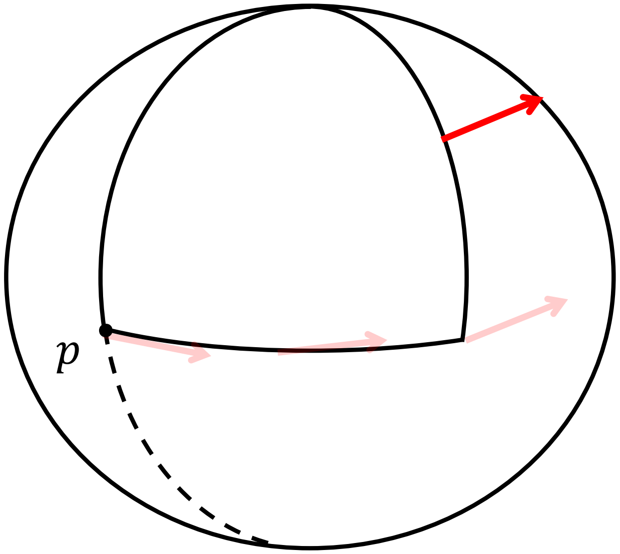
- *In a curved space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*

2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*





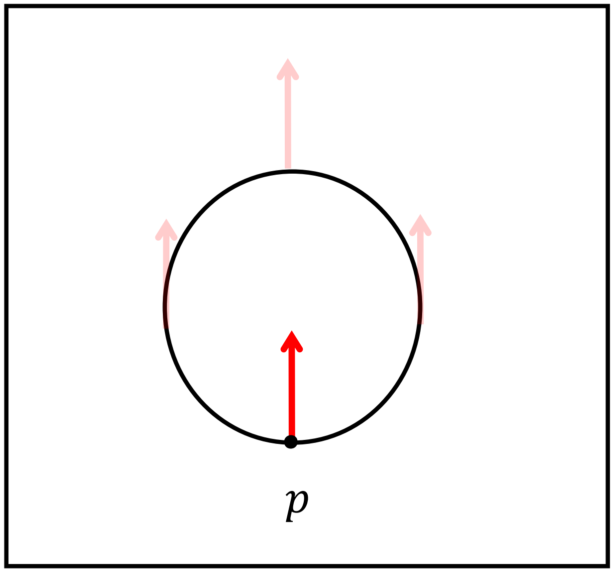
- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*
- *It ends back pointing in same direction.*

- *In a curved space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*

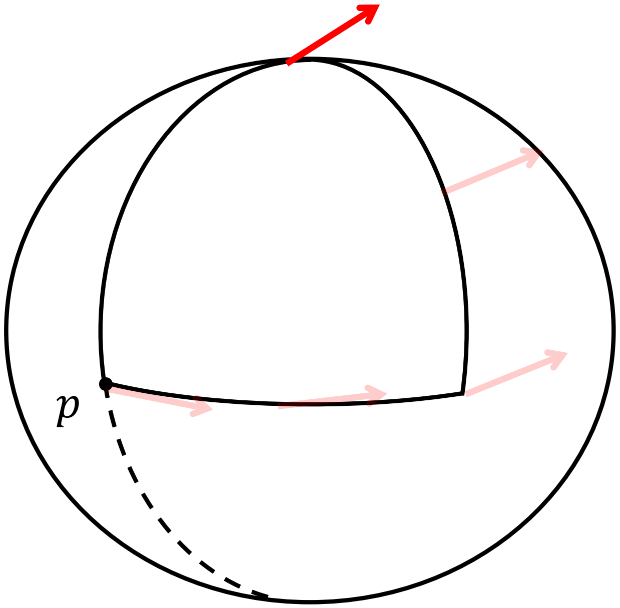
2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*



- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*
- *It ends back pointing in same direction.*

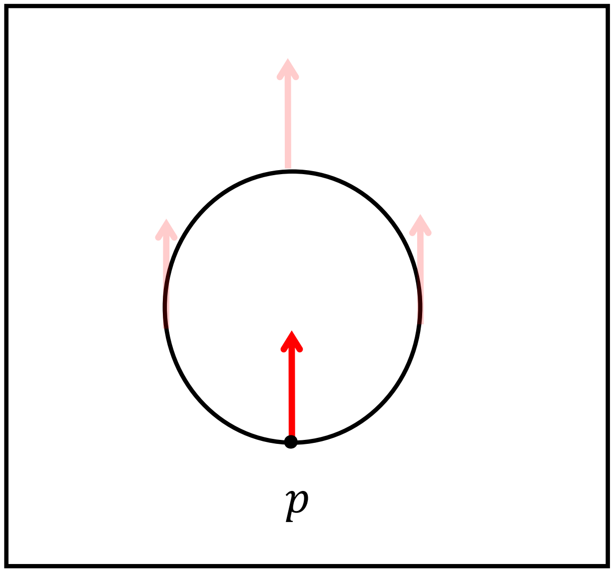


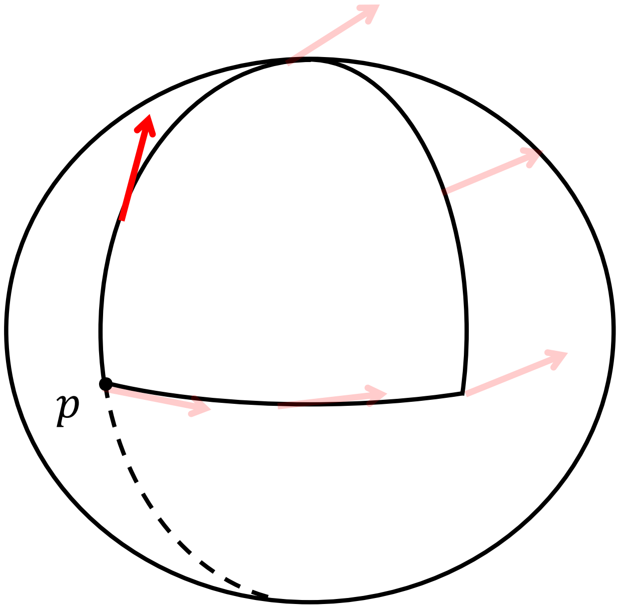
- *In a curved space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*

2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*





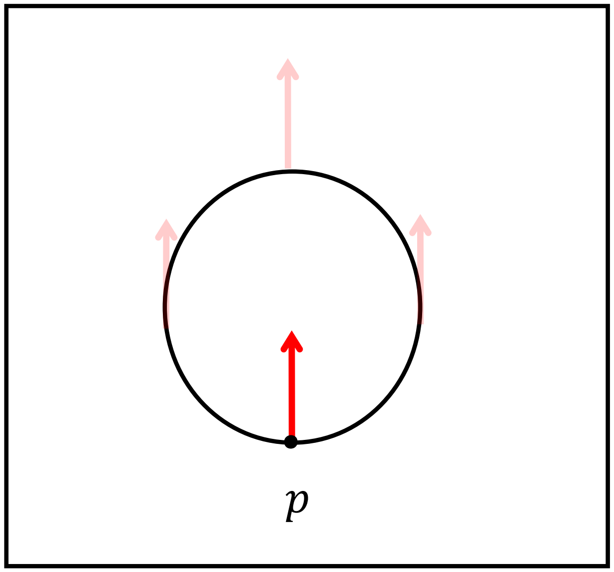
- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*
- *It ends back pointing in same direction.*

- *In a curved space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*

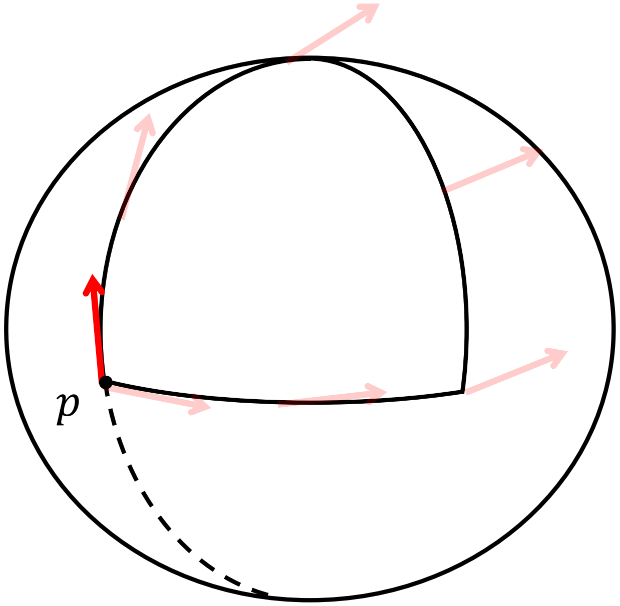
2. *Curvature Tensor*. Gives us a *necessary* and *sufficient* condition for flatness.

Idea: Parallel-transport a tangent vector around a closed loop.

- *If the space is curved, it will come back pointing in a different direction!*

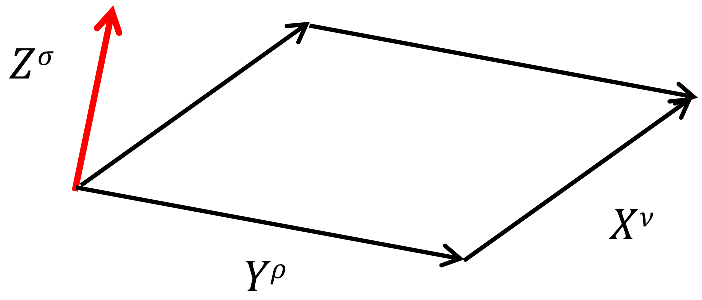


- *In a flat space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*
- *It ends back pointing in same direction.*

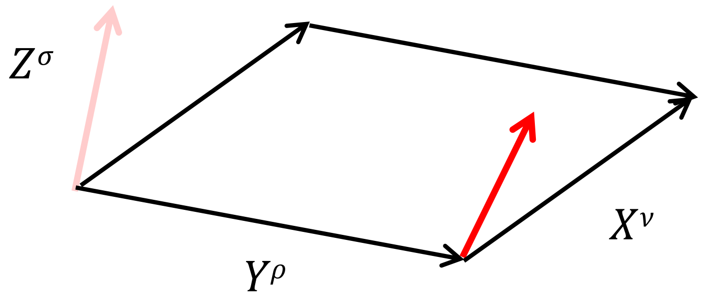


- *In a curved space, start at point p and transport the vector around the loop in such a way that it always points in the same direction, tangent to the space.*
- *It ends back pointing in a different direction.*

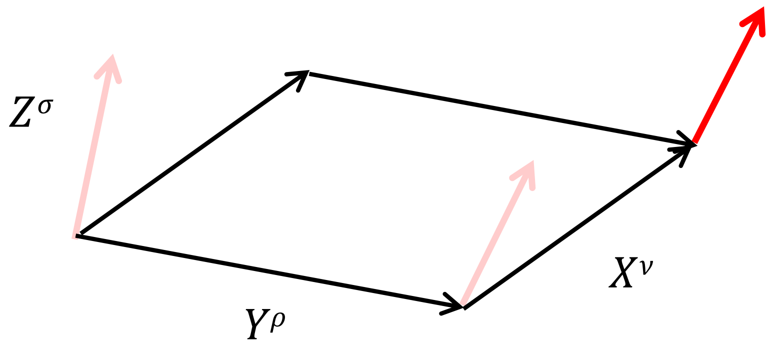
- Define a 4-indexed quantity, the *curvature tensor* $R^{\sigma}{}_{\mu\nu\rho}$, that measures this change.
- It acts on three vectors X^{ν}, Y^{ρ}, Z^{μ} and outputs the amount of change experienced by Z^{μ} upon parallel-transport around an infinitesimal curve defined by X^{ν} and Y^{ρ}:



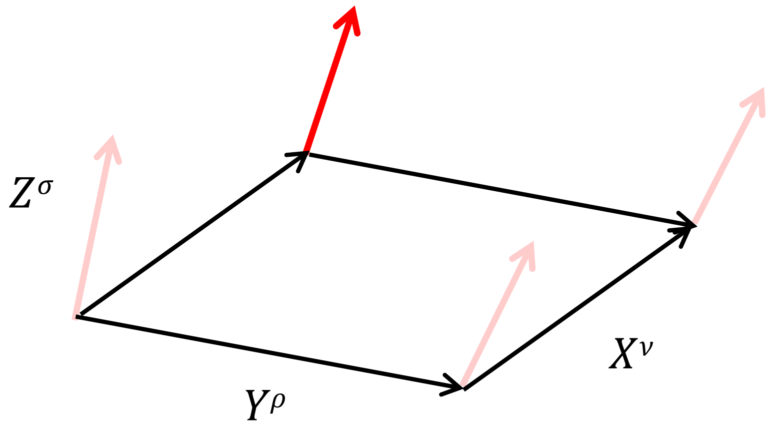
- Define a 4-indexed quantity, the *curvature tensor* $R^{\sigma}{}_{\mu\nu\rho}$, that measures this change.
- It acts on three vectors X^{ν}, Y^{ρ}, Z^{μ} and outputs the amount of change experienced by Z^{μ} upon parallel-transport around an infinitesimal curve defined by X^{ν} and Y^{ρ}:



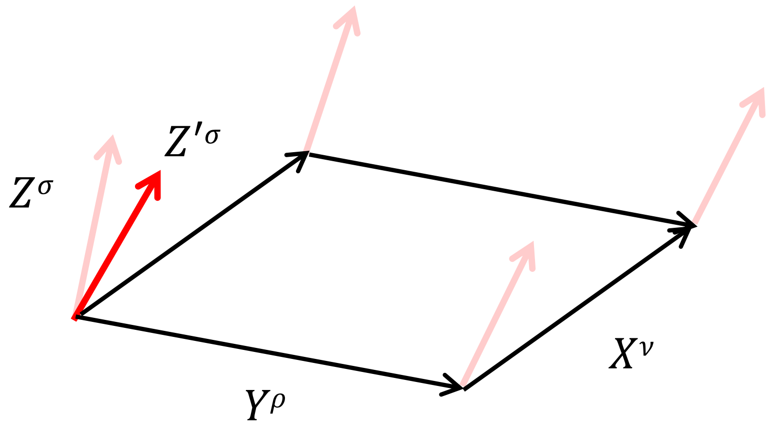
- Define a 4-indexed quantity, the *curvature tensor* $R^{\sigma}{}_{\mu\nu\rho}$, that measures this change.
- It acts on three vectors X^ν, Y^ρ, Z^μ and outputs the amount of change experienced by Z^μ upon parallel-transport around an infinitesimal curve defined by X^ν and Y^ρ:



- Define a 4-indexed quantity, the *curvature tensor* $R^{\sigma}_{\ \mu\nu\rho}$, that measures this change.
- It acts on three vectors X^{ν}, Y^{ρ}, Z^{μ} and outputs the amount of change experienced by Z^{μ} upon parallel-transport around an infinitesimal curve defined by X^{ν} and Y^{ρ}:



- Define a 4-indexed quantity, the *curvature tensor* $R^{\sigma}_{\ \mu\nu\rho}$, that measures this change.
- It acts on three vectors X^{ν}, Y^{ρ}, Z^{μ} and outputs the amount of change experienced by Z^{μ} upon parallel-transport around an infinitesimal curve defined by X^{ν} and Y^{ρ}:



$$\begin{aligned} R^{\sigma}_{\ \mu\nu\rho} X^{\nu} Y^{\rho} Z^{\mu} &= \delta Z^{\sigma} \\ &= Z^{\sigma} - Z'^{\sigma} \\ &= \textit{change in } Z^{\sigma} \textit{ upon parallel transport} \\ &\quad \textit{around loop defined by } X^{\nu} \text{ and } Y^{\rho}. \end{aligned}$$

Properties of curvature tensor:

(a) $R^{\sigma}_{\ \mu\nu\rho} = 0$ *if and only if* the space is flat.

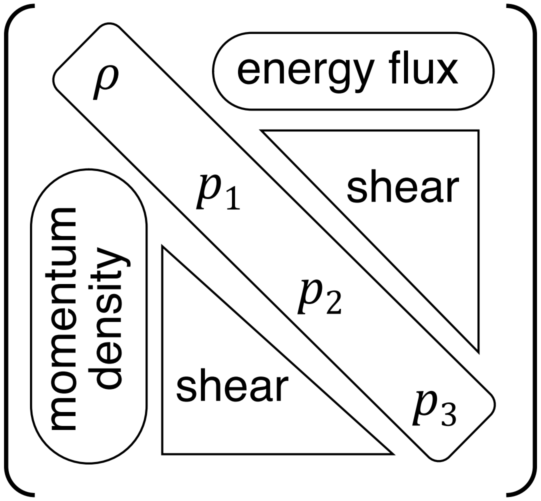
(b) $R^{\sigma}_{\ \mu\nu\rho}$ depends explicitly on the metric $g_{\mu\nu}$.

- To geometrize gravity, we need to relate the source of gravity (energy/mass) to the curvature tensor!
- We want there to be curvature in the presence of gravity, and no curvature in its absence.
- *Require*: (*curvature of spacetime*) $\propto$ (*matter density*).

Can represent this mathematically by curvature tensor $R^{\sigma}_{\mu\nu\rho}$

Can represent this mathematically by "energy-momentum tensor" $T_{\mu\nu}$

$T_{\mu\nu} =$



$T^{00} = \rho = \textit{energy density}$

$T^{0i} = \textit{energy flux components}$

$T^{i0} = \textit{momentum density components}$

$T^{ii} = p_i = \textit{pressure components}$

$T^{ij} = \textit{shear components}\ (i \neq j)$

$i, j = 1, 2, 3$

- *Problem*: The curvature tensor $R^{\sigma}_{\mu\nu\rho}$ has 4 indices and the energy-momentum tensor $T_{\mu\nu}$ has 2. Only tensors of the same "rank" can be equated!

- *Eventual solution*: Construct a 2-index tensor $G_{\mu\nu}$ out of $R^{\sigma}_{\mu\nu\rho}$ and set it equal to $T_{\mu\nu}$ with an appropriate proportionality constant.

- *Result*: The Einstein equations:

$$G_{\mu\nu}(g_{\mu\nu}) = \kappa T_{\mu\nu}$$

"Einstein" tensor

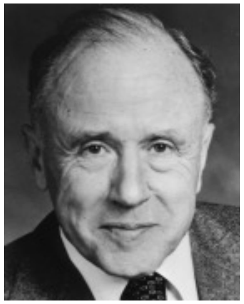
Einstein tensor as a function of $g_{\mu\nu}$:

$$G_{\mu\nu}(g_{\mu\nu}) = R_{\mu\nu} - ½ R g_{\mu\nu}$$

"Ricci" tensor

"Ricci" scalar

- The constant $\kappa = 8\pi G$ (G = Newtonian grav. constant) guarantees that these equations reproduce Newton's Law of Gravity in the Newtonian limit.
- Represent 16 differential equations of the metric $g_{\mu\nu}$, 6 of which are dependent on the rest; so 10 non-linear partial differential equations!
- *What they mean*:



John Wheeler
(1911-2008)

"...spacetime geometry tells matter how to move, matter tells spacetime how to curve"

- To solve the Einstein equations, one must make *initial assumptions*
  - *Either about spacetime geometry (e.g., isotropic; asymptotically flat, etc.).*
  - *Or about the matter distribution (e.g., evenly distributed, clumped in one spot, no negative energy, etc).*

A general relativistic spacetime = a 4-dim collection of points with the following additional structure: Between any two points, there is a spacetime interval given by $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$, where $g_{\mu\nu}$ is a *pseudo-Riemannian* metric that satisfies the Einstein equations.

"reduces to the Minkowski metric at any point"