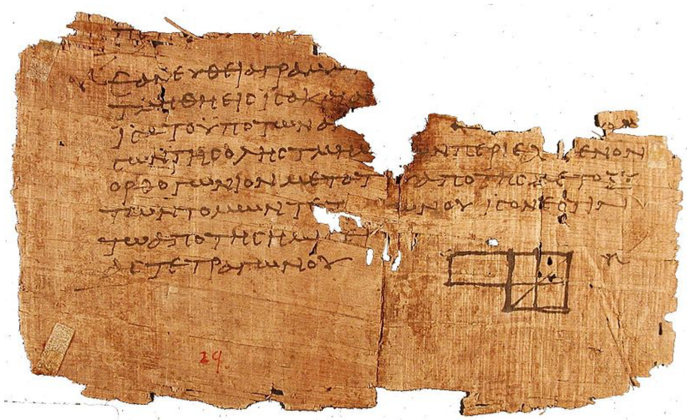
08. Non-Euclidean Geometry

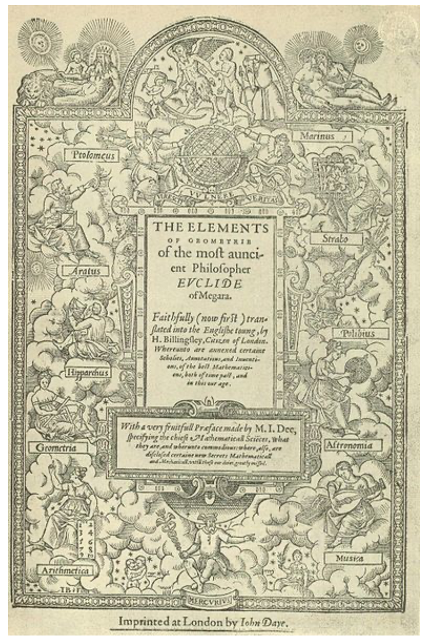
Topics:
1. Euclidean Geometry
2. Non-Euclidean Geometry

1. Euclidean Geometry

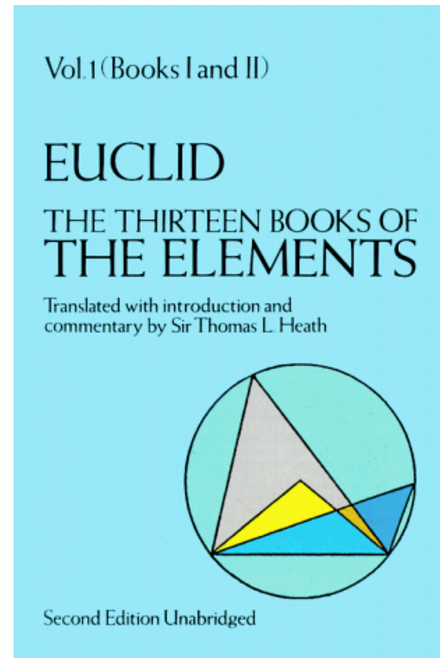
- *The Elements*. ~300 B.C.



~100 A.D.
Earliest existing copy



1570 A.D.
First English translation

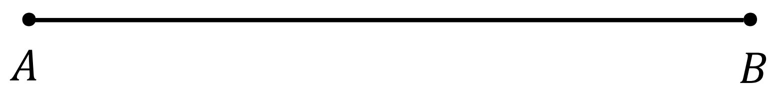


1956
Dover Edition

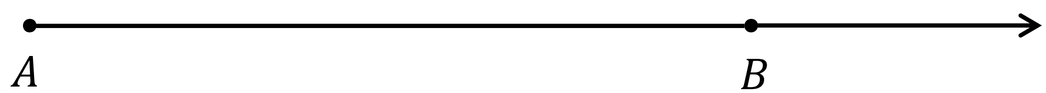
- 13 books of propositions, based on 5 postulates.

Euclid's 5 Postulates

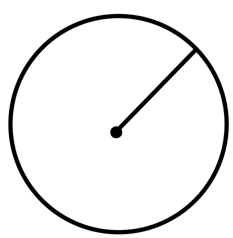
1. A straight line can be drawn from any point to any point.



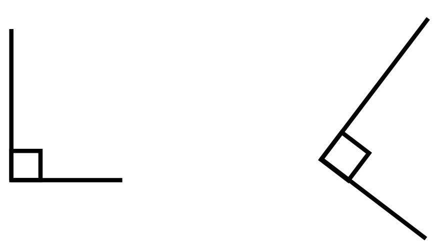
2. A finite straight line can be produced continuously in a straight line.



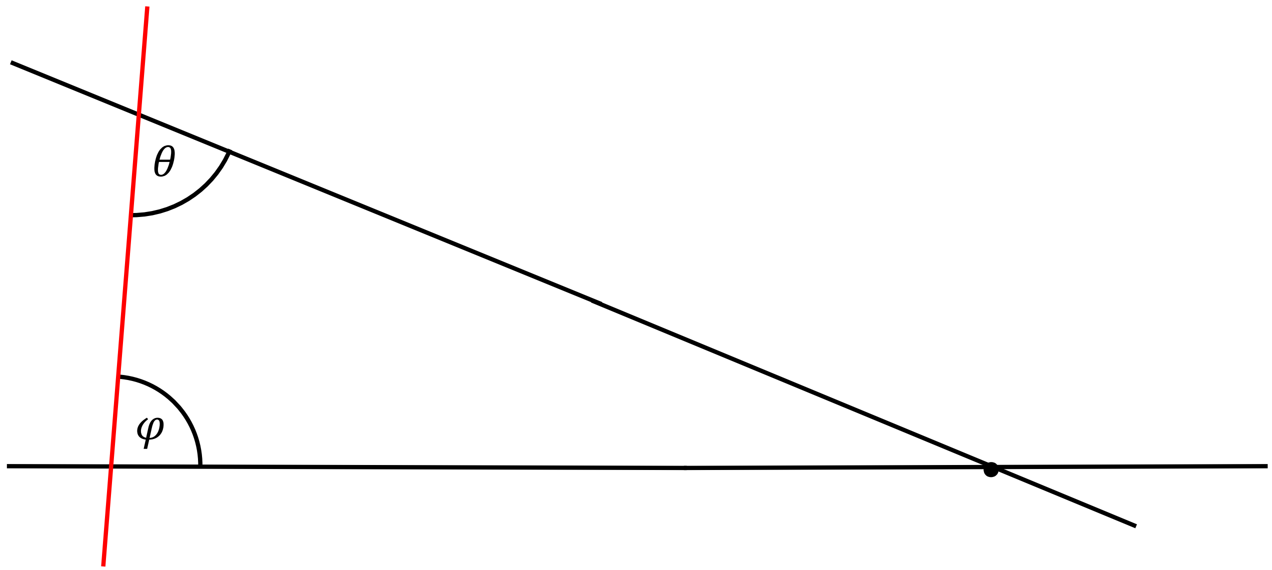
3. A circle may be described with any center and distance.



4. All right angles are equal to one another.



5. If a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the angles are together less than two right angles.

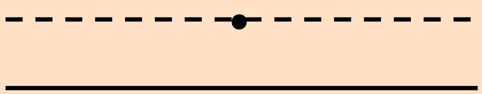


$\theta + \varphi < 180^\circ$

- *Euclid's Accomplishment*: showed that all geometric claims then known follow from these 5 postulates.
- Is 5th Postulate necessary? (1st cent.-19th cent.)
- *Basic strategy*: Attempt to show that replacing 5th Postulate with alternative leads to contradiction.

- Equivalent to 5th Postulate (Playfair 1795):

5′. Through a given point, exactly one line can be drawn parallel to a given line (that does not contain the point).

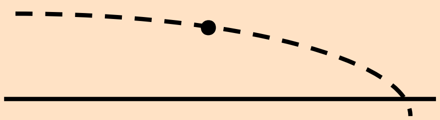


John Playfair
(1748-1819)

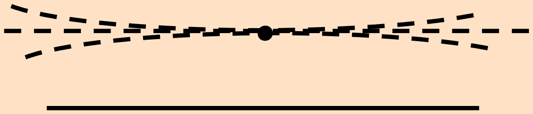
Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in either direction, do not meet one another in either direction. (*The Elements*: Book I, Def. 23)

- Only two logically possible alternatives:

5^{none}. Through a given point, no lines can be drawn parallel to a given line.

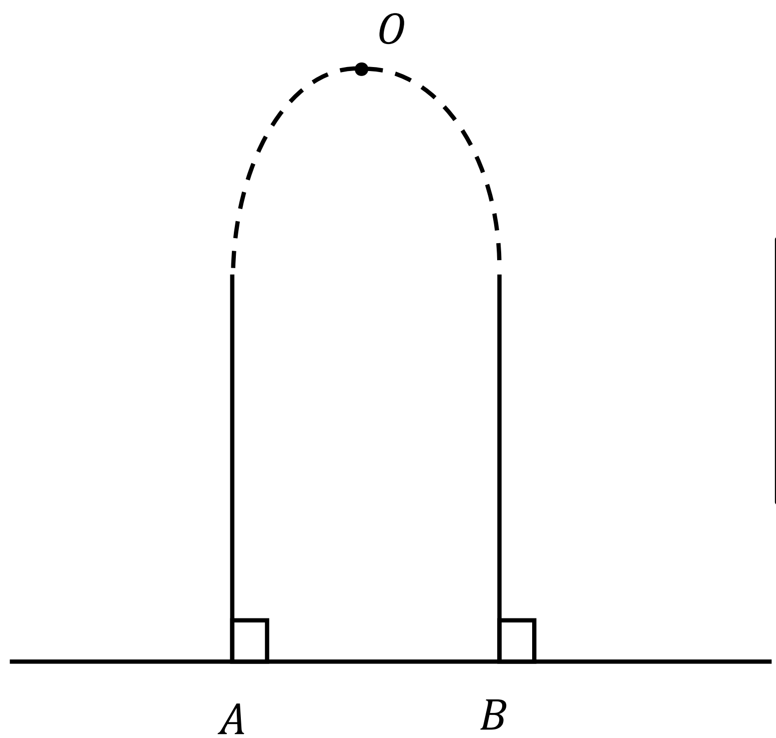


5^{many}. Through a given point, more than one line can be drawn parallel to a given line.



2. Non-Euclidean Geometry

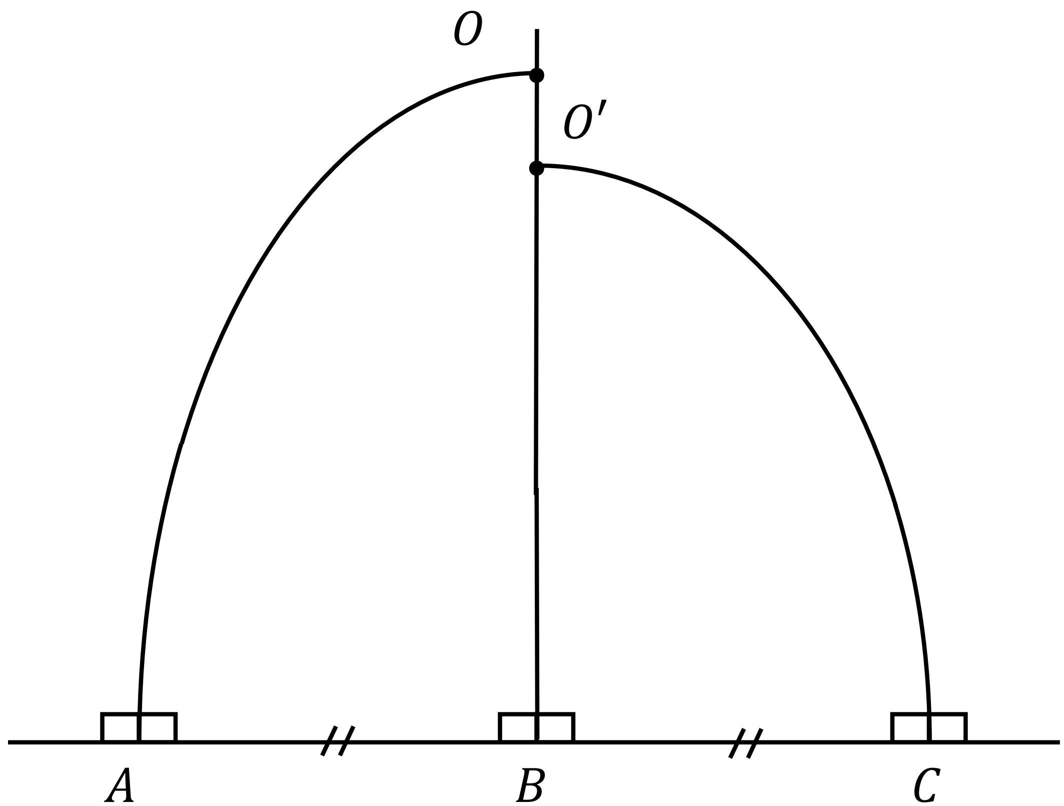
Case of 5^{none}



Result #1: The sum of the interior angles of a triangle may be more than 2 right angles.

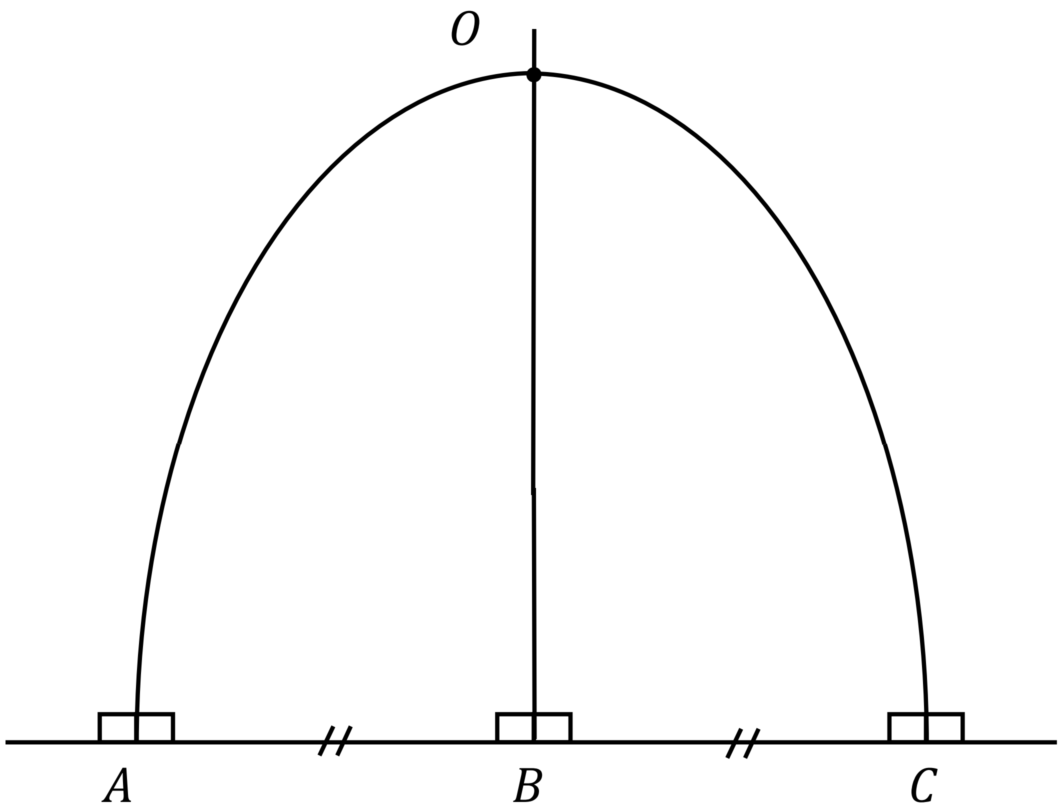
- Pick *any* straight line.
- Erect perpendiculars.
- Perpendiculars *must* meet at some point, call it O.
- *Thus*: Angles of $\triangle OAB$ sum to *more* than 2 right angles!

Case of 5[none]



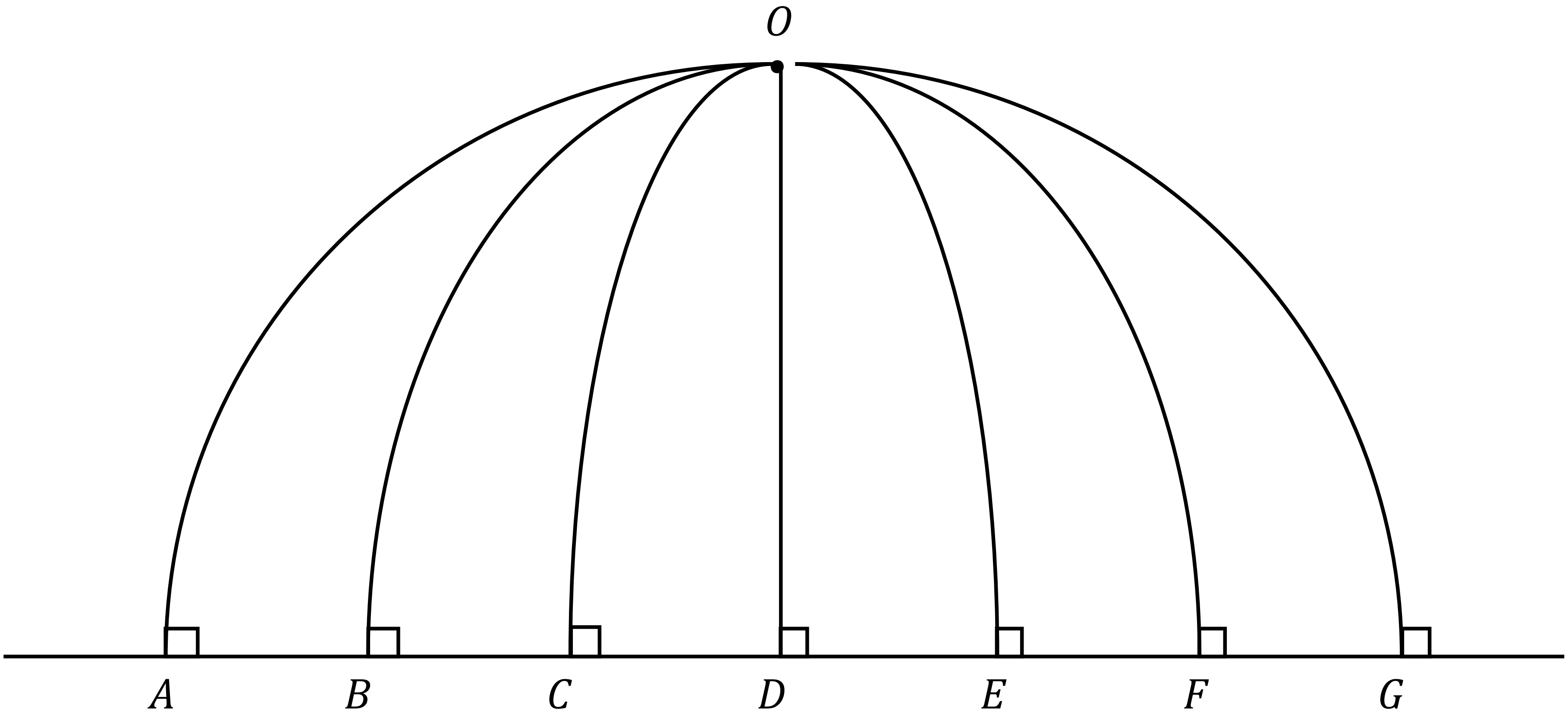
- Take same straight line. Mark off equal intervals $AB = BC$.
- Perpendiculars from A and B must meet at some point O, and perpendiculars from B and C must meet at some point O', not *necessarily* identical to O.
- *But*: $\triangle OBA$ is congruent to $\triangle O'BC$ ("angle-side-angle") and $AB = BC$.
- *So*: Side OB = side $O'B$. Thus O must indeed be identical to O'.

Case of 5^{none}



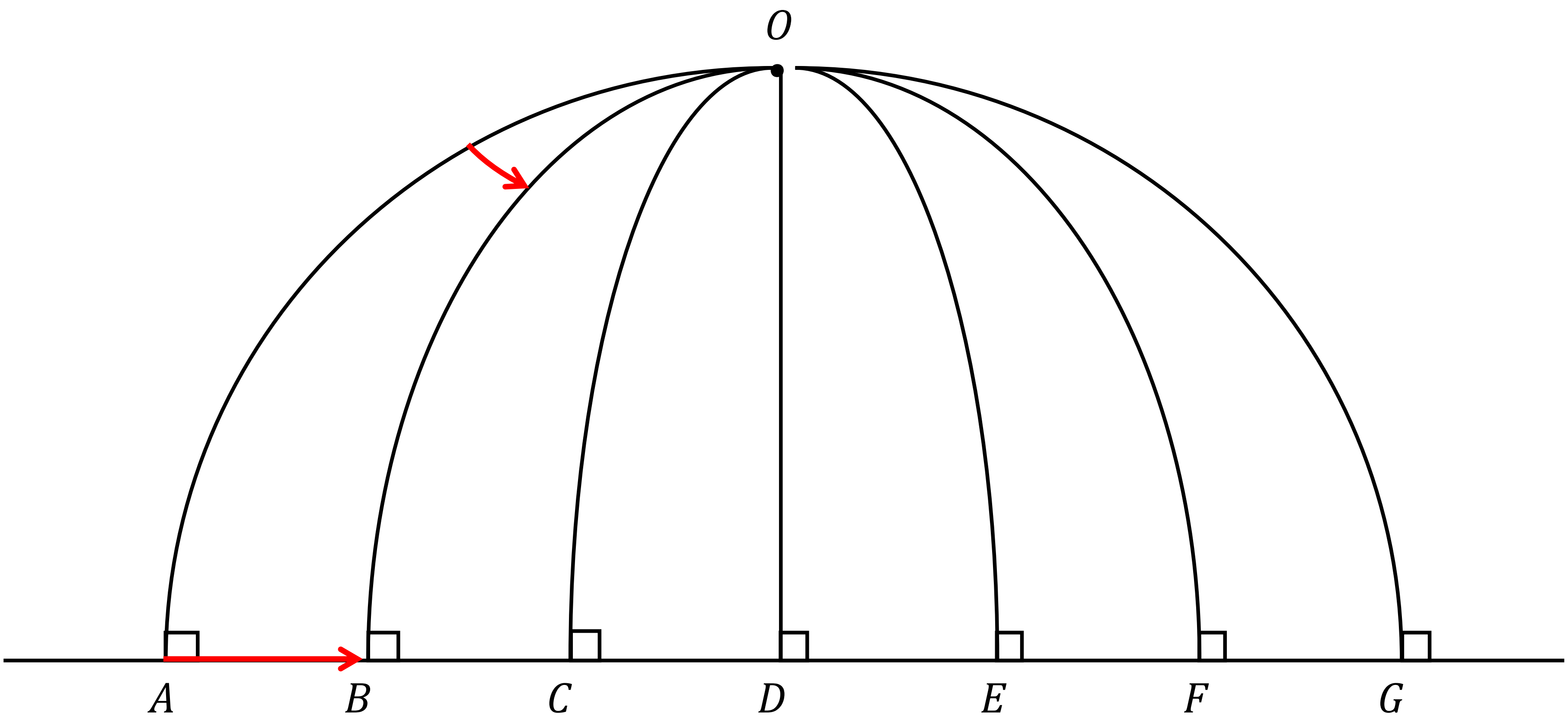
- Take same straight line. Mark off equal intervals $AB = BC$.
- Perpendiculars from A and B must meet at some point O, and perpendiculars from B and C must meet at some point O', not *necessarily* identical to O.
- *But*: $\triangle OBA$ is congruent to $\triangle O'BC$ ("angle-side-angle") and $AB = BC$.
- *So*: Side OB = side $O'B$. Thus O must indeed be identical to O'.

Case of 5^{none}



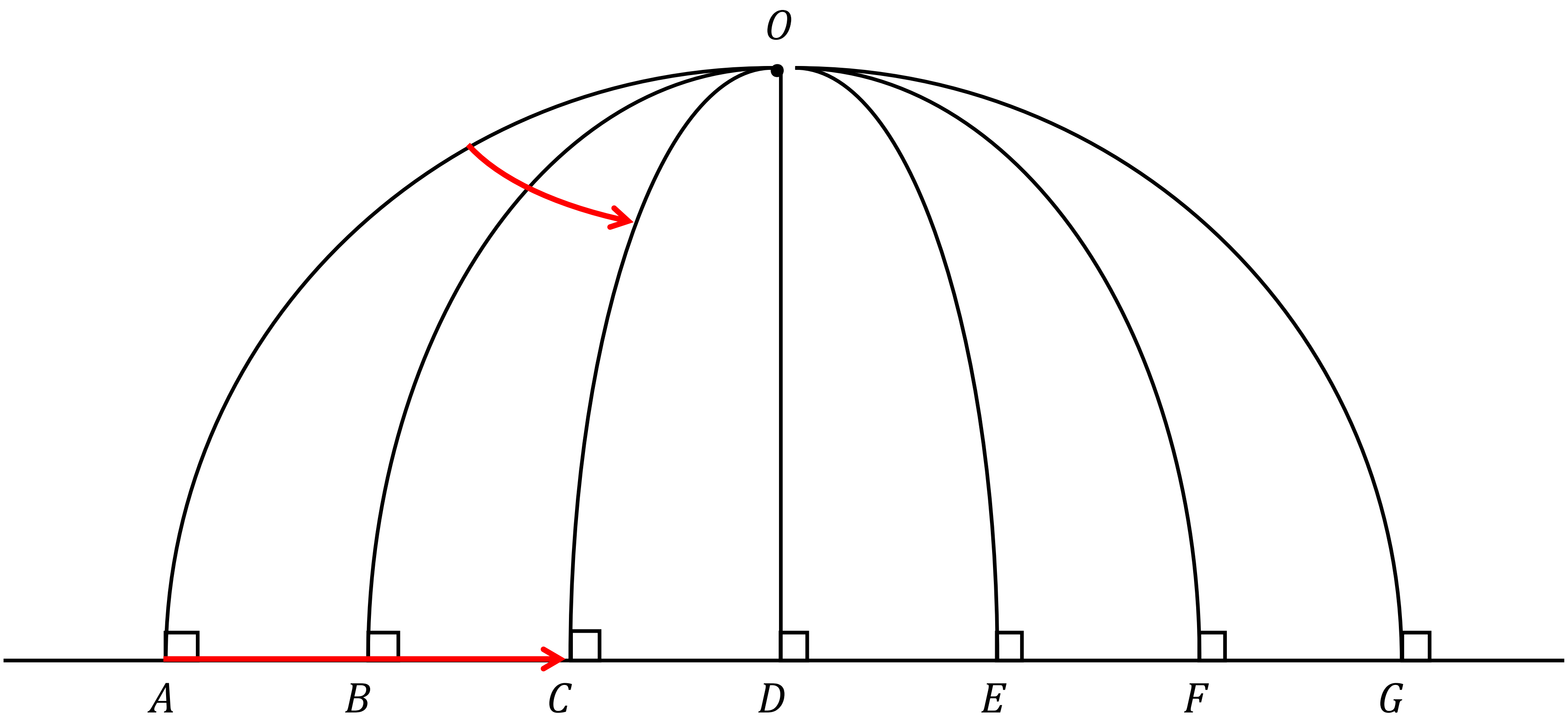
- Repeat construction.
- All triangles are congruent: $\triangle OAB \cong \triangle OBC \cong \triangle OCD \cong$ ...
- All angles at O are equal: $\measuredangle AOB = \measuredangle BOC = \measuredangle COD =$ ...
- All lines from O are equal: $OA = OB = OC = OD = \ldots$

Case of 5[none]



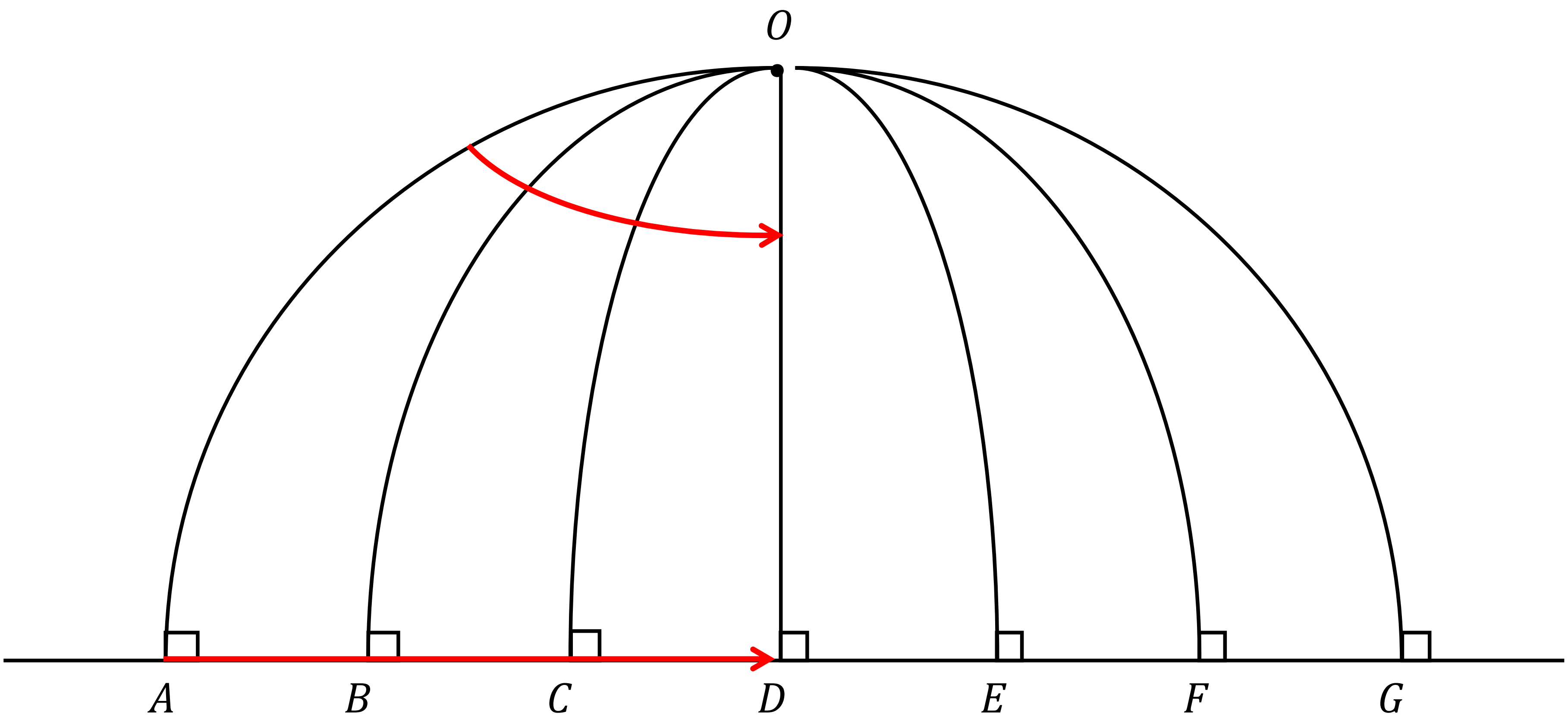
- Distance in direction of AG is proportional to angle subtended at O.

Case of 5^{none}



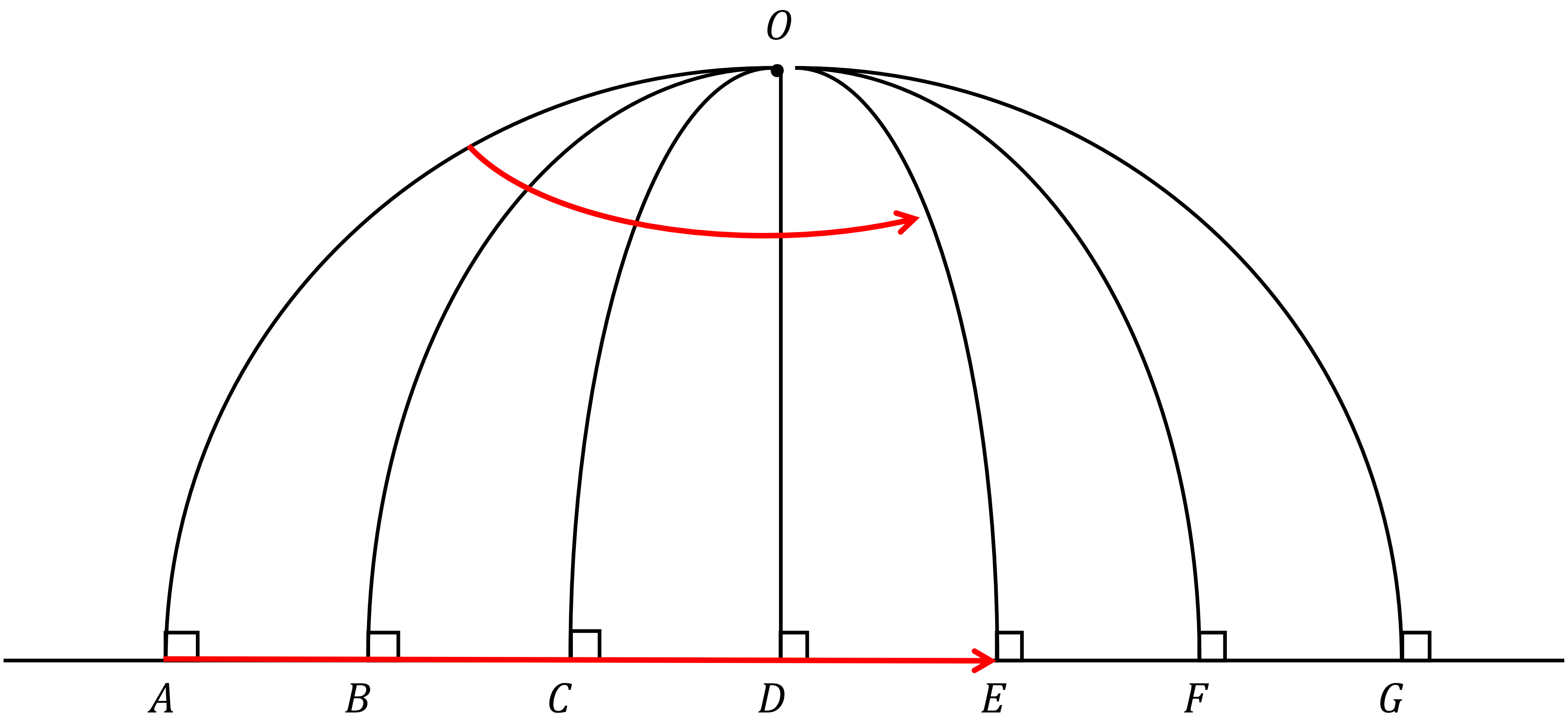
- Distance in direction of AG is proportional to angle subtended at O.

Case of 5^{none}



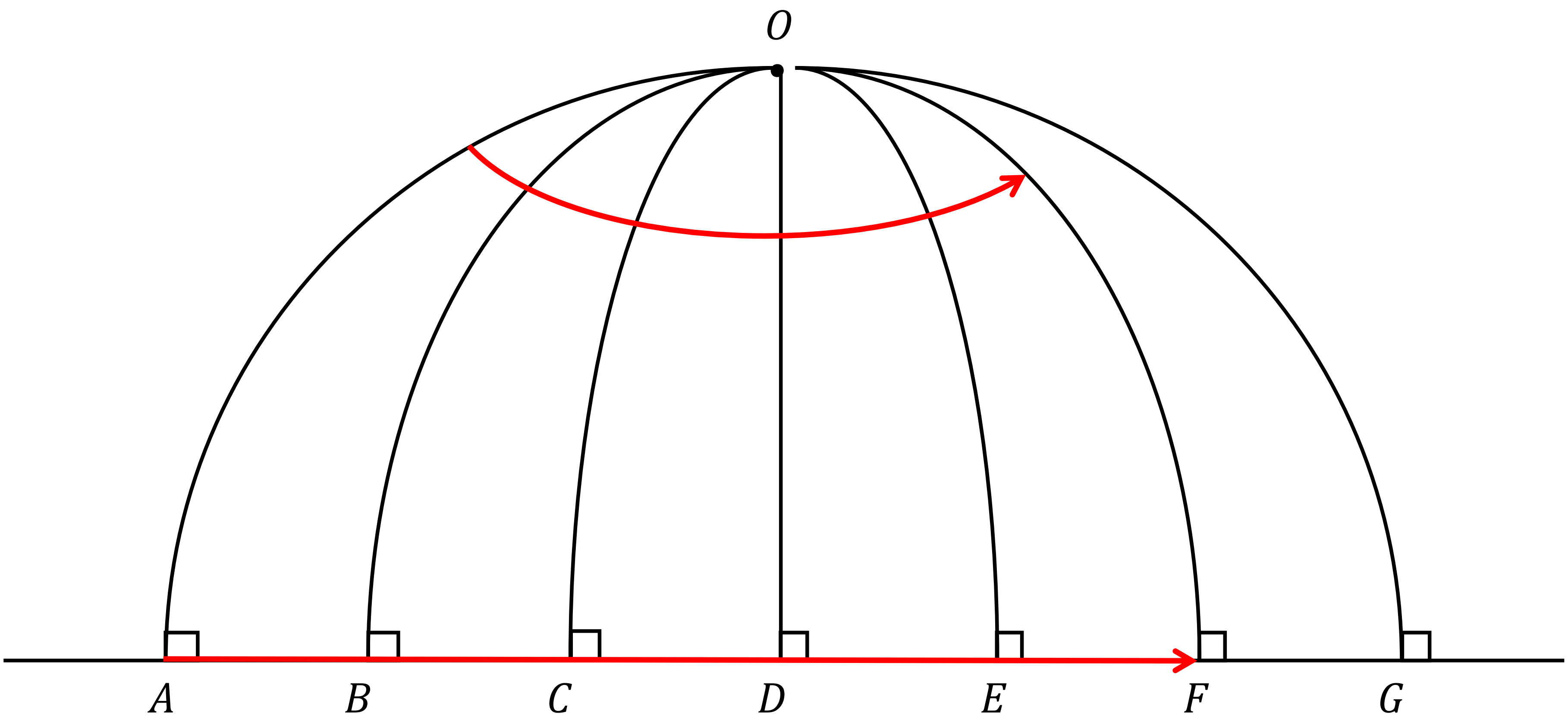
- Distance in direction of AG is proportional to angle subtended at O.

Case of 5^{none}



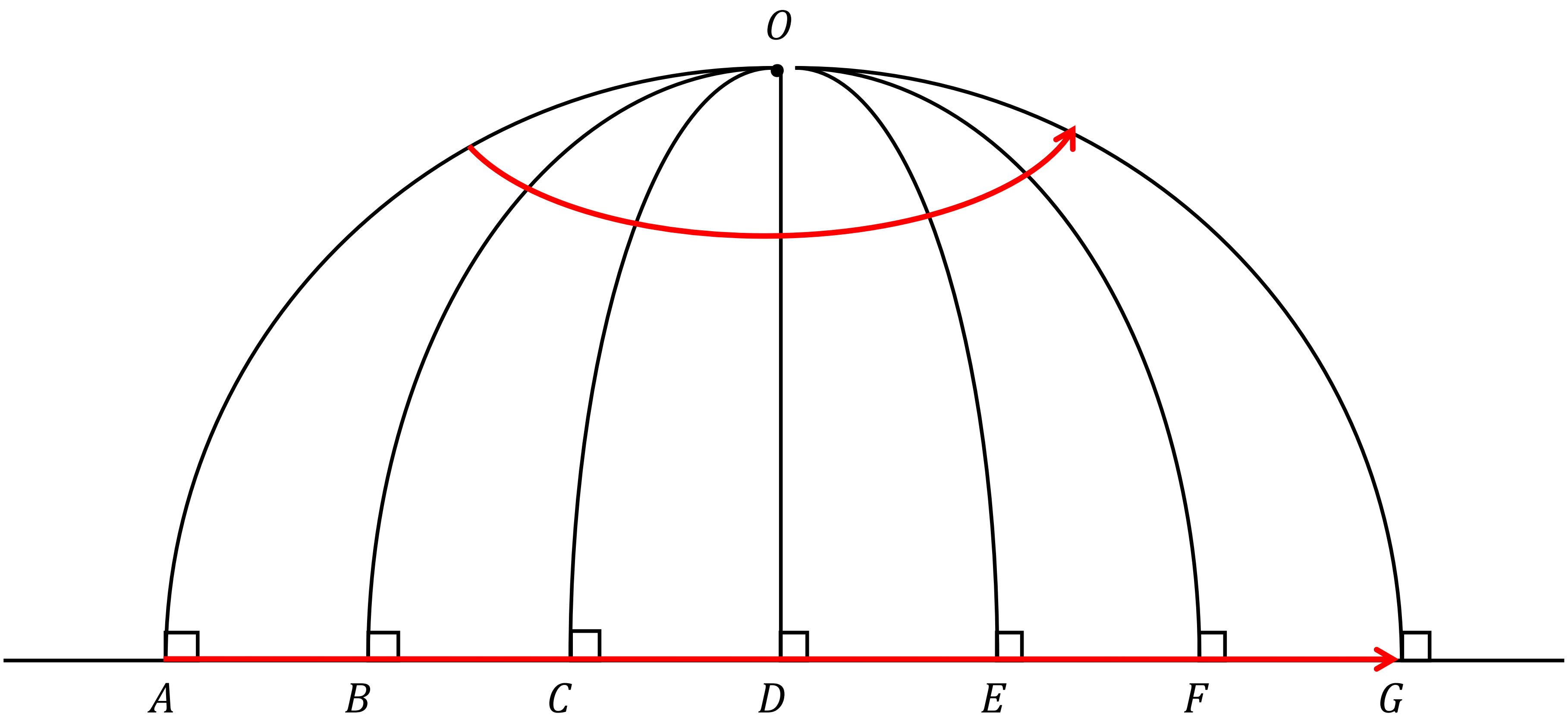
- Distance in direction of AG is proportional to angle subtended at O.

Case of 5^{none}



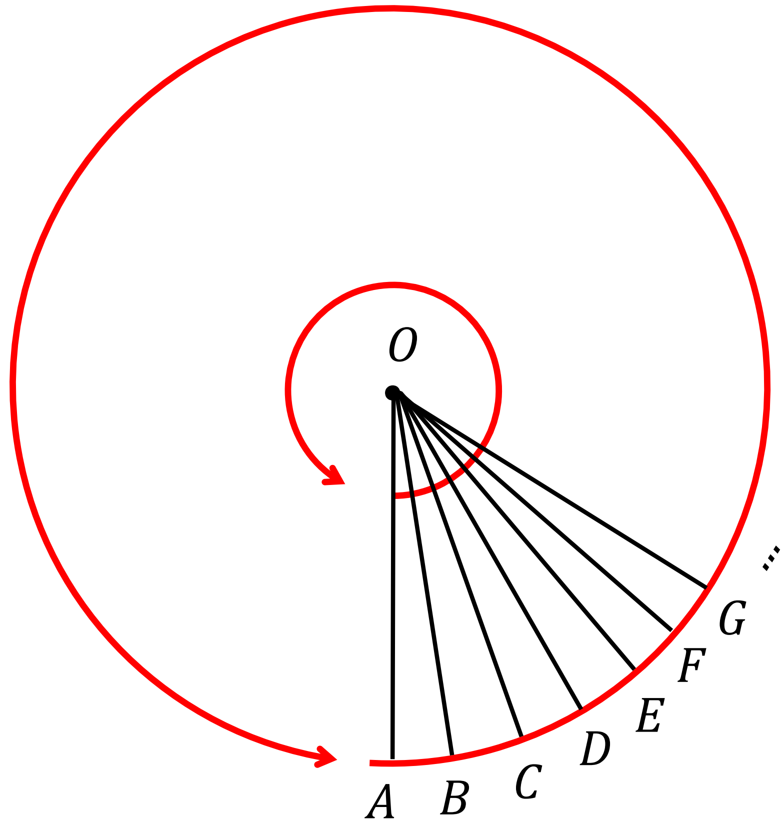
- Distance in direction of AG is proportional to angle subtended at O.

Case of 5^{none}



- Distance in direction of AG is proportional to angle subtended at O.

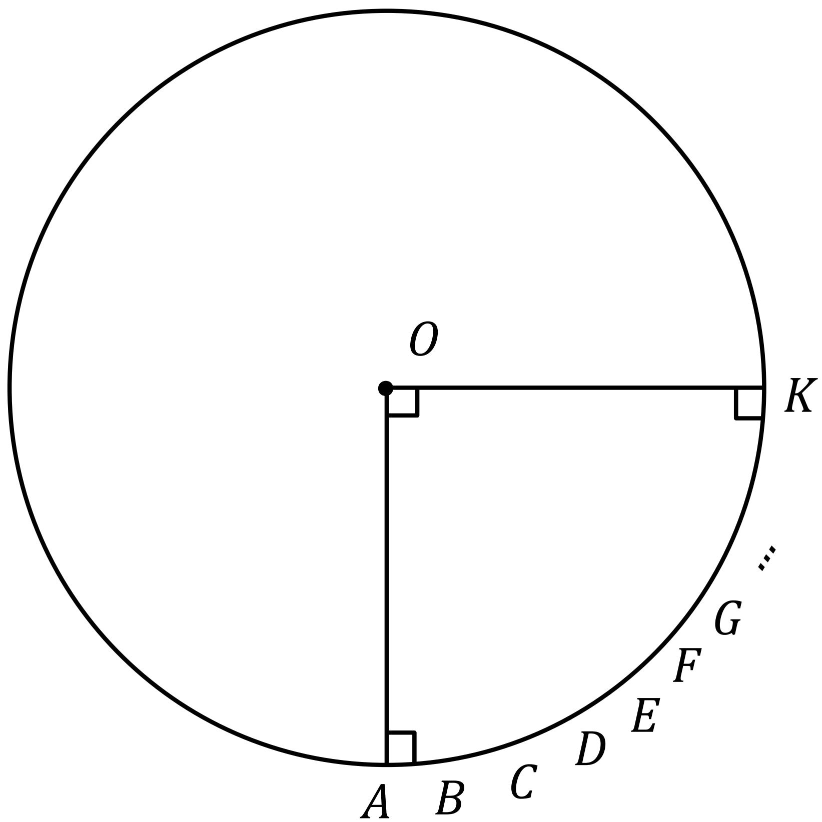
Case of 5^{none}



- Distance in direction of AG is proportional to angle subtended at O.
- Maximum angle subtended at O is 360°, which must correspond to a maximum distance in direction of AG.

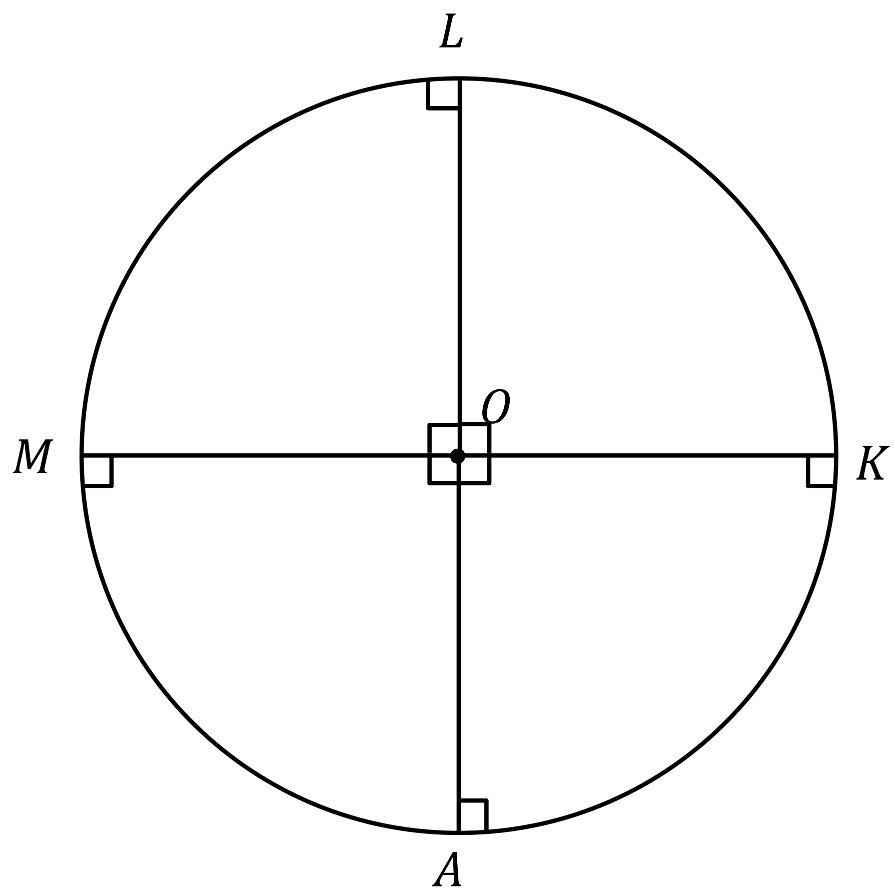
Result #2: All straight lines eventually close on themselves!

Case of 5^{none}



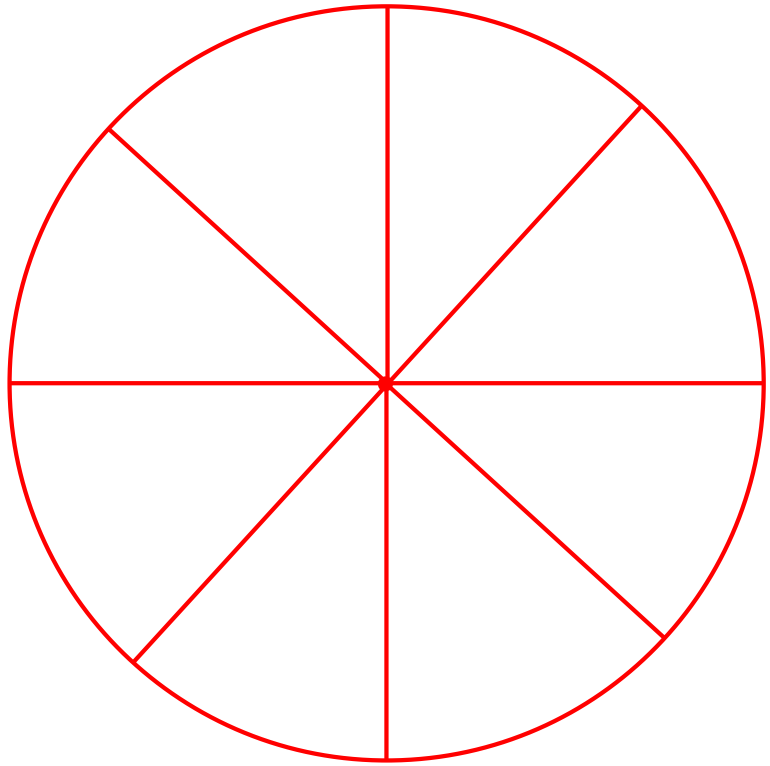
- *Note*: There exists a point K on the (straight!) line AG such that $\measuredangle AOK$ = 1 right angle.
- Sum of angles of $\triangle OAK$ = 3 right angles!
- For *small* triangles, sum of angles ≈ 2 right angles.

Case of 5^{none}



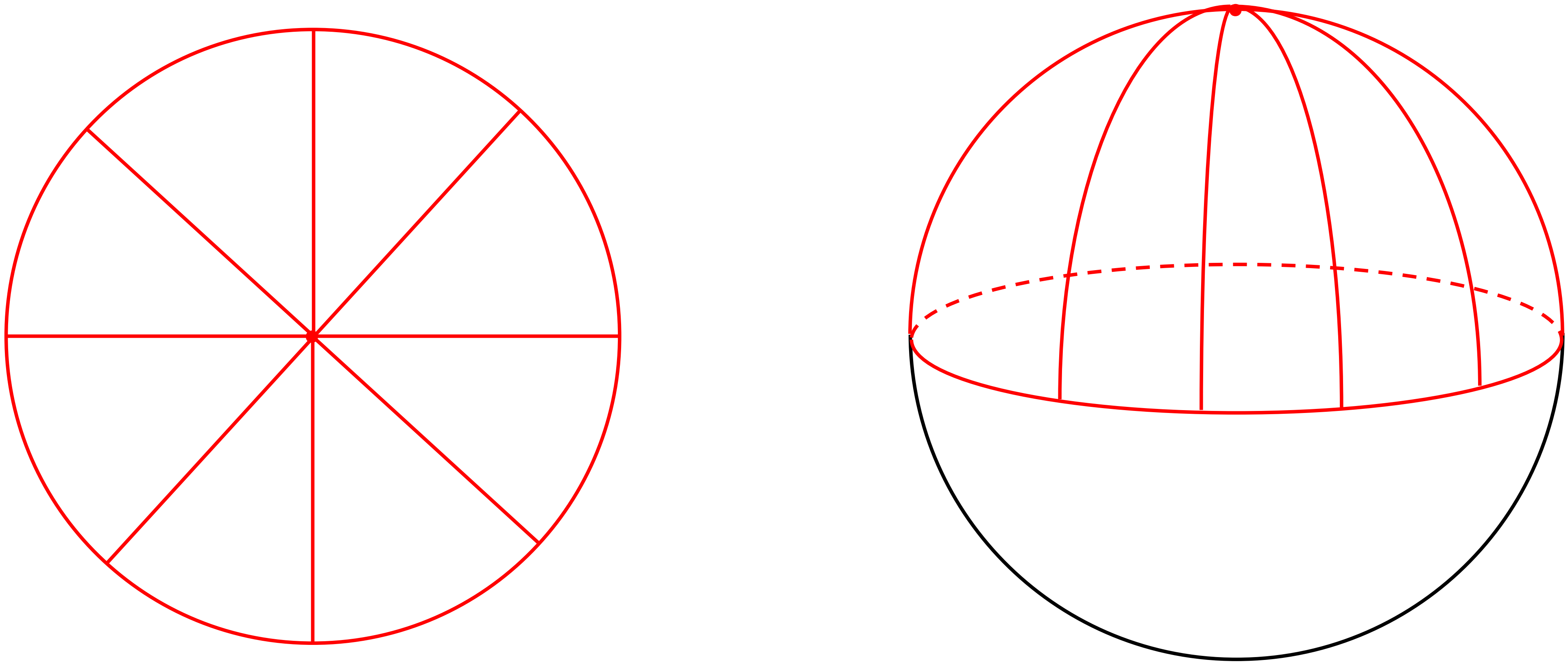
- $\triangle OAK$ is equilateral (all angles equal). So $OA = AK$.
- Same for $\triangle OKL$, $\triangle OLM$, $\triangle OMA$. So $radius\ OA = AK = KL = LM = MA.$
- So $circumference = AK + KL + LM + MA = 4 \times radius < 2\pi \times radius$
- Can also show: $area = 8/\pi \times (radius)^2 < \pi \times (radius)^2$

Case of 5 ^{none}



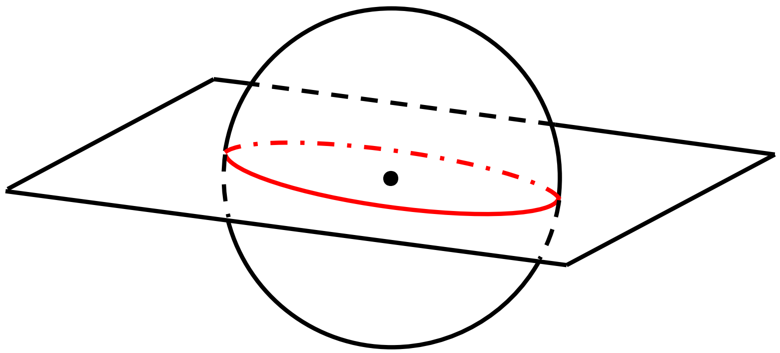
- Our construction maps 1-1 to the top hemisphere of a 3-dim sphere!

Case of 5^{none}



- Our construction maps 1-1 to the top hemisphere of a 3-dim sphere!
- 5^{none} geometry = *spherical geometry* = 2-dim geometry of surface of a 3-dim sphere.
- Generalized 5^{none} geometry = n-dim geometry of surface of $(n+1)$-dim sphere.
- Euclidean geometry is "flat". Spherical geometry is "positively curved".

Def. 1. A *great circle* on a sphere of radius R and center C, is any circle with radius R and center C.



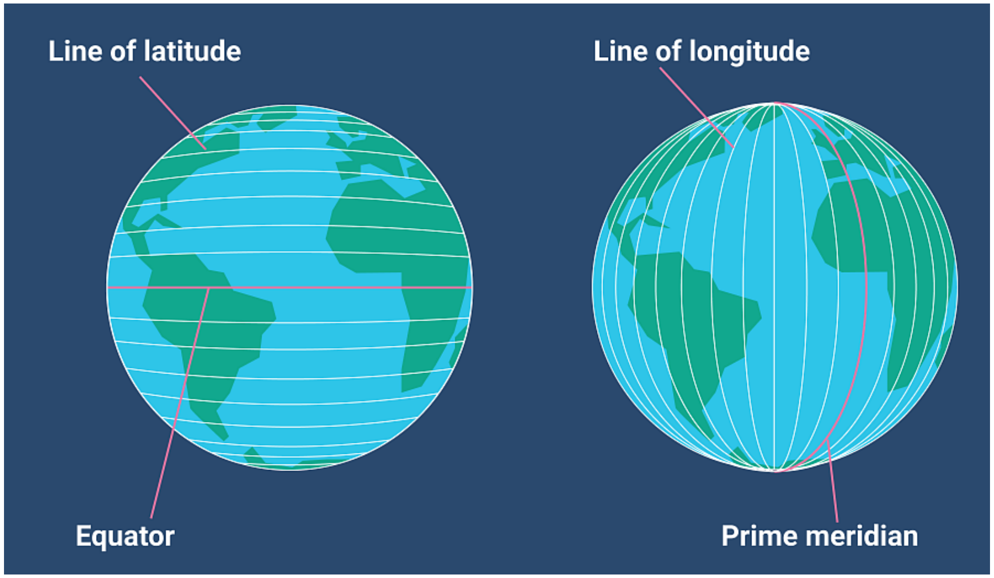
Pass plane through center: where it intersects sphere defines a great circle

Def. 2. A *geodesic* is the shortest distance between two points.

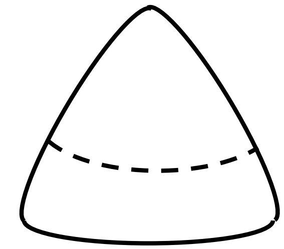
<u>*Claim*</u>:

(a) On the surface of a sphere, the geodesics are given by great circles.

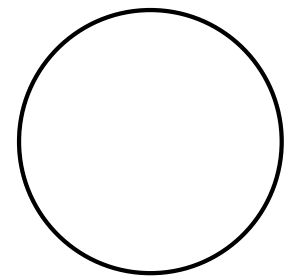
(b) Any two great circles on the surface of a sphere intersect at two points.



On the surface of a sphere:

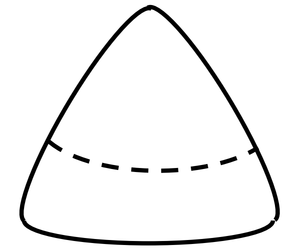


- There are no parallel straight lines.
- The sum of angles of a triangle $>$ 2 right angles.
- The *circumference* of any circle $< 2\pi \times$ *radius*.



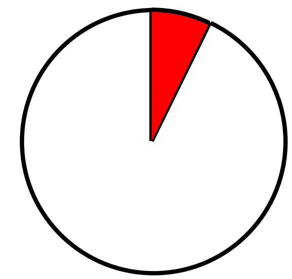
Tighten up a Euclidean circle -- remove wedges from it.

On the surface of a sphere:



- There are no parallel straight lines.
- The sum of angles of a triangle $>$ 2 right angles.
- The *circumference* of any circle $< 2\pi \times$ *radius*.

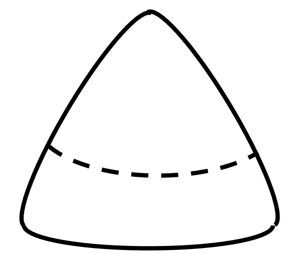
Tighten up a Euclidean circle --
remove wedges from it.



On the surface of a sphere:

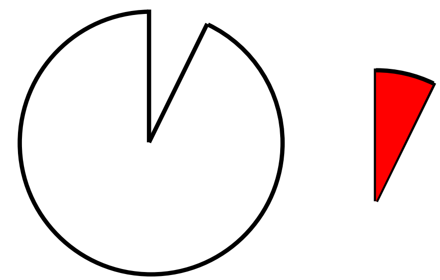
- There are no parallel straight lines.

- The sum of angles of a triangle $>$ 2 right angles.

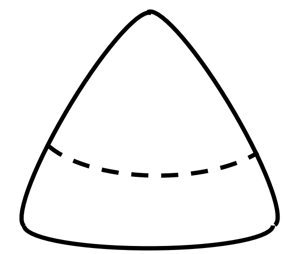


- The *circumference* of any circle $< 2\pi \times$ *radius*.

Tighten up a Euclidean circle -- remove wedges from it.

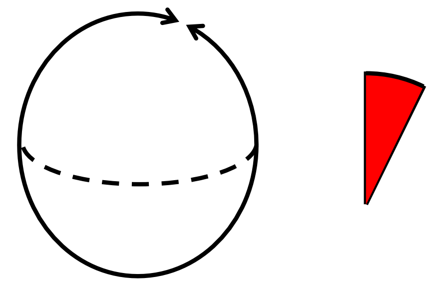


On the surface of a sphere:



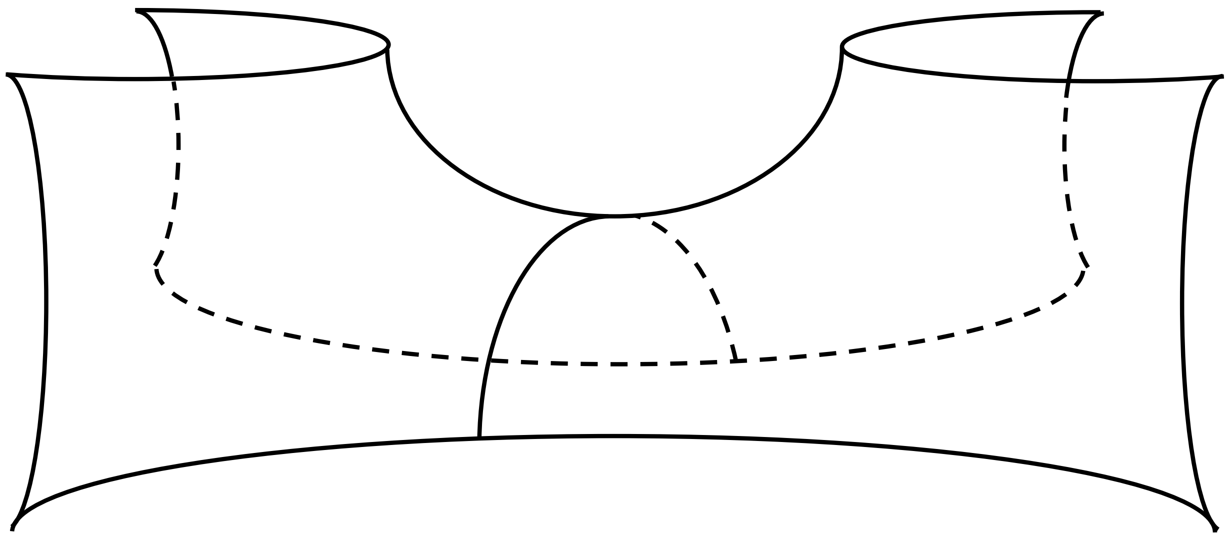
- There are no parallel straight lines.
- The sum of angles of a triangle $>$ 2 right angles.
- The *circumference* of any circle $< 2\pi \times$ *radius*.

Tighten up a Euclidean circle --
remove wedges from it.



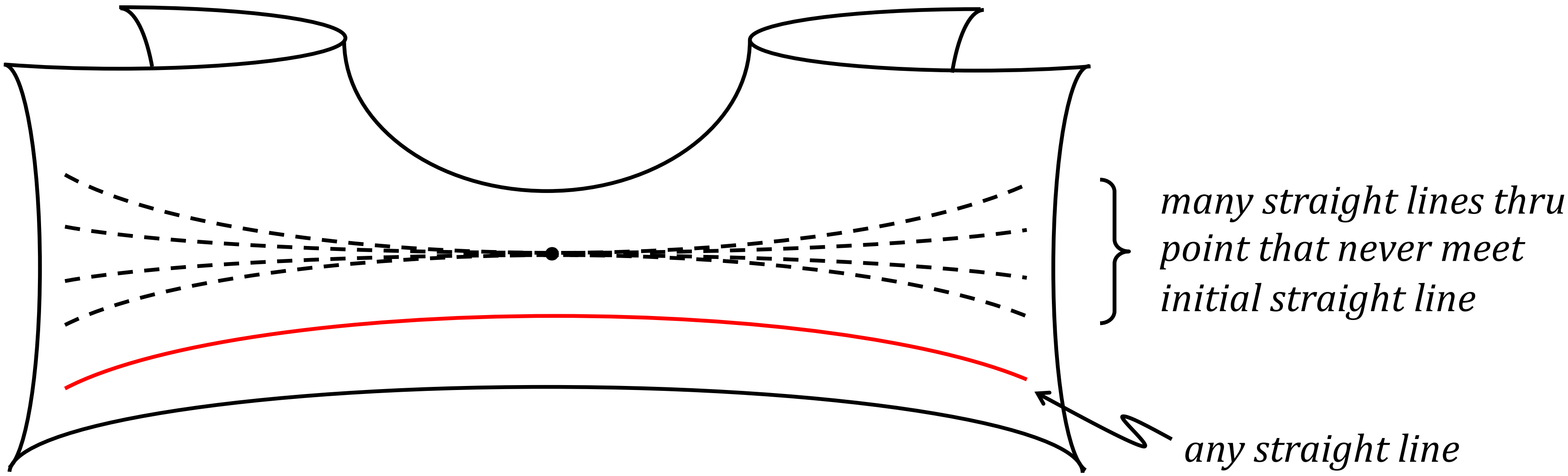
Case of 5^{many}

- 5^{many} geometry $=$ *hyperbolic geometry* $=$ geometry of surfaces of *negative* curvature.
- *Example*: surface of a saddle (hyperbolic paraboloid)



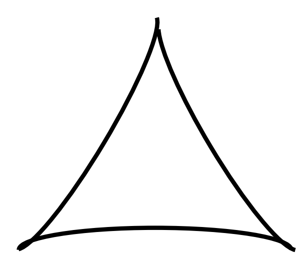
Case of 5^{many}

- 5^{many} geometry = *hyperbolic geometry* = geometry of surfaces of *negative* curvature.
- *Example*: surface of a saddle (hyperbolic paraboloid)



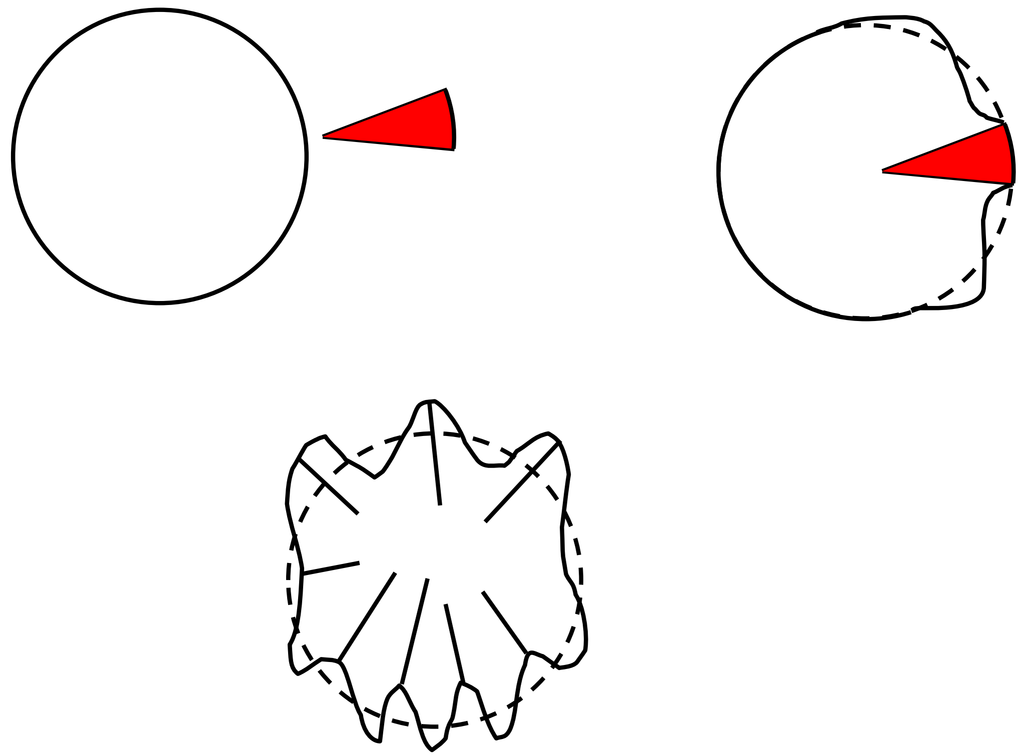
On the surface of a saddle

- There are indefinitely many lines through a given point that are parallel to any given straight line.
- The sum of angles of a triangle < 2 right angles.



- The *circumference* of a circle $> 2\pi \times$ *radius*.

Loosen up a Euclidean circle --
add wedges to it.

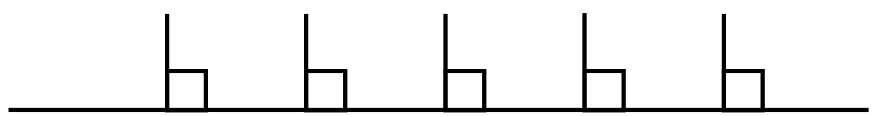


Euclidean circle (dotted line) with circumference $= 2\pi R$.

Wavey hyperbolic circle with circumference $> 2\pi R$.

Question: How could we determine what type of space we live in?

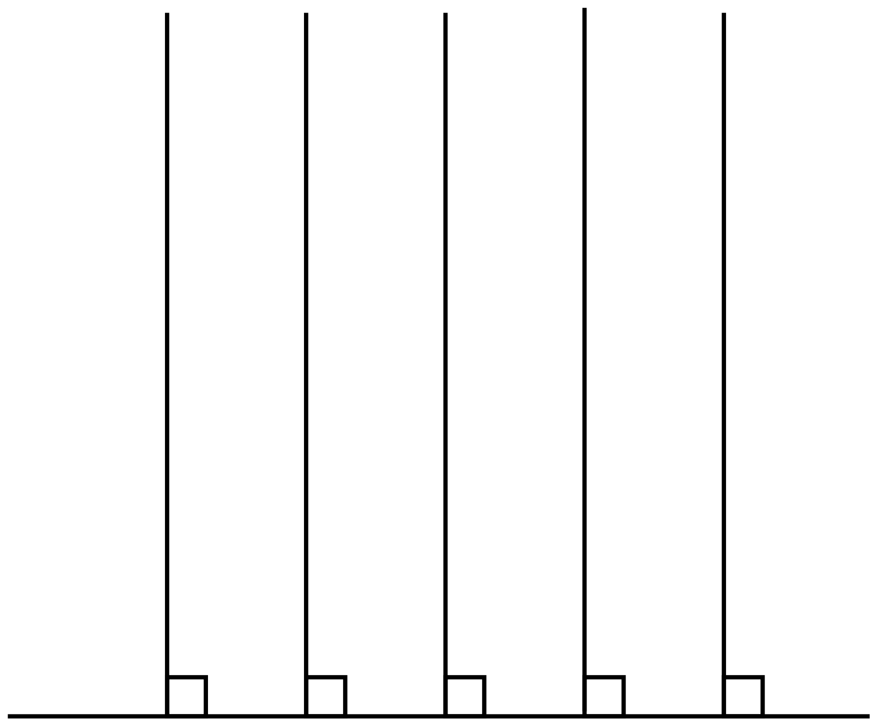
Answer: Use "geodesic deviation" to detect intrinsic curvature.



- Draw base line.
- Erect perpendiculars.
- Determine whether geodesics deviate in other regions of space.

<u>Question</u>: How could we determine what type of space we live in?

<u>Answer</u>: Use "geodesic deviation" to detect intrinsic curvature.

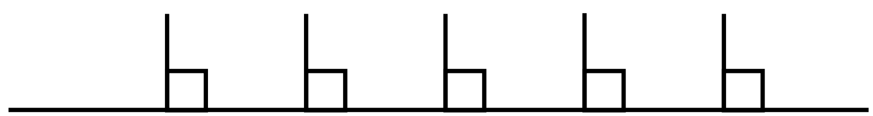


zero curvature

- Draw base line.
- Erect perpendiculars.
- Determine whether geodesics deviate in other regions of space.

Question: How could we determine what type of space we live in?

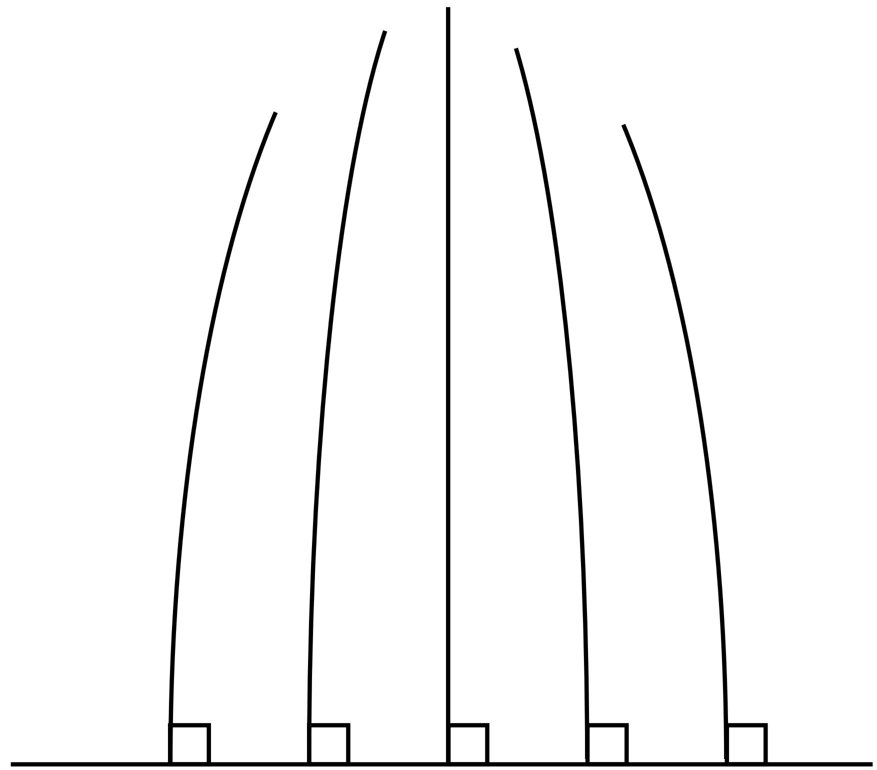
Answer: Use "geodesic deviation" to detect intrinsic curvature.



- Draw base line.
- Erect perpendiculars.
- Determine whether geodesics deviate in other regions of space.

Question: How could we determine what type of space we live in?

Answer: Use "geodesic deviation" to detect intrinsic curvature.

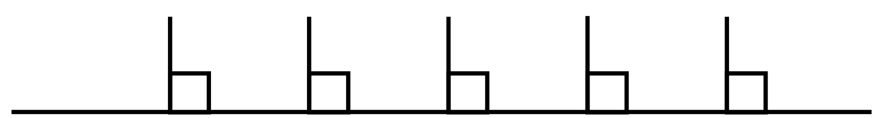


positive curvature

- Draw base line.
- Erect perpendiculars.
- Determine whether geodesics deviate in other regions of space.

Question: How could we determine what type of space we live in?

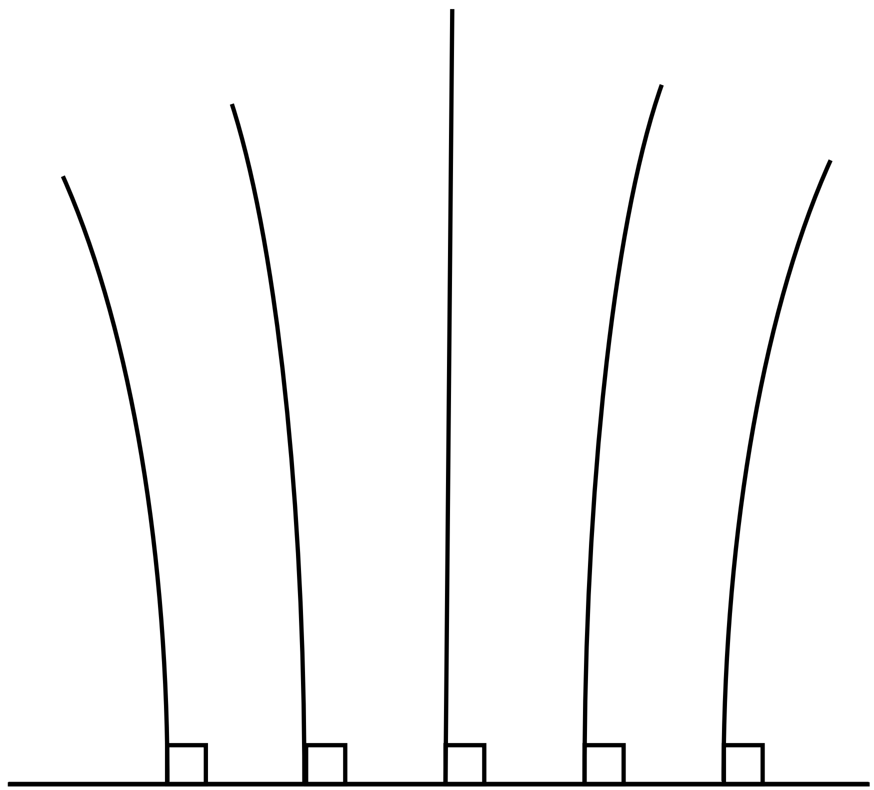
Answer: Use "geodesic deviation" to detect intrinsic curvature.



- Draw base line.
- Erect perpendiculars.
- Determine whether geodesics deviate in other regions of space.

<u>*Question*</u>: How could we determine what type of space we live in?

<u>*Answer*</u>: Use "geodesic deviation" to detect intrinsic curvature.



negative curvature

- Draw base line.
- Erect perpendiculars.
- Determine whether geodesics deviate in other regions of space.

- 1868: Eugenio Beltrami demonstrates that hyperbolic geometry is logically consistent.
- 1871: Felix Klein demonstrates that elliptical (spherical) geometry is logically consistent.

Eugenio Beltrami
(1835-1900)

Felix Klein
(1849-1925)

- <u>*So*</u>: Euclidean geometry is not a *necessary precondition* for a consistent description of the spatial aspects of the physical world.

Immanuel Kant
(1724-1804)

"Space is a necessary a priori representation, which underlies all outer intuitions. We can never represent to ourselves the absence of space, though we can quite well think it as empty of objects. It must therefore be regarded as the condition of the possibility of appearances, and not as a determination dependent on them." (1781)

- Euclidean geometry is a *consistent* axiomatic system: Given Euclid's 5 postulates, all other Euclidean claims can be derived, and no contradictory claims can be derived.
- <u>*Kant*</u>: Euclidean geometry is necessary and universal; a necessary precondition for experiencing the world.

- Can this still be maintained?
  - *Perhaps humans are predisposed to perceive the world in Euclidean terms, even though the world might not be Euclidean.*
- *<u>But</u>*: Kant's claim is stronger than this. Statements in Euclidean geometry are necessary and universal truths about the world.
- *<u>And</u>*: With the development of consistent non-Euclidean geometries, this can no longer be the case.
  - *Is Euclid's 5th Postulate a necessary and universal truth?*
  - *Not if hyperbolic or elliptic geometry is true of the world*.
- What geometry is true of the world is now a matter of empirical inquiry, and no longer a matter of pure reason alone.
- One can thus distinguish between:

  (a) pure geometery: statements are *analytic a priori*.

  (b) applied geometry: statements are *synthetic a posteriori*.