05. Dynamics--Momentum

Topics:
1. Three Basic Concepts
2. Newtonian Momentum
3. Relativistic Momentum

Theory of motion = kinematics + dynamics

- Kinematics = Description of motion in the absence of forces.
  - *spacetime diagrams*
  - *relativity of simultaneity*
  - *time dilation*
  - *length contraction*
  - *structure of spacetime (Minkowski metric)*

- Dynamics = Description of motion in the presence of forces.

Let's now consider aspects of the dynamics of special relativity.

- End up with the famous expression $E = mc^2$

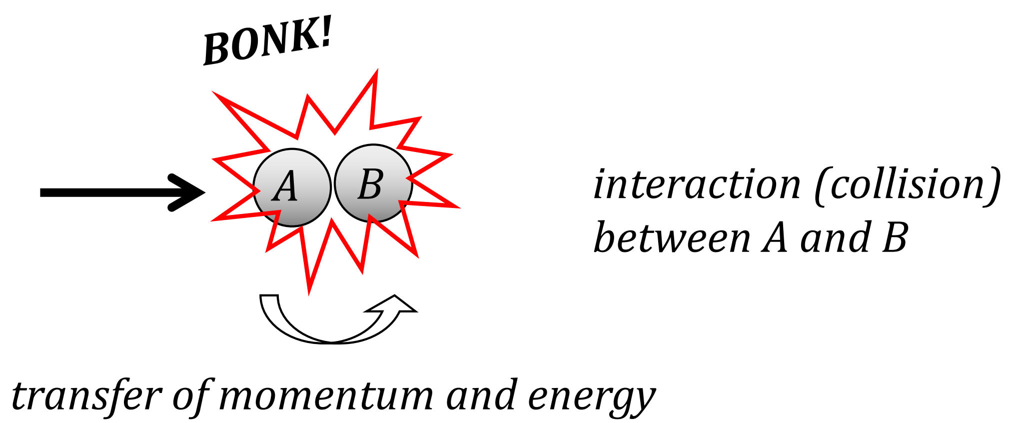
1. Three Basic Concepts of Newtonian Dynamics

(i) *Momentum* = quantity of motion of an object = (*mass*) × (*velocity*).

(ii) *Energy* = ability of an object to do work.

Fundamental principle of Newtonian dynamics:
When objects interact, they exchange energy and momentum.

Conservation of energy and momentum in Newtonian dynamics:
In all interactions, the total amount of energy and the total amount of momentum stay the same.



1. Three Basic Concepts of Newtonian Dynamics

(i) *Momentum* = quantity of motion of an object = (*mass*) × (*velocity*).

(ii) *Energy* = ability of an object to do work.

(iii) *Force* = the cause of changes in momentum and energy.

Two simple relations between Energy, Momentum, and Force:

$$\left[\text{Momentum gained}\right] = \left[\text{Force}\right] \times \left[\begin{array}{l}\text{Time through}\\ \text{which force acts}\end{array}\right]$$

$$\left[\text{Energy gained}\right] = \left[\text{Force}\right] \times \left[\begin{array}{l}\text{Distance through}\\ \text{which force acts}\end{array}\right]$$

2. Newtonian Momentum

- Newtonian 3-momentum: $\vec{\mathbf{p}} = m\vec{\mathbf{v}} = m\left(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}\right)$

> *Newton's 2nd Law of Motion*:
>
> The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

- *Restate as*: *The rate of change of momentum is equal to the impressed force.*

small change in momentum → $d\vec{\mathbf{p}}$

small change in time → dt

$$\frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}}$$

Note: $\dfrac{d\vec{\mathbf{p}}}{dt} = \dfrac{d(m\vec{\mathbf{v}})}{dt} = m\dfrac{d\vec{\mathbf{v}}}{dt} = m\vec{\mathbf{a}}$, where $\vec{\mathbf{a}}$ is acceleration.

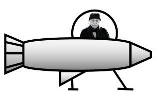
- *So*: Can also express Newton's 2nd Law as $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$

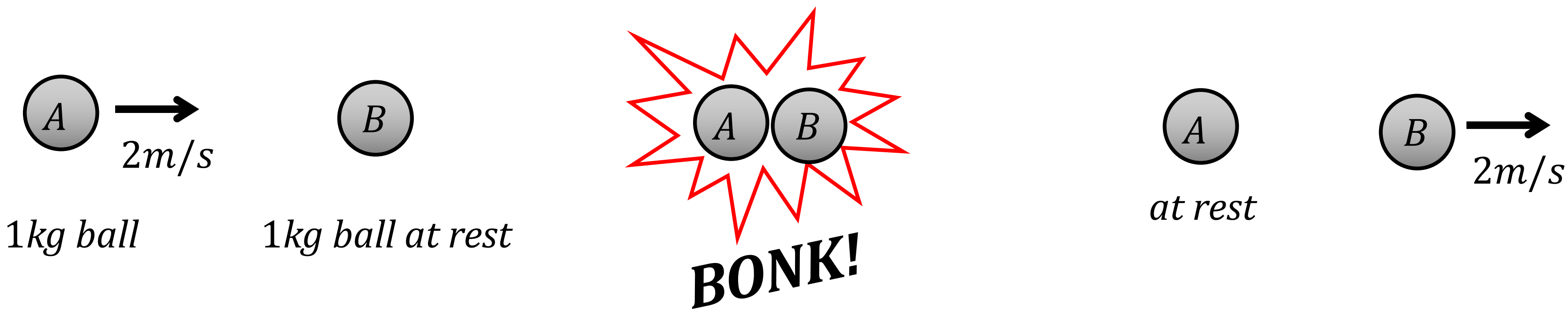
Conservation of Newtonian Momentum

- A closed system is one on which no external forces are acting: $\vec{\mathbf{F}} = 0$
- *Claim*: The total momentum of a closed system is conserved.

  Proof: $\vec{\mathbf{F}} = \dfrac{d\vec{\mathbf{p}}}{dt} = 0$, thus $\vec{\mathbf{p}}$ doesn't change over time.

Example:





- Initial momentum $= \vec{\mathbf{p}}_A + \vec{\mathbf{p}}_B = (2kg \cdot m/s)\hat{\mathbf{x}} + 0 = (2kg \cdot m/s)\hat{\mathbf{x}}$
- Final momentum $= \vec{\mathbf{p}}_A + \vec{\mathbf{p}}_B = 0 + (2kg \cdot m/s)\hat{\mathbf{x}} = (2kg \cdot m/s)\hat{\mathbf{x}}$

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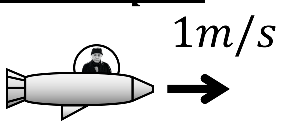
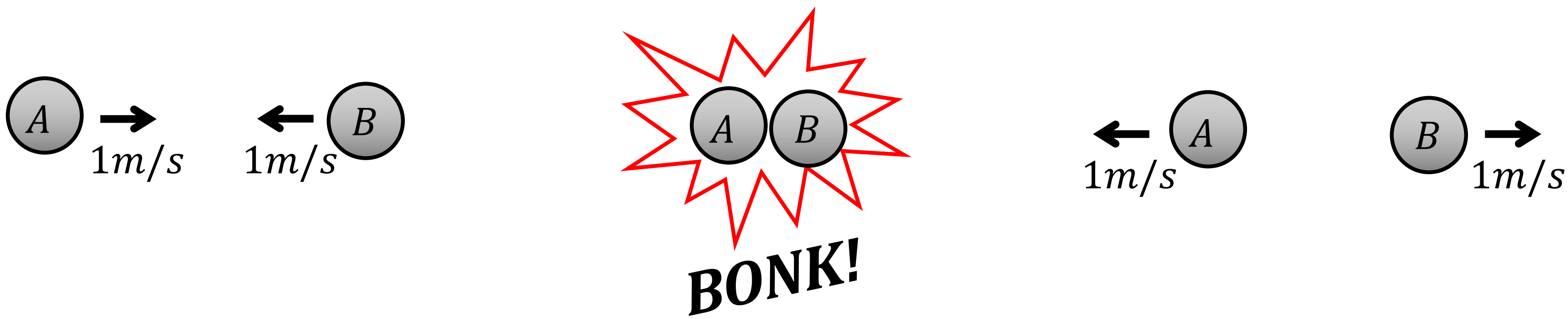
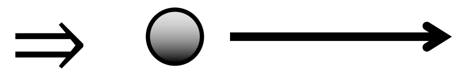




- Initial momentum $= \vec{\mathbf{p}}_A + \vec{\mathbf{p}}_B = (1kg \cdot m/s)\hat{\mathbf{x}} + -(1kg \cdot m/s)\hat{\mathbf{x}} = 0$
- Final momentum $= \vec{\mathbf{p}}_A + \vec{\mathbf{p}}_B = -(1kg \cdot m/s)\hat{\mathbf{x}} + (1kg \cdot m/s)\hat{\mathbf{x}} = 0$

Claim: Newtonian momentum can be used to produce arbitrarily large velocities.

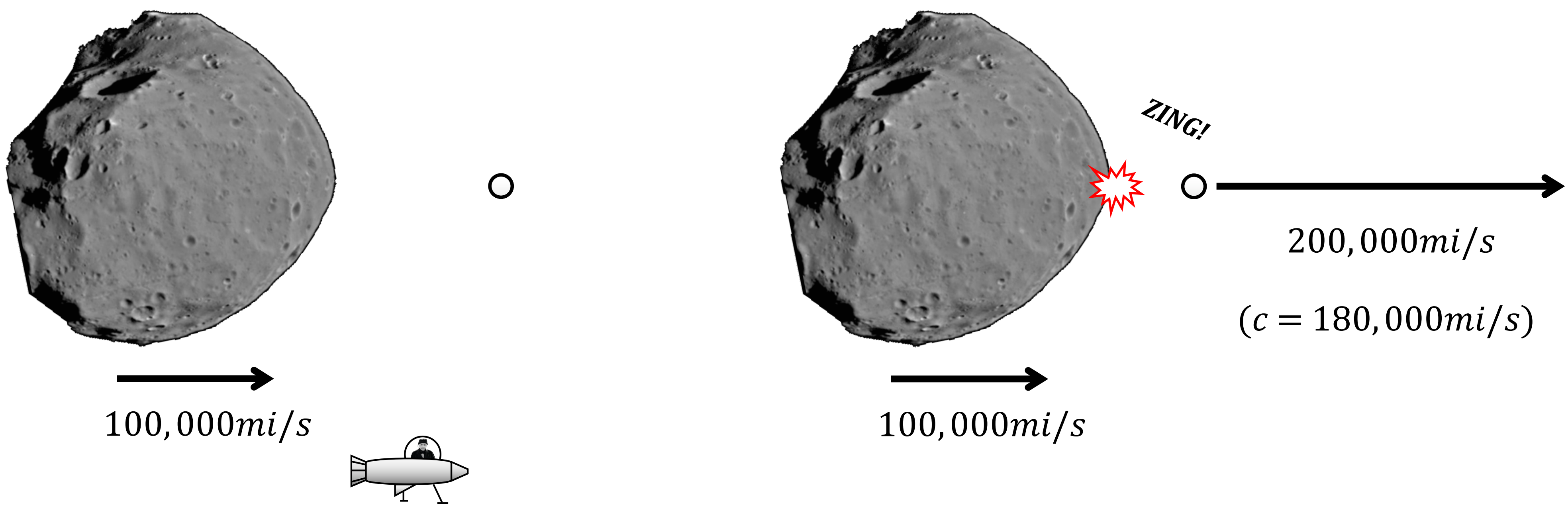


Apply huge force to object

Transfer correspondingly huge amount of momentum to object.

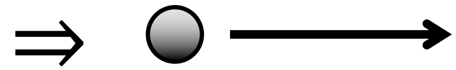
- $(momentum) = (mass) \times (velocity)$.
- No upper limit for Newtonian momentum; Newtonian mass is constant.
- Thus no upper limit for Newtonian velocities! Possible velocities > *c!*

Example:



observer at rest

Claim: Newtonian momentum can be used to produce arbitrarily large velocities.



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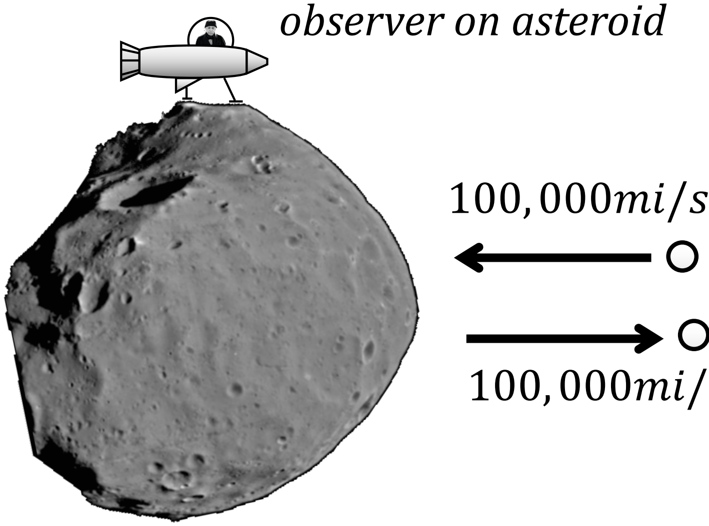

- In asteroid frame, ping-pong ball approaches at $100{,}000 mi/s$, and ricochets off at effectively the same speed.

Why can't this happen in special relativity?

Kinematical Explanation

- In special relativity, we can't simply add velocities together.
  - *The final speed of the ball is not simply* $100{,}000 mi/s + 100{,}000 mi/s = 200{,}000 mi/s.$

- Recall the rule for velocity composition in special relativity:

$$v = \frac{v_0 + v'}{1 + \frac{v_0 v'}{c^2}}$$

v = final speed of ball *w.r.t.* stationary frame

v' = final speed of ball *w.r.t.* moving frame (asteroid)

v_0 = final speed of moving frame (asteroid) *w.r.t.* stationary frame

- According to this rule, the final speed of the ball is:

$$v = \frac{100{,}000 mi/s + 100{,}000 mi/s}{1 + \frac{100{,}000 mi/s \times 100{,}000 mi/s}{(186{,}000 mi/s)^2}} = 155{,}000 mi/s$$

Why can't this happen in special relativity?

Dynamical Explanation

$$(momentum) = (mass) \times (velocity)$$

- Suppose we retain the assumption of no upper limit on momentum.
- In special relativity, there's an upper limit on velocity for things like ping-pong balls; namely, c.
- Thus, we have to give up the Newtonian assumption that mass is constant!

We have to re-define Newtonian momentum for special relativity!

3. Relativistic Momentum

Requirements on Relativistic Momentum

(1) Agrees with Newtonian momentum $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, m constant, when $v \ll c$.

(2) Becomes *infinite* as *v* approaches *c*.

(2) prohibits being able to use momentum transfers to push the speed of an object over c.

Definition (relativistic 3-momentum):

$\vec{\mathbf{p}}_{SR} = m\gamma\vec{\mathbf{v}}$

Claim: Relativistic 3-momentum satisfies Properties (1) and (2).

- Recall that $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ has the following properties: $\begin{cases} \gamma \approx 1, & \text{when } v \ll c \\ \gamma \to \infty, & \text{when } v \to c \end{cases}$
- *So*: When $v \ll c$, $m\gamma\vec{\mathbf{v}} \approx m\vec{\mathbf{v}}$
- *And*: When $v \to c$, $m\gamma\vec{\mathbf{v}} \to \infty$

Terminology

$m\gamma\vec{\mathbf{v}}$ = *relativistic momentum*

$m\gamma$ = *relativistic mass* $(v \neq 0)$

m = *rest mass* $(v = 0)$

- Relativistic mass is *not* constant: It *varies with velocity*.
  - *As $v \to c$, relativistic mass $\to \infty$.*

- Why can't we boost a ping-pong ball past c?
  - *Because as $v \to c$, the ping-pong's relativistic mass approaches ∞!*

- What about objects that travel at $v = c$?

$$\vec{\mathbf{p}}_{SR} = m\gamma\vec{\mathbf{v}} \overset{?}{\to} \infty$$

  - *Such objects (light signals, etc.) do possess finite momentum.*
  - *So their rest mass m must be zero!*