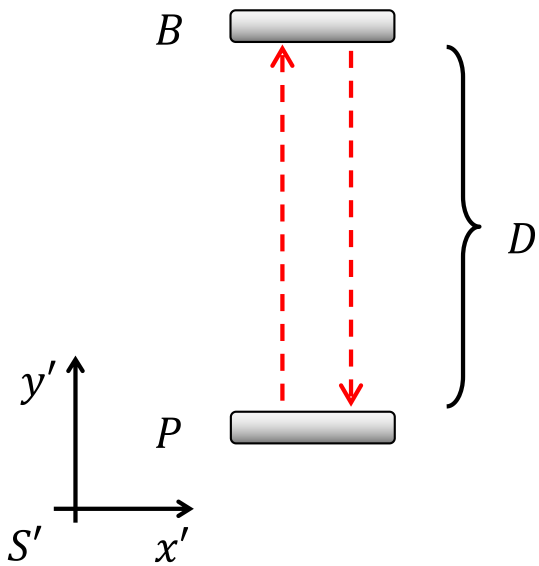
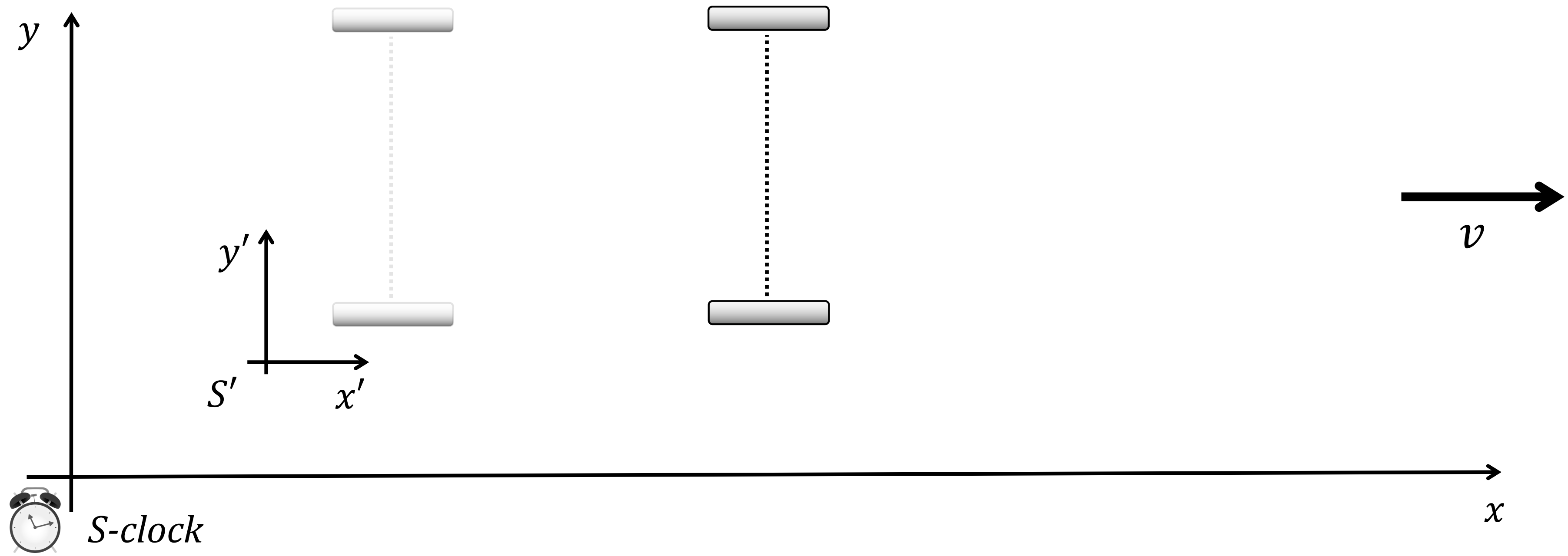
04. Kinematical Effects
1. Time Dilation



- Consider a "light clock" that defines a reference frame S'.
- One "tick" of this clock (time for light to travel PBP): $\Delta t' = \dfrac{2D}{c}$.

- Now set S'-light clock in motion *w.r.t.* reference frame S.



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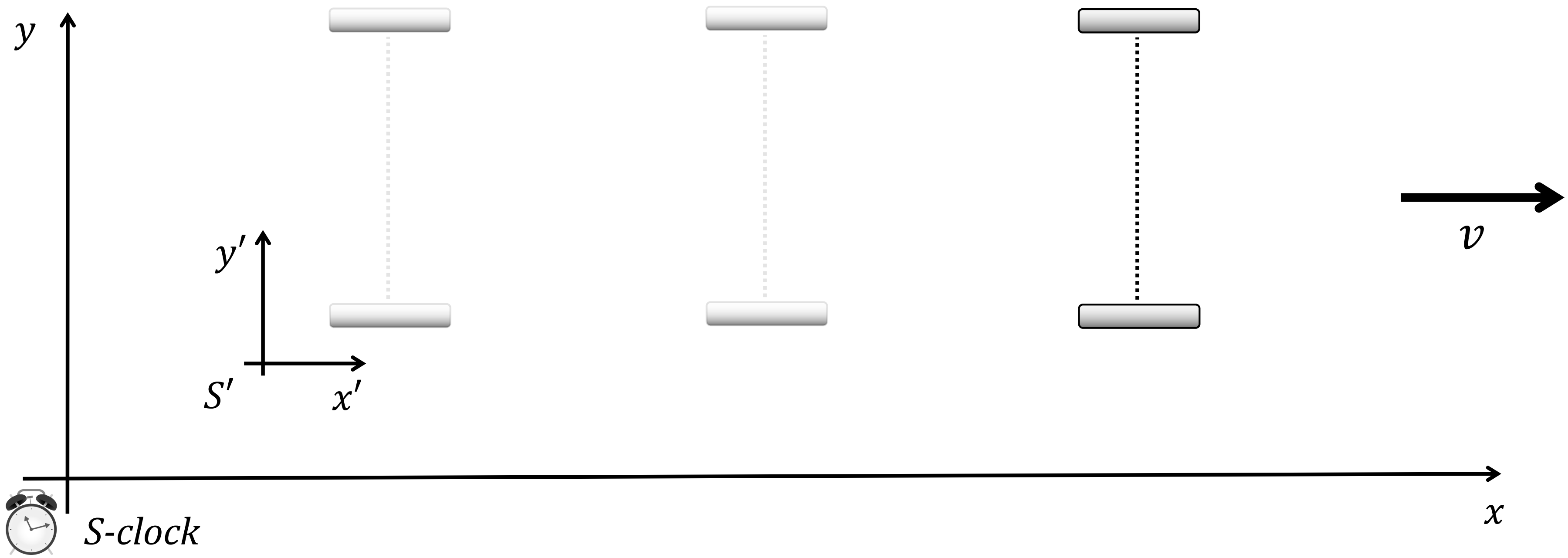

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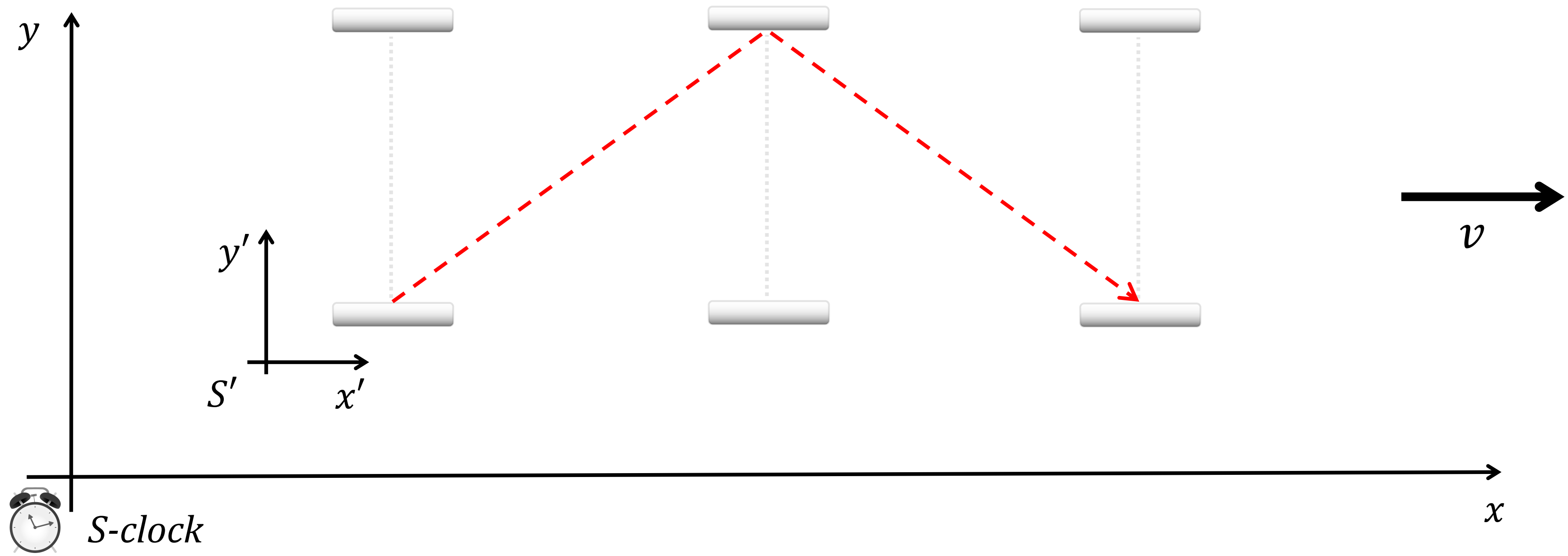
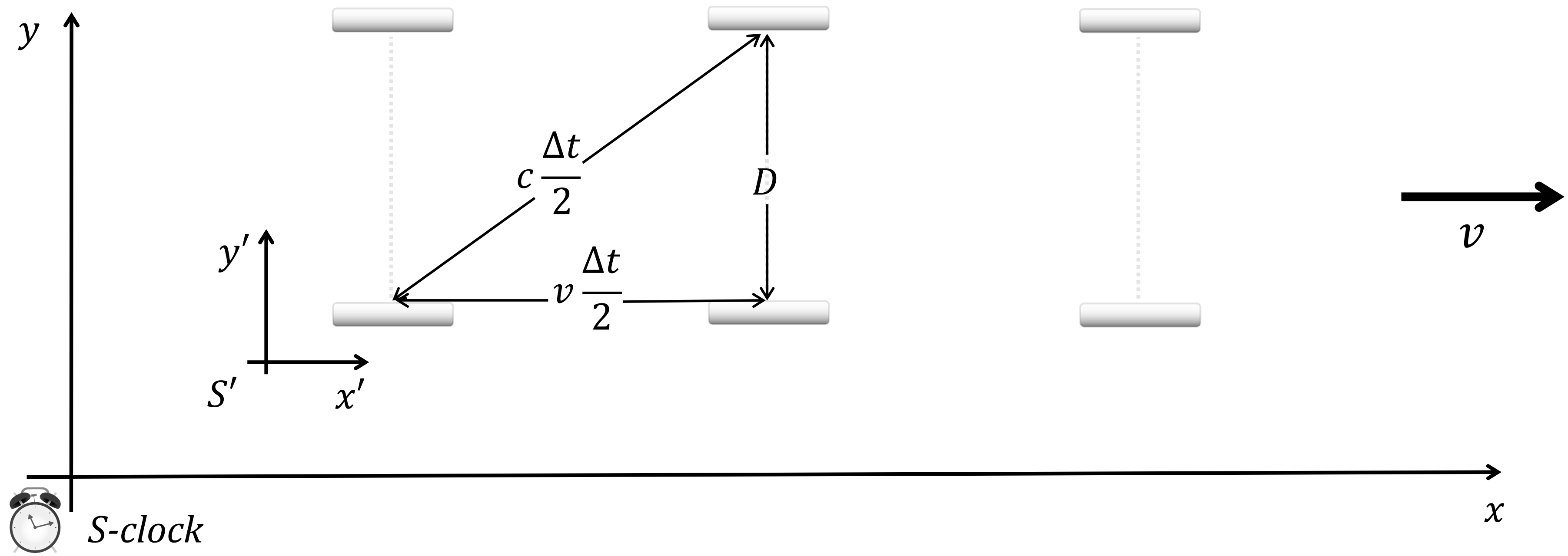


- Now set S'-light clock in motion *w.r.t.* reference frame S.
- What is time Δt for 1 tick of light clock *w.r.t.* S-clock at rest in frame S?



- In S-frame: $\left(c\frac{\Delta t}{2}\right)^2 = D^2 + \left(v\frac{\Delta t}{2}\right)^2$. So: $\Delta t = \frac{2D}{\sqrt{c^2 - v^2}} = \frac{2D}{c}\gamma = \Delta t'\gamma.$

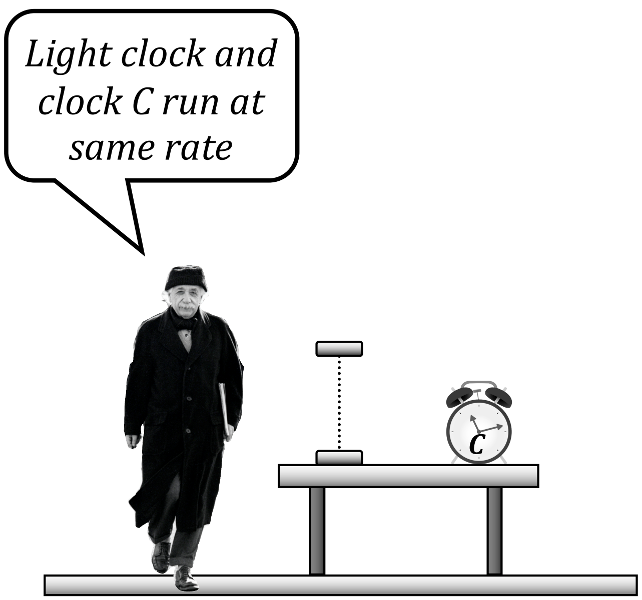
- The speed of the light signal c is the same in both S and S' (*Light Postulate*).
- With respect to S, light signal of the moving S'-clock has to travel a *greater* distance. *So it's tick will come after the tick of the stationary S-clock.*
- With respect to S, the moving S'-clock ticks slower: $\Delta t' = \frac{\Delta t}{\gamma}$

> *Time dilation*: A clock moving at constant velocity runs at a slower rate with respect to a stationary reference frame.

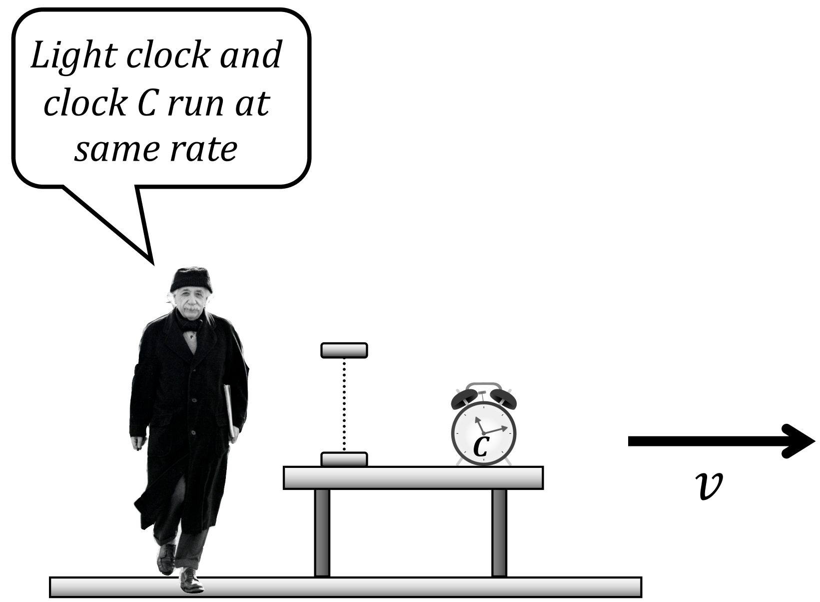
Claim: Time dilation holds for *any* type of time measuring device. *(Not just for light clocks!)*

Proof:

- Suppose there was a clock C that did not experience time dilation.
- Then we could use C to physically distinguish between inertial frames.



lab at rest

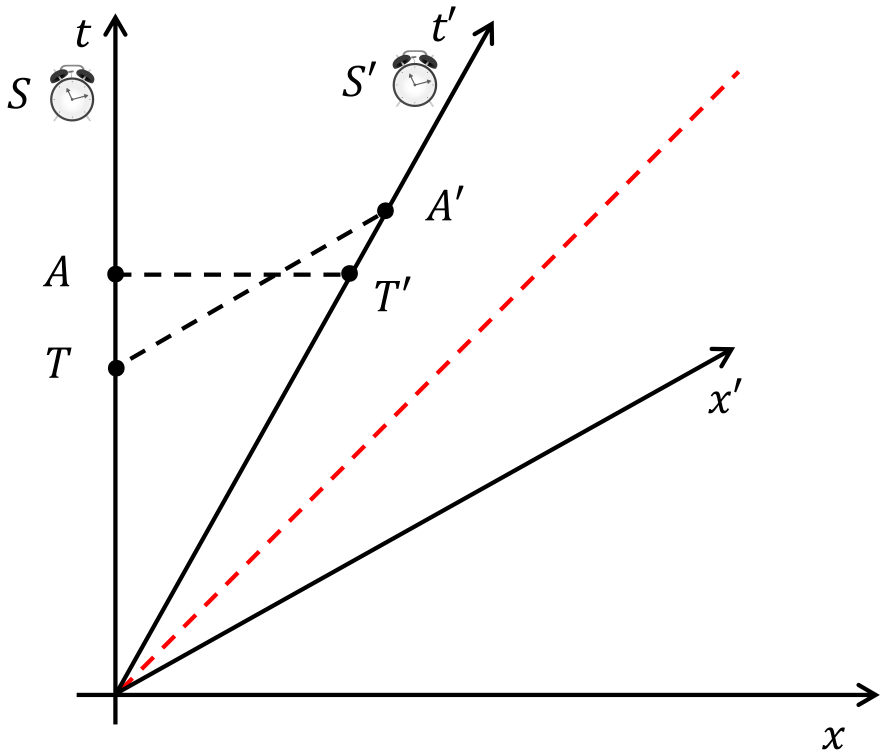


lab in constant motion

Albert must say the same thing in the uniformly moving lab (otherwise Principle of Relativity would be violated).

Reciprocity

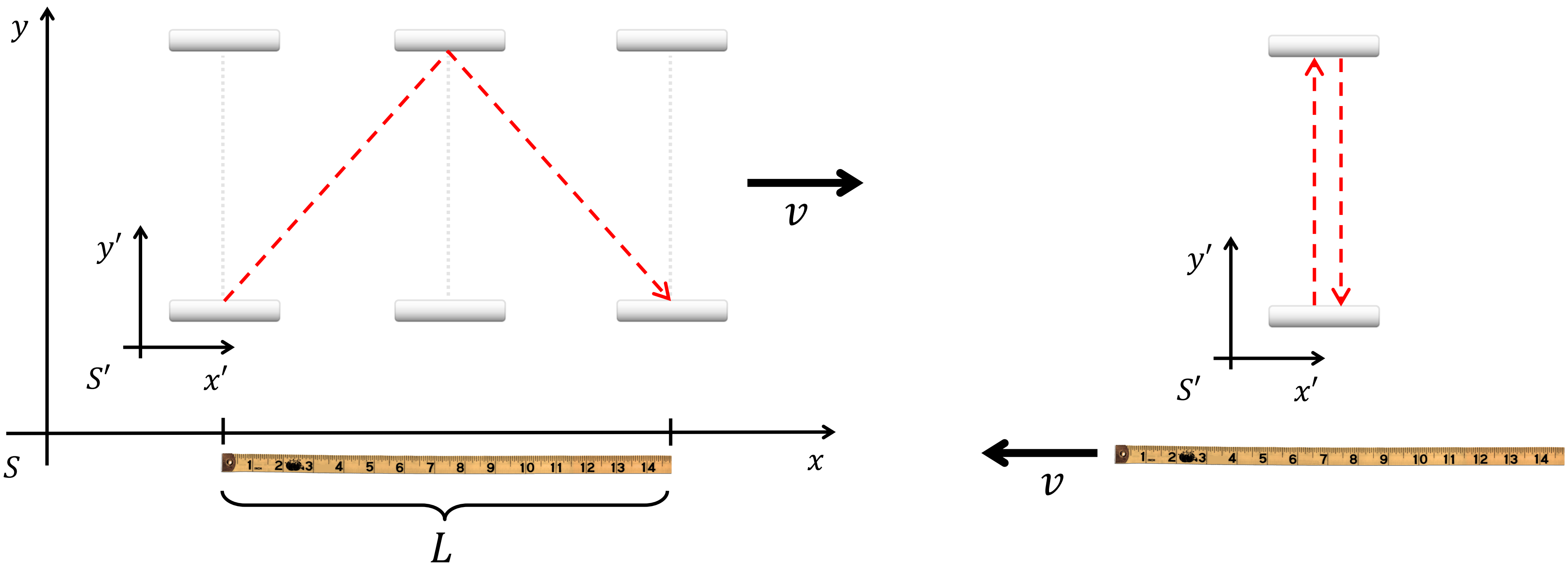
- S says S'-clock ticks slower.
- S' says S-clock ticks slower.



$T = 1$ tick of S clock
$T' = 1$ tick of S' clock

- If T' happens after T with respect to S...
- ... then T' happens before T with respect to S'.

2. Length Contraction

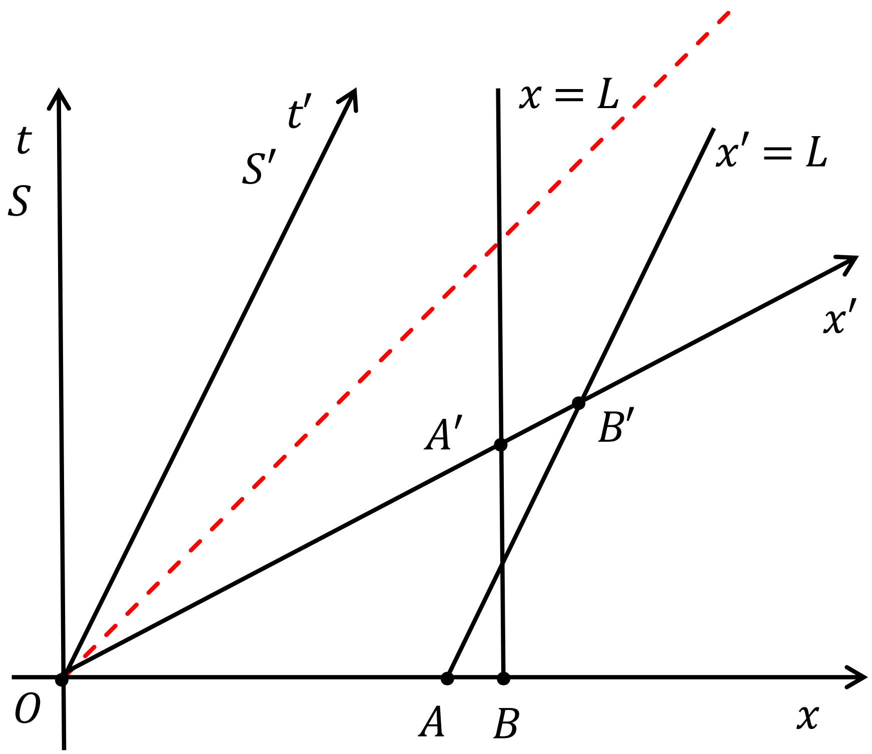


- In S-frame, distance S'-clock travels in one tick is $L = v\Delta t$.
  - *Think of L as a ruler at rest w.r.t. S.*
  - *S' sees this ruler traveling at constant speed v to the left.*
  - *It passes in time $\Delta t'$ (1 tick of S'-clock) and so has length $L' = v\Delta t'$ according to S'.*
- We thus have $L = v\Delta t = v\gamma\Delta t' = L'\gamma$.

<u>*Length contraction*</u>: The length of a constantly moving object parallel to its direction of motion is shorter with respect to a stationary reference frame

Reciprocity

- S says S'-ruler has shrunk.
- S' says S-ruler has shrunk



- Let R be a ruler at rest with respect to S of length $x = L$.
- Let R' be a ruler in motion with respect S of same "rest" length $x' = L$.
- If R' is shorter than R with respect S, then $OA < OB$.
- So $OA' < OB'$. Thus R is shorter than R' with respect S'.

Why does S see R′ shorter than R?

- To measure the length of the moving R', a stationary S observer must make *simultaneous* marks where the endpoints of R' are as it passes by.

Why does S see R′ shorter than R?

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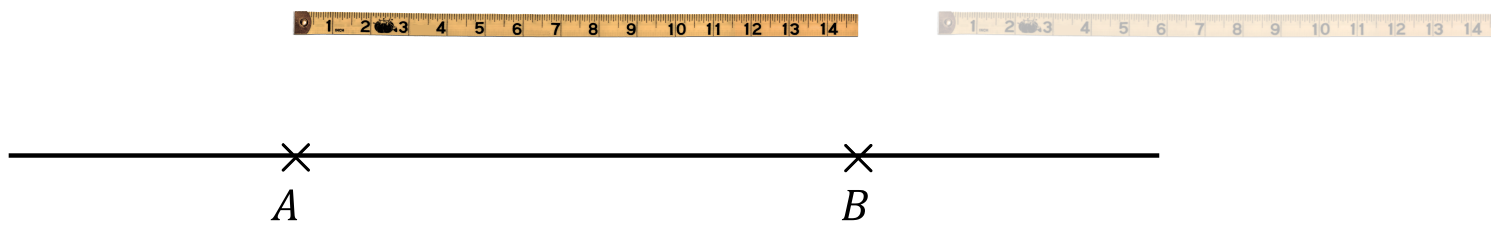

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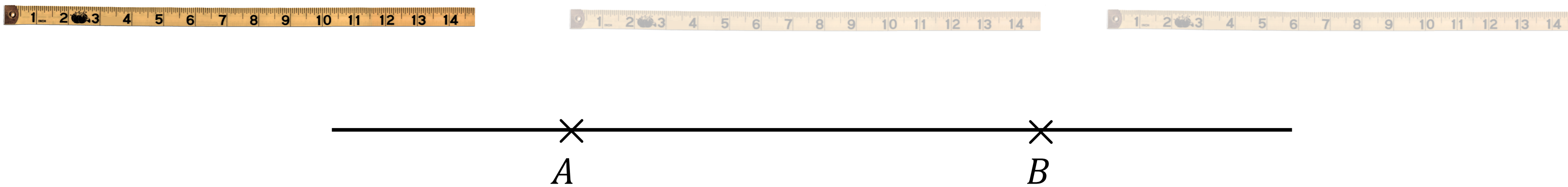

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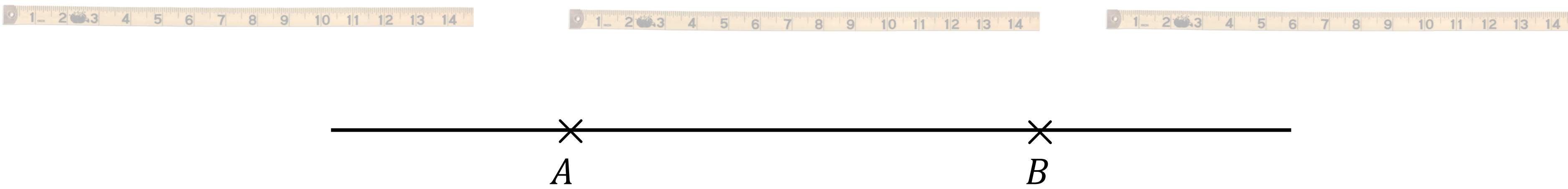
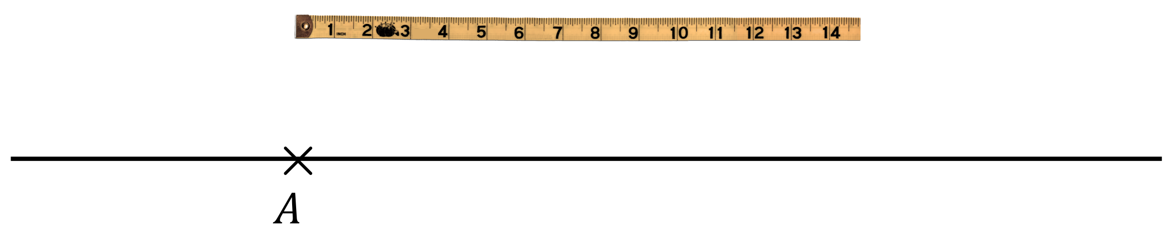


- <u>*But*</u>: Two events simultaneous with respect to S will not be simultaneous with respect to the moving ruler's frame S'!
- From point of veiw of S', S does not place the marks A, B at the same instant. Rather, with respect to S', S marks A before B.



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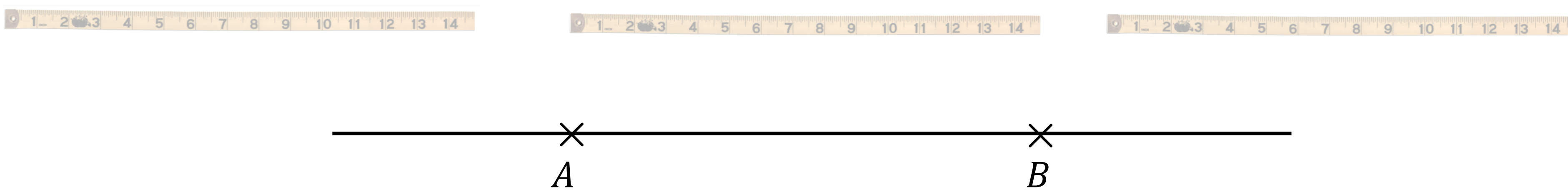

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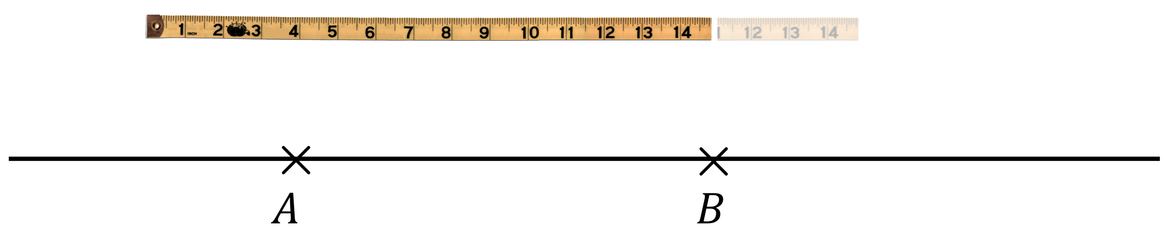
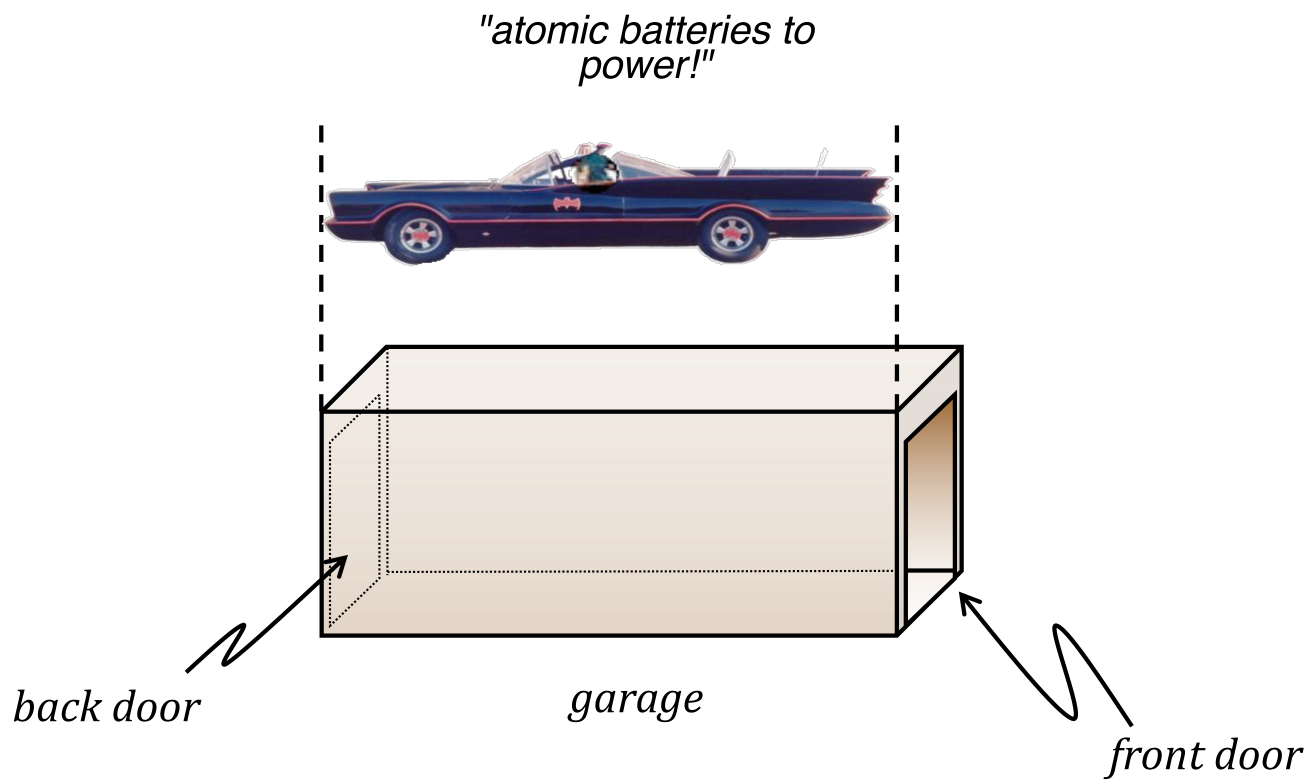


Length contraction is a consequence of the relativity of simultaneity.

3. "Paradoxes"

(a) The Batmobile "Paradox"

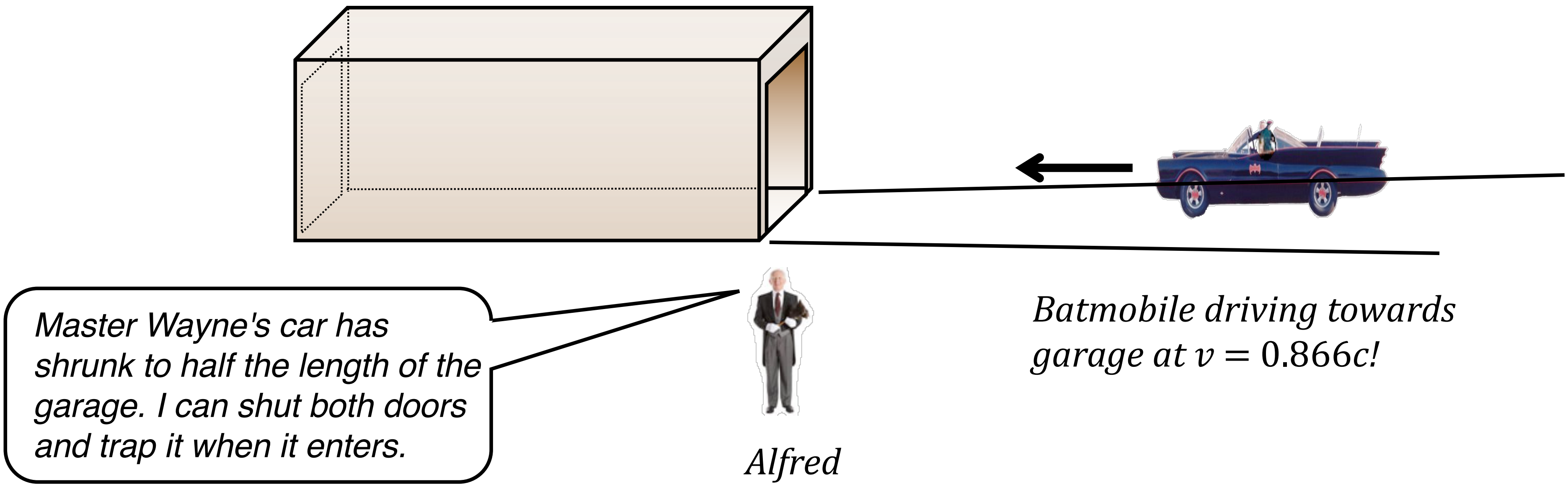
Setup:

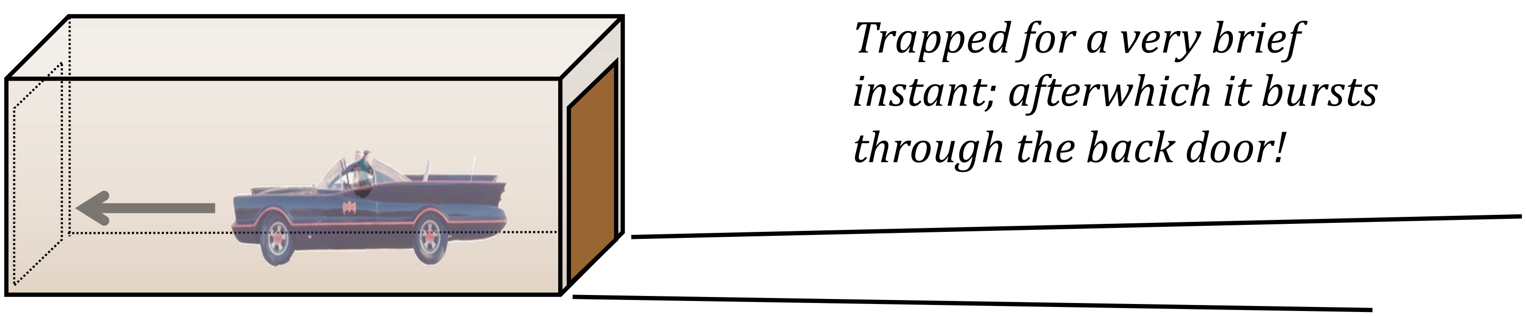


- Suppose the Batmobile and the Bat-garage are exactly the same length at rest *w.r.t.* each other.

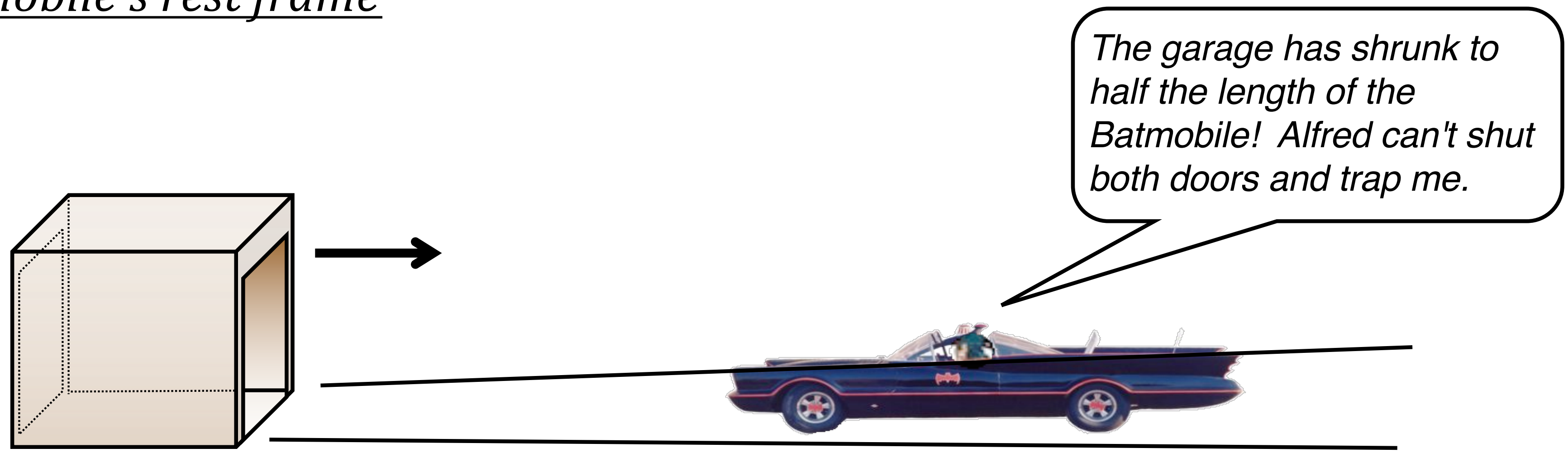
- Suppose the Batmobile is moving at a very fast constant speed with respect to the Bat-garage.

<u>In the garage's rest frame</u>



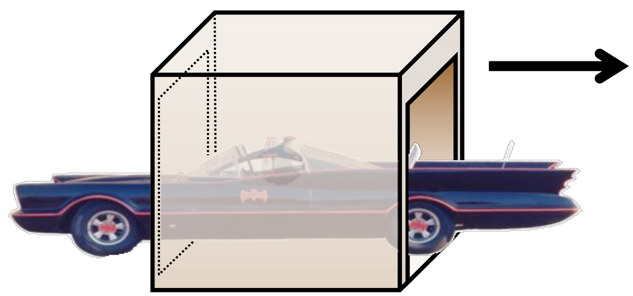


In the Batmobile's rest frame



Batman

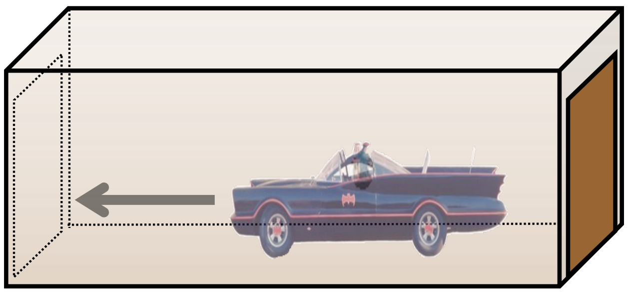
Garage moving towards Batmobile at $v = 0.866c$!



Before the rear of the Batmobile has entered the garage, it's front end has burst through the closed back door!

Who's right? Alfred or Batman? (Does the Batmobile get trapped or not?)

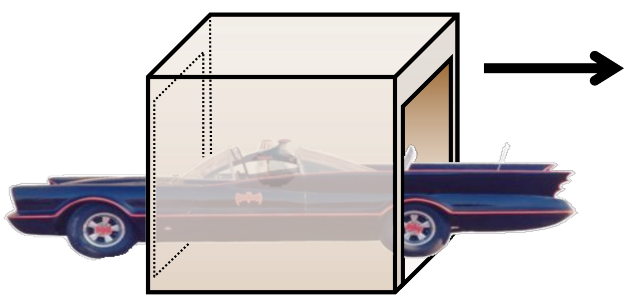
- Both are right!
- For Alfred, there *is* a time during which the Batmobile is *completely* inside the garage with both doors closed:



Alfred

As soon as the rear of the Batmobile clears the front door, I ***simultaneously*** *close both front and back doors, thereby trapping Master Wayne briefly inside, before he plows through the back door.*

- For Batman, the front and rear doors are not closed simultaneously! So there is never a time during which the Batmobile is completely inside the garage with both doors closed.



Batman

The back door closes first, then the front of the Batmobile plows through it, then just as the rear of the Batmobile clears the front door, Alfred closes it.

(b) The Twin "Paradox"

Set-up:

- Two twins: S (stay-at-home) and S' (traveling).
- Let S' travel away at constant speed v, close to c, for time $\Delta t'$ before returning at same speed v.

- *We know*: $\Delta t' = \frac{\Delta t}{\gamma}$, where $\Delta t =$ time elapsed at turnaround with respect to S.
- *And*: $2\Delta t$ = time of trip with respect to S
  $2\Delta t'$ = time of trip with respect to S'

- *So*: $2\Delta t' = \frac{2\Delta t}{\gamma} < 2\Delta t$.

- The traveling twin S' has aged less than S at the end of the trip!

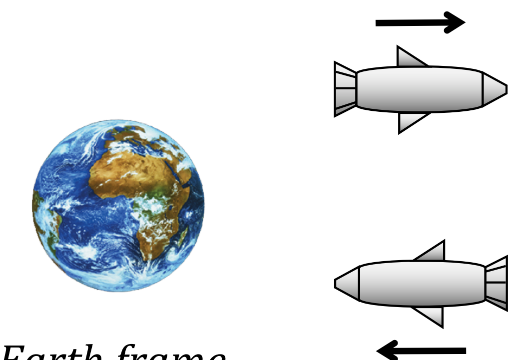
Rocket Time	*Earth Time*	*Distance S' travels*
1 *yr*	1.18 *yr*	0.56 *light-yr*
5 *yr*	83.7 *yr*	82.7 *light-yr*
10 *yr*	14,433 *yr*	14,432 *light-yr*
25 *yr*	7.4×10^{10} *yr*	7.4×10^{10} *light-yr*

<u>Does this violate the Principle of Relativity (PR)?</u>

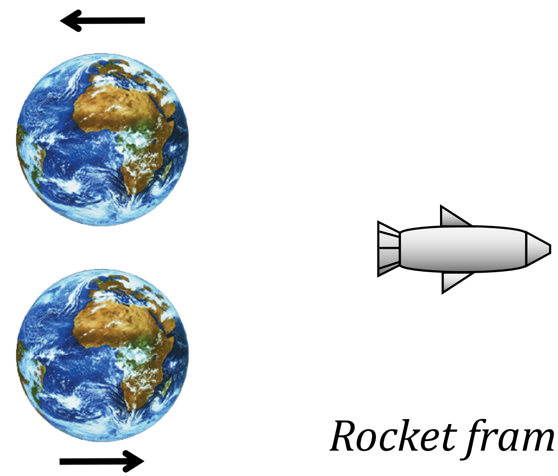
(a) S' is *younger* according to S.

(b) S is *older* according to S'.

- Does't PR require that the following reference frames be equivalent?



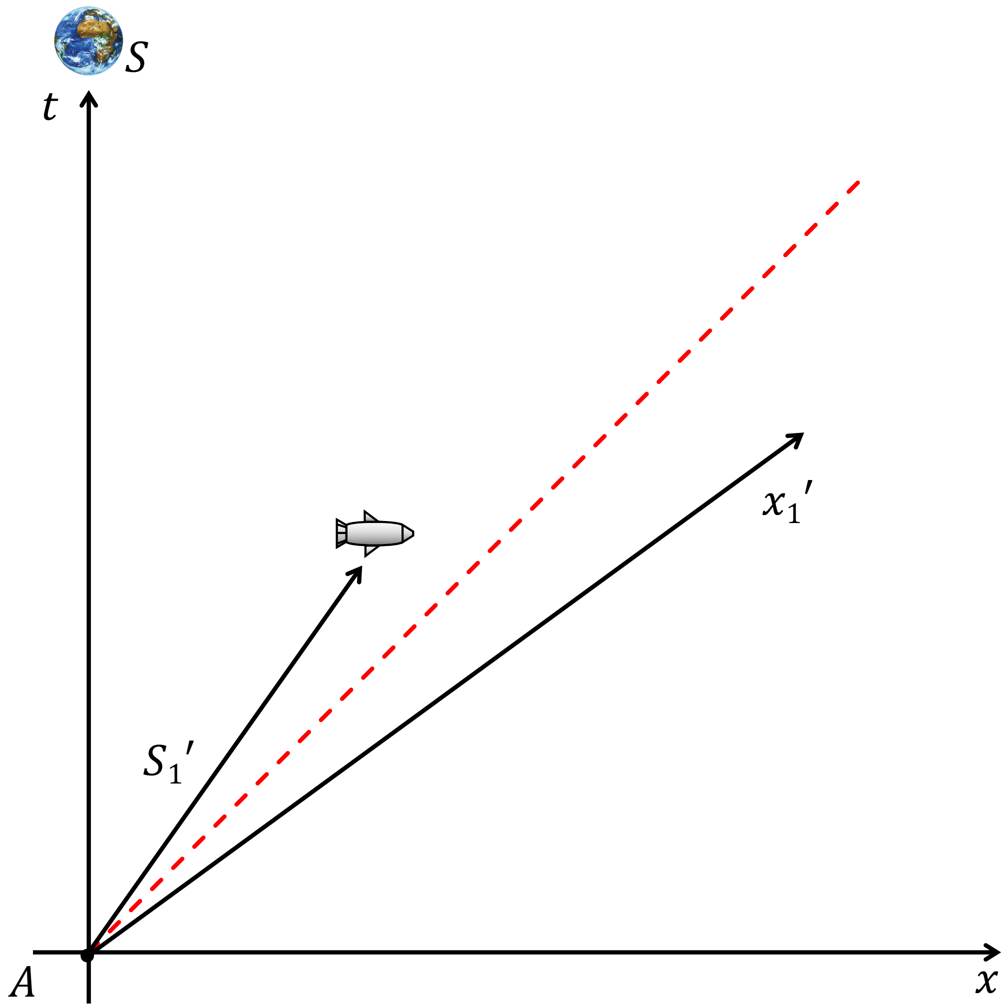
Earth frame

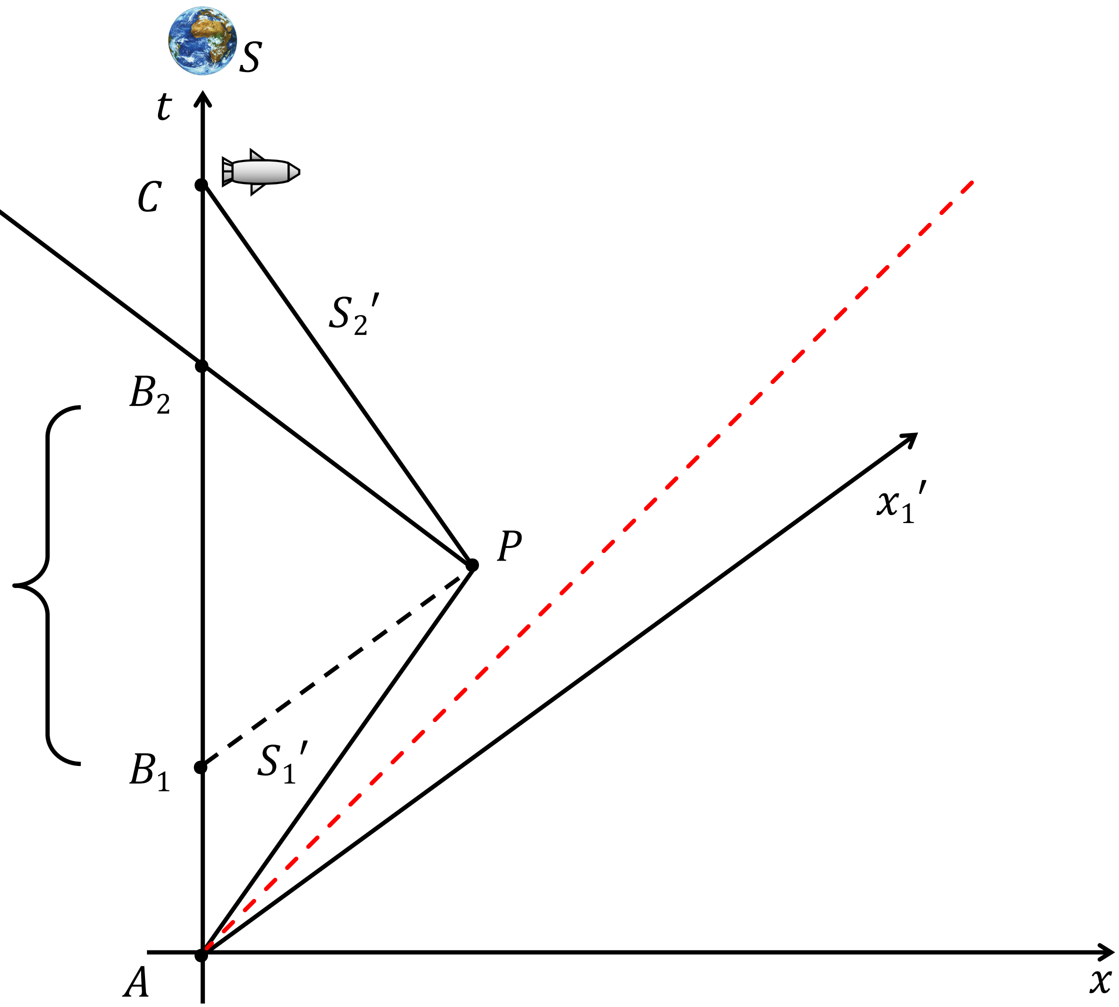


Rocket frame

- *<u>NO violation of PR</u>*: Reciprocity holds only between *single* inertial frames.
  - *S defines one inertial frame.*
  - *S' defines two intertial frames: one going and one returning.*

- *<u>Thus</u>*: S' uses two *different* standards of simultaneity during the entire trip. This accounts for why she is younger than S.


S
t
x₁'



- In the S_1' frame, P and B_1 are simultaneous.
- In the S_2' frame, P and B_2 are simultaneous.
- The chunk of S-time between B_1 and B_2 is missing from S'-time.

4. The Minkowski Metric

Spacetime of Special Relativity = Minkowski spacetime

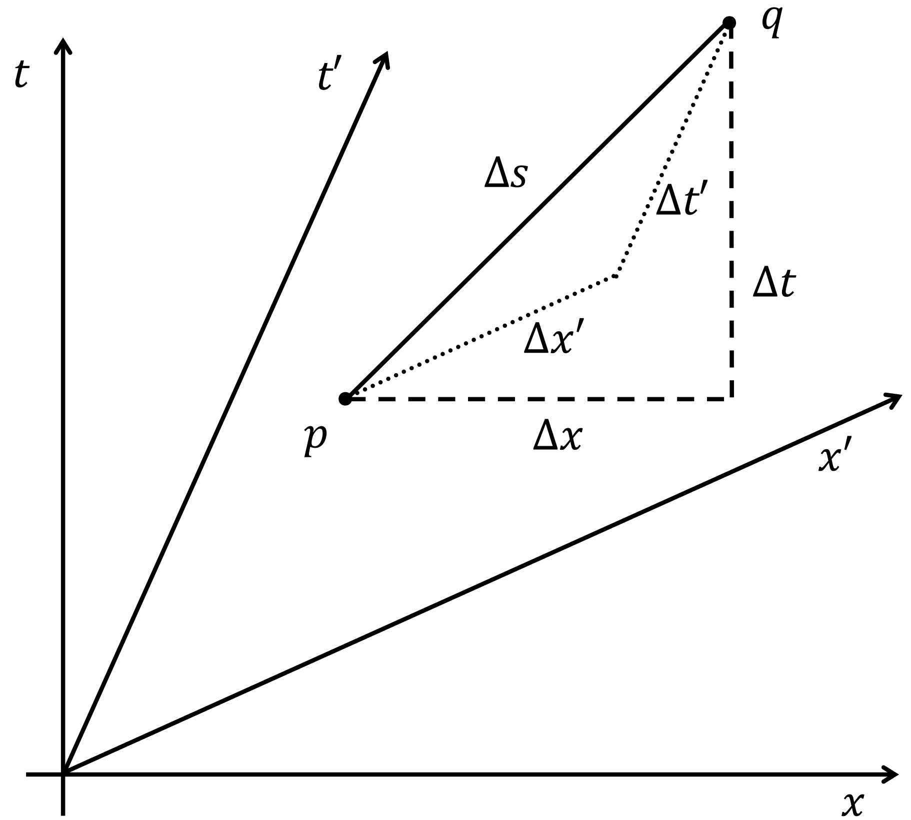
Hermann Minkowski (1864-1909)

<u>*Minkowski spacetime*</u> is a 4-dim collection of points such that between any two points p, q with coordinates (t, x, y, z) and $(t+\Delta t, x+\Delta x, y+\Delta y, z+\Delta z)$, there is a definite spacetime interval given by

$$\Delta s = \sqrt{-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

- Similar to Euclidean *spatial* interval: $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$.
  - *<u>But</u>: Includes the time coordinate difference, too! And it's negative!*
- Why this particular form?
  - *This particular form is invariant under Lorentz transformations!*
- What spacetime property remains the same in all inertial reference frames (*i.e.*, frames related by Lorentz transformations)?
  - *The Minkowski spacetime interval!*

In Minkowski spacetime:



$$\Delta s = \sqrt{-(c\Delta t)^2 + (\Delta x)^2}$$

$$= \sqrt{-(c\Delta t')^2 + (\Delta x')^2}$$

- All inertial frames agree on the *spacetime* distance between any two events P and Q.
- They disagree on the *temporal* distance between p and q (time dilation) and on the *spatial* distance (length contraction).
  - *They disagree on how they split* Δs *into temporal and spatial parts.*

The Minkowski spacetime interval is encoded in the *Minkowski metric* $\eta_{\mu\nu}$.

$$
\begin{aligned}
(\Delta s)^2 &= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\
&= \eta_{00}\Delta x^0 \Delta x^0 + \eta_{01}\Delta x^0 \Delta x^1 + \cdots + \eta_{33}\Delta x^3 \Delta x^3 \\
&= \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu
\end{aligned}
$$

$$
\text{where} \quad \begin{cases} \Delta x^0 = c\Delta t \\ \Delta x^1 = \Delta x \\ \Delta x^2 = \Delta y \\ \Delta x^3 = \Delta z \end{cases} \qquad \text{and} \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

- Infinitesimally, we can write (suppressing the summation sign):

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

The Minkowski interval $(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ has three forms:

(a) *Timelike*. $(\Delta s)^2 < 0$, or: $$\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{\Delta t} < c$$

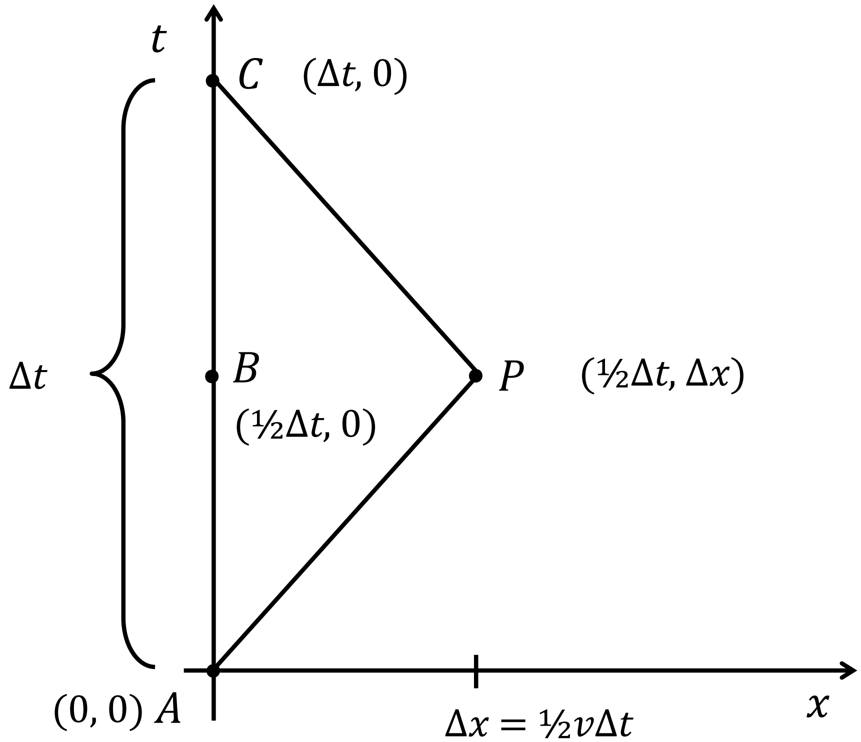
(b) *Lightlike*. $(\Delta s)^2 = 0$, or: $$\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{\Delta t} = c$$

(c) *Spacelike*. $(\Delta s)^2 > 0$, or: $$\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}{\Delta t} > c$$

Three different types of worldline in Minkowski spacetime!

- The worldlines of physical objects have negative length!
  - *Only strange from a Euclidean perspective.*
- The worldlines of light signals have zero length!
  - *Only strange from a Euclidean perspective.*

Twin Paradox again



- *ABC* is Stay-at-Home Twin worldline.
- *APC* is Traveling Twin worldline.
- Traveling Twin has velocity v.

$$\Delta s_{ABC} = \sqrt{-(c\Delta t)^2} = ic\Delta t$$

$$\begin{aligned}\Delta s_{APC} &= 2\Delta s_{AP} \\ &= 2\sqrt{-(c\tfrac{1}{2}\Delta t)^2 + (\tfrac{1}{2}v\Delta t)^2} \\ &= ic\Delta t\sqrt{1 - \frac{v^2}{c^2}} \\ &< \Delta s_{ABC}\end{aligned}$$

The Minkowski length of APC is less than the Minkowski length of ABC!