Time Dilation as a Real Effect

Muon detection experiment

Muons are elementary particles with the following properties:

(1) Created in upper atmosphere at altitudes of about 9000 meters.
(2) Average life span is $2 \times 10^{-6} s = 2ms$ (*note*: ms = "millisecond")
(3) Typical speed is $0.998c$

So we would expect that they could only travel at most

$$0.998c \times (2 \times 10^{-6} s) \approx 600m$$

But they can be observed at ground level.

Why? In the rest frame of the Earth, the lifespan of a traveling muon experiences time dilation:

$$t = \gamma t'$$

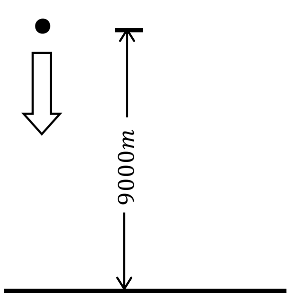
t = lifespan of muon with respect to Earth
t′ = lifespan of traveling muon

where the dilation factor γ is given by

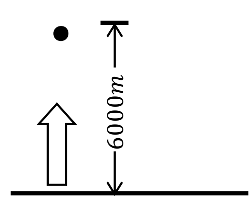
$$\gamma = \frac{1}{\sqrt{1 - (0.998c)^2/c^2}} \approx 15$$

So: In the Earth's reference frame, a typical muon lives on the average $(2 \times 15)ms = 30ms$.
So: With respect to the Earth, a typical muon can travel $0.998c \times 30ms \approx 9000m$.
So: Special relativity predicts we should be able to observe muons at ground level, and we do.

Note: In the traveling muon's reference frame, it is at rest and the Earth is rushing up to meet it at $0.998c$. The distance between it and the Earth thus is shorter than $9000m$ by length contraction. With respect to the muon, this distance is $9000m/15 = 600m$.



Earth frame



muon frame

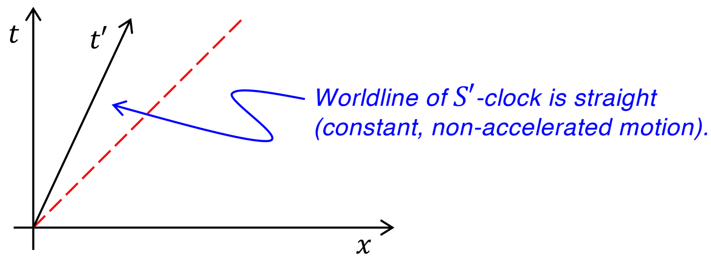
Pragmatics of Space Travel

Let's investigate how time dilation effects space travel.

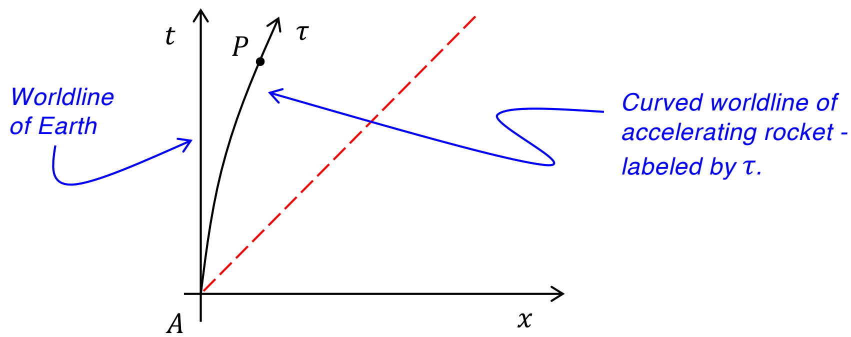
First, note that the time dilation formula is *only* for inertial reference frames:

$$t = \gamma t'$$

$t =$ time of stationary S-clock
$t' =$ time of S'-clock in *constant, non-accelerating, motion* with respect to S.



What about a rocket that accelerates away from the earth?

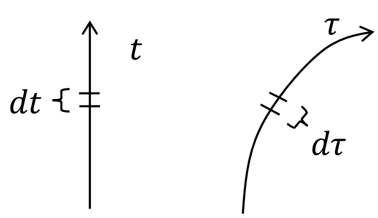


ASIDE: τ is called the "proper time" of the rocket's worldline. It's the time read by a clock that moves with the rocket. In general, any timelike worldline (straight or curved) has a proper time associated with it; namely, the time as read by a clock moving along it.

Here we use τ instead of t' just to indicate that the moving clock's worldline is curved. For such an *accelerating* clock, $t \neq \gamma\tau$.

BUT: We can look at very small (infinitesimal) t-intervals dt and τ-intervals $d\tau$. If they are small enough, they will be effectively "straight" and so be related by

$$dt = \gamma d\tau$$



To relate the t-time between events A and P to the τ-time between them, we now sum all the infinitesimal pieces dt that correspond to $d\tau$ pieces between A and P:

$$t = dt_1 + dt_2 + \cdots + dt_N$$ (for N pieces)

$$= \gamma_1 d\tau_1 + \gamma_2 d\tau_2 + \cdots + \gamma_N d\tau_N$$

$$= \int_A^P \gamma(\tau) d\tau$$ (An integral is just a sum of infinitely many infinitesimal pieces)

where $\gamma(\tau) = \frac{1}{\sqrt{1-\frac{v(\tau)^2}{c^2}}}$ and $v(\tau)$ is $\frac{1}{\text{slope}}$ of the τ-clock's worldline at value τ.

We can now decide on the "shape" of the rocket's trajectory; *i.e.*, how much acceleration we want to give it. Let's accelerate it at constant acceleration $g = 9.8m/s^2$. This is the acceleration due to gravity on the Earth (so the astronaut will feel comfortable). So the velocity $v(\tau)$ of the rocket is changing at a constant rate. Note that, from the Earth's perspective, we can't simply add on the changes to $v(\tau)$--we have to be careful about velocity composition. Given this care (see technical derivation on next page), the solution to the integral for t is:

$$t = \frac{c}{2g}\left(e^{g\tau/c} - e^{-g\tau/c}\right)$$

We can now use this formula to relate Earth time t to rocket time τ. To do this, we will make a simplifying approximation. First note that for large τ, the first exponential $e^{g\tau/c}$ dominates the second $e^{-g\tau/c}$ (which is just $1/e^{g\tau/c}$). In other words, as we increase τ, $e^{g\tau/c}$ gets larger and larger, while $e^{-g\tau/c}$ gets correspondingly smaller and smaller. So let's restrict ourselves to large values of τ (*i.e.*, very long rocket trips), thereby letting us drop the second exponential. Let's also use units in which time is measured in years, and distance in light-years (so $c = 1$, and, if you work it out, $g = 1.03c/yr$). Our formula for t is now:

$$t \approx \frac{e^{g\tau}}{2g}$$

Earth time t versus rocket time τ, for large values of τ and $g = 1.03c/yr$

The formula for the x-coordinate of the Earth is the same (see technical derivation on next page for details). The x-coordinate gives the distance traveled by the rocket with respect to the Earth. Here are some sample values:

Rocket time τ	Earth time t	Distance x of rocket from Earth
$1\ yr$	$1.18\ yr$	$0.56\ light\text{-}yr$
$5\ yr$	$83.7\ yr$	$82.7\ light\text{-}yr$
$10\ yr$	$14,433\ yr$	$14,432\ light\text{-}yr$
$25\ yr$	$7.4 \times 10^{10}\ yr$	$7.4 \times 10^{10}\ light\text{-}yr$

Technical derivation of formula for Earth time t

The rocket's speed $v(\tau)$ is being constantly "boosted" by a fixed amount g. This boosting is not simply a matter of adding amounts onto $v(\tau)$ in a linear manner, so we shouldn't just set $v(\tau) = g\tau$. Recall that velocities cannot be simply added together. What can be added together is a quantity called the rapidity r that is related to the velocity v by:

$$v = c\tanh\left(\frac{r}{c}\right)$$

The hyperbolic tangent function $\tanh(x)$ ranges from -1 to $+1$. The formula thus guarantees that the velocity v can only vary from $-c$ (when $r = -c$) to $+c$ (when $r = c$). While velocities cannot be simply added together, rapidities can. If r' is the rapidity associated with the velocity v', then the combined rapidity r'' is the sum $r'' = r + r'$, which follows from the identity

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$$

The upshot of this is that the rapidity r of the rocket is linearly increasing by the formula $r = g\tau$. Now note that, from the definition of r above, it follows that

$$\gamma = \cosh\left(\frac{r}{c}\right)$$

(You can get this from the identities $\tanh(x) = \sinh(x)/\cosh(x)$ and $\cosh^2(x) - \sinh^2(x) = 1$.) So our integral for t becomes

$$t = \int \cosh\left(\frac{g\tau}{c}\right) d\tau$$

And the solution to this integral is

$$t = \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right) = \frac{c}{2g}\left(e^{g\tau/c} - e^{-g\tau/c}\right)$$

Note: The formula for the x-coordinate of the Earth can be obtained in a similar manner. (This gives the distance of the trip with respect to the Earth.) We know that $x = vt = \int v\gamma d\tau$. From the definition of the rapidity above, it can also be shown that $\gamma = (c/v)\sinh(r/c)$. So:

$$x = c\int \sinh\left(\frac{g\tau}{c}\right) d\tau = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right) = \frac{c^2}{2g}\left(e^{g\tau/c} + e^{-g\tau/c}\right)$$

For large values of τ, and in units in which $c = 1$, the formulas for t and x are the same (we can disregard the second exponential). When τ is small, t and x will differ.