08. The KS Theorem, The Measurement Problem

How Should Superpositions be Interpreted? Part 2.

> *Value Definiteness (VD)*
> The properties of a quantum system are determinate (possess values) at all times, even when the system is in a superposed state.

Isn't this how classical properties behave?

(A) Literally (QM description is complete):

EPR say: non-local!

Options:

A1. *Standard Claim*: Deny VD. The properties of a quantum system in a superposed state are *indeterminate* (do not possess values).

(B) Non-literally (QM description is incomplete):

Options:

Bell says: No! Conflicts with experiment.

B1. *Local Hidden Variables with VD.*

B2. *Non-local Hidden Variables with VD.*

Why not (B2)? Non-locality isn't all that spooky...

But: The KS Theorem says "No" to VD.

1. The Kochen-Specker Theorem

Summary of KS Theorem

- A *mathematical claim* about the nature of *Hilbert spaces* (the special type of vector spaces that are the most general representation of the state space for a quantum system).

Due specifically to the structure of Hilbert spaces.

Not a problem for classical systems with state spaces represented by point sets.

- *So*: *KS* Theorem just *reconfirms* that our original choice of using Hilbert spaces to represent quantum state spaces is correct, if we want to be able to represent quantum properties that are fundamentally different from classical properties.

- *KS Theorem says*: *If* properties are represented as operators on a Hilbert space in a 1-1 fashion (*i.e.*, each property is represented by a unique operator), then these properties cannot all be said to simultaneously have values.

More precisely...

Kochen-Specker Theorem

For Hilbert spaces of dimension ≥ 3, (1) and (2) are contradictory:

(1) *Value Definiteness*: Any set of properties represented by operators A, B, C, ... on $\mathcal{H}$ simultaneously have values $v(A)$, $v(B)$, $v(C)$, ...

(2) *Value Constraints*:

- (a) (*Sum Rule*) If A, B, C, are *compatible* and $C = A + B$, then $v(C) = v(A) + v(B)$.
- (b) (*Product Rule*) If A, B, C, are *compatible* and $C = AB$, then $v(C) = v(A)v(B)$.

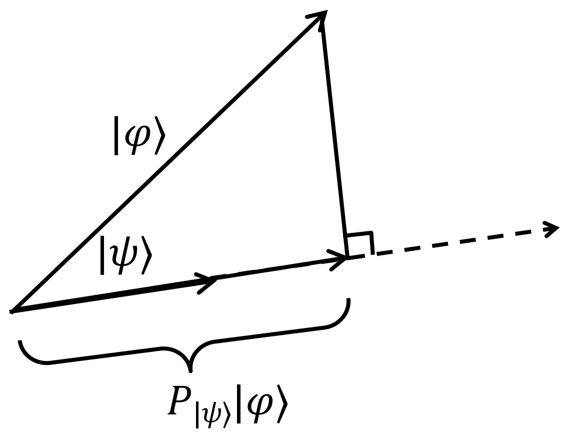
- *Compatibility* means *A, B, C* all have a set of eigenvectors in common.
- *Value Constraints* is a consequence (in part) of requiring a 1-1 correspondence between properties and operators ("*non-contexuality*").

To prove the KS Theorem, we first need the notion of a projection operator...

Def. A *projection operator* $P_{|\psi\rangle}$ *associated with a unit vector* $|\psi\rangle$ maps any vector $|\varphi\rangle$ to another one $P_{|\psi\rangle}|\varphi\rangle$ defined by,

$$P_{|\psi\rangle}|\varphi\rangle \equiv \langle\psi|\varphi\rangle|\psi\rangle$$

that is the result of *projecting* $|\varphi\rangle$ in the direction of $|\psi\rangle$.



- $P_{|\psi\rangle}$ is sometimes written as $|\psi\rangle\langle\psi|$.
- <u>*So*</u>: $P_{|\psi\rangle}|\varphi\rangle = |\psi\rangle\langle\psi|\varphi\rangle = \langle\psi|\varphi\rangle|\psi\rangle$.
- <u>*Note*</u>: $\langle\psi|\varphi\rangle$ is a *number* and $|\psi\rangle$ is a *vector*, so $\langle\psi|\varphi\rangle|\psi\rangle$ is a vector.

Two important Characteristics of Projection Operators

$(P_{|\psi\rangle})^2 = P_{|\psi\rangle}$ (*idempotency*)

$P_{|a_1\rangle} + P_{|a_2\rangle} + \cdots + P_{|a_N\rangle} = I_N$ (*resolution of the identity*)

where $|a_1\rangle, |a_2\rangle, \ldots, |a_N\rangle$ form an orthonormal basis for an N-dimensional Hilbert space $\mathcal{H}$ with identity operator I_N.

Proof of Idempotency:

$$(P_{|\psi\rangle})^2|\varphi\rangle = P_{|\psi\rangle}(P_{|\psi\rangle}|\varphi\rangle) = P_{|\psi\rangle}(|\psi\rangle\langle\psi|\varphi\rangle) = |\psi\rangle\langle\psi|\psi\rangle\langle\psi|\varphi\rangle = \langle\psi|\varphi\rangle|\psi\rangle$$

Proof of Resolution of the Identity:

- *Suppose*: $|B\rangle = b_1|a_1\rangle + \cdots + b_N|a_N\rangle$ is any vector in $\mathcal{H}$.
- *Then*: $(P_{|a_1\rangle} + P_{|a_2\rangle} + \cdots + P_{|a_N\rangle})|B\rangle$

$$= (P_{|a_1\rangle} + P_{|a_2\rangle} + \cdots + P_{|a_N\rangle})(b_1|a_1\rangle + \cdots + b_N|a_N\rangle)$$

$$= b_1|a_1\rangle\langle a_1|a_1\rangle + \cdots + b_N|a_N\rangle\langle a_N|a_N\rangle, \qquad \text{since } \langle a_i|a_j\rangle = 0, \text{ unless } i = j$$

$$= b_1|a_1\rangle + b_2|a_2\rangle + \cdots + b_N|a_N\rangle$$

$$= |B\rangle$$

What property does a projection operator represent?

Recall: Any orthonormal basis $|a_1\rangle, \ldots, |a_N\rangle$ is a set of eigenvectors of some (complete) operator A. For these eigenvectors:

(a) $A|a_i\rangle = a_i|a_i\rangle$	$|a_i\rangle$ is an eigenvector of A with eigenvalue a_i
(b) $P_{|a_i\rangle}|a_i\rangle = |a_i\rangle\langle a_i|a_i\rangle = |a_i\rangle$	$|a_i\rangle$ is an eigenvector of $P_{|a_i\rangle}$ with eigenvalue $+1$
(c) $P_{|a_i\rangle}|a_j\rangle = |a_i\rangle\langle a_i|a_j\rangle = 0$ $= 0|a_j\rangle$, for $i \neq j$	$|a_j\rangle$ is an eigenvector of $P_{|a_i\rangle}$ with eigenvalue 0

- *So*: Any eigenvector of $P_{|a_i\rangle}$ with eigenvalue $+1$ represents a state that possesses the value a_i of the property represented by A.
- *And*: Any eigenvector of $P_{|a_i\rangle}$ with eigenvalue 0, represents a state that possesses some value, other than a_i, of the property represented by A.

Def. Let $P_{|a_i\rangle}$ be a projection operator, where $|a_i\rangle$ is an eigenvector of the operator A. Then $P_{|a_i\rangle}$ represents the property "The value of A is a_i".

Only two values of this property in a given state:
(i) $+1$, which means the state has the value a_i of A.
(ii) 0, which means the state does not have the value a_i of A.

The significance of projection operators to the KS Theorem

Value Constraints entails a particular constraint on projection operators:

<u>*Projection Operator Value Constraint (PVC)*</u>

Let $|a_1\rangle, \ldots, |a_N\rangle$ be an orthonormal basis for an N-dim Hilbert space. Then,

$v(P_{|a_1\rangle}) + v(P_{|a_2\rangle}) + \cdots + v(P_{|a_N\rangle}) = 1$, where $v(P_{|a_i\rangle}) = 1$ or 0, for $i = 1 \ldots N$.

<u>*Proof*</u>: Let $|a_1\rangle, \ldots, |a_N\rangle$ be an orthonormal basis for an N-dim Hilbert space with identity operator I_N.

- <u>*Then*</u>: $P_{|a_1\rangle} + P_{|a_2\rangle} + \cdots + P_{|a_N\rangle} = I_N$ (*resolution of identity*)
- <u>*And*</u>: $v(P_{|a_1\rangle}) + \cdots + v(P_{|a_N\rangle}) = v(I_N)$ (*sum rule*)
- <u>*Note*</u>: For any operator $\mathcal{O}$,

  $v(\mathcal{O}) = v(I_N\mathcal{O})$

  $= v(I_N)v(\mathcal{O})$ (*product rule*)
- <u>*So*</u>: $v(I_N) = 1$
- <u>*So*</u>: $v(P_{|a_1\rangle}) + \cdots + v(P_{|a_N\rangle}) = 1$
- <u>*Now*</u>: $v(P_{|a_i\rangle})v(P_{|a_i\rangle}) = v(P_{|a_i\rangle}{}^2)$ (*product rule)*

  $= v(P_{|a_i\rangle})$ (*idempotency*)
- <u>*Thus*</u>: $v(P_{|a_i\rangle}) = 1$ or 0

So: The *KS* Theorem simply says that (*VD*) and (*PVC*) are contradictory.

- *(PVC) says*: The operator A with eigenvectors $|a_i\rangle$ has a definite value (*i.e.*, just the value a_i for which $v(P_{|a_i\rangle}) = 1$).
- *(VD) says*: *All* operators have definite values; not just A, but even those that are incompatible (*i.e.*, don't have the same eigenvectors) with A.
- *Thus*: (*VD*) requires that (*PVC*) holds, not only for the $|a_i\rangle$ orthonormal basis, but for *all* orthonormal bases.

The KS Theorem is a consequence of the claim:

(*PVC*) cannot hold for *all* orthonormal bases of Hilbert spaces with dim ≥ 3.

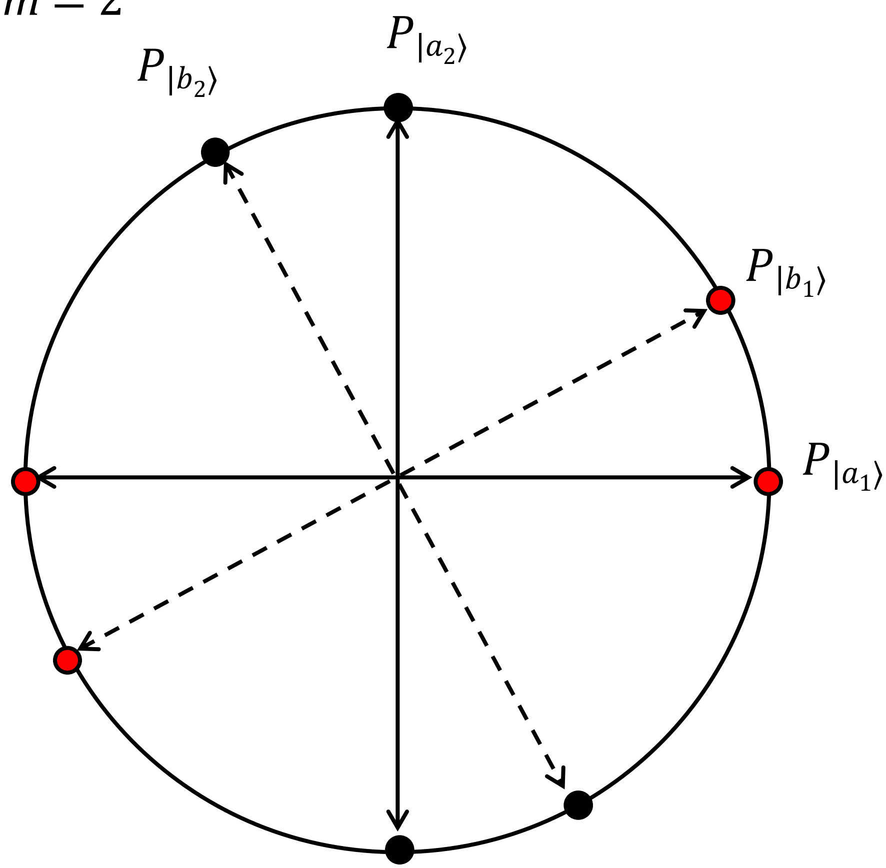
Proof Sketch: First implement (*PVC*) by the following:

- Label each $P_{|a_i\rangle}$ either black or red, depending on whether $v(P_{|a_i\rangle}) = 1$ or 0.
- *(PVC) says*: For the set of basis vectors corresponding to $P_{|a_i\rangle}$, one is labeled red and all the others are labeled black.
- (*VD*) now requires us to do this for *all* sets of bases.

Proof Sketch (cont.): Now demonstrate that you can't color all sets of basis vectors of a Hilbert space of dim ≥ 3 such that one member of each set is red and all the other members of the set are black. Let's see how this works for dimension 2, but doesn't work for dimension 3.

- *Note*: $P_{|a_i\rangle}$ corresponds to the *ray* through $|a_i\rangle$. So we'll color *basis rays*.

$dim = 2$



- There are an infinite number of pairs of orthogonal rays $\{P_{|a_1\rangle}, P_{|a_2\rangle}\}$, $\{P_{|b_1\rangle}, P_{|b_2\rangle}\}$, ..., *etc*, obtained by rotating $\{P_{|a_1\rangle}, P_{|a_2\rangle}\}$ by some angle θ, $0 < \theta \leq 90°$.
- For each set, can consistently color one red and the other black. If we continue coloring, we'll color the entire circle such that each alternating quadrant is black or red.

Proof Sketch (cont.): Now demonstrate that you can't color all sets of basis vectors of a Hilbert space of dim ≥ 3 such that one member of each set is red and all the other members of the set are black. Let's see how this works for dimension 2, but doesn't work for dimension 3.

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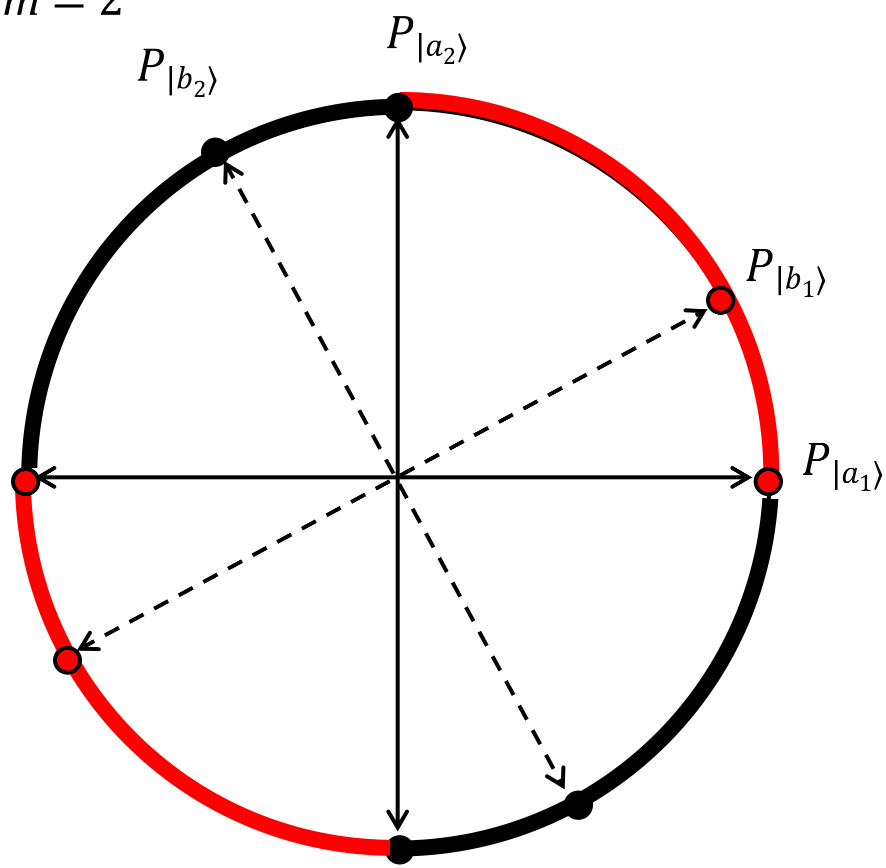

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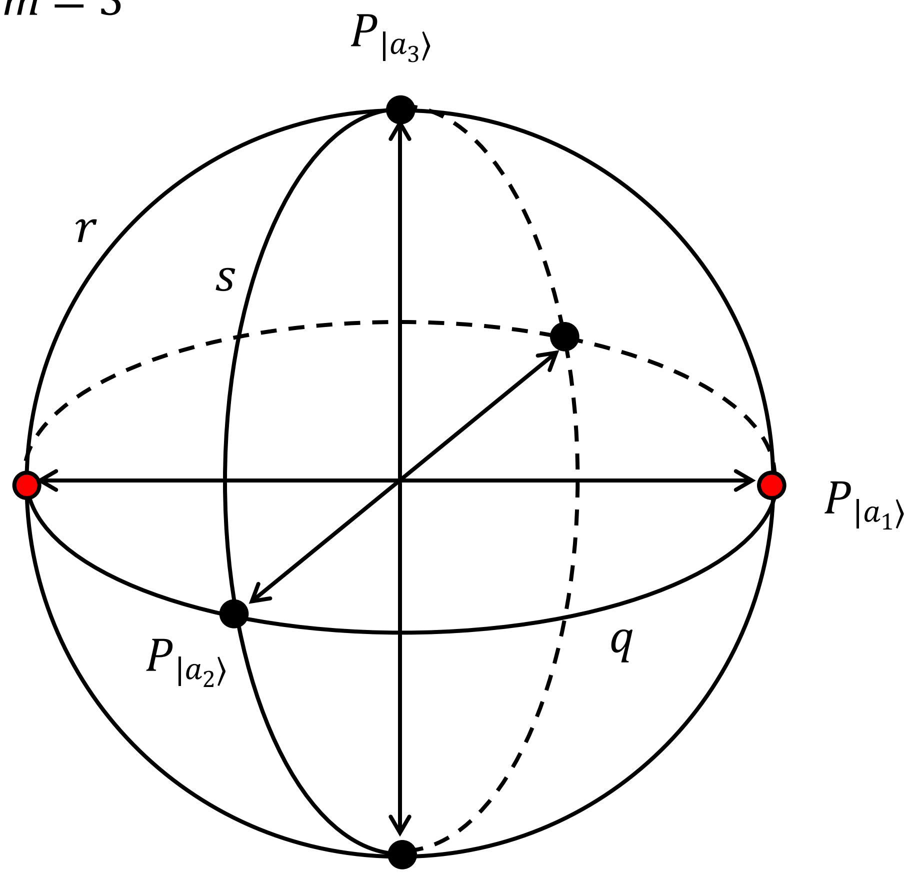

<u>Proof Sketch (cont.)</u>: Now demonstrate that you can't color all sets of basis vectors of a Hilbert space of dim ≥ 3 such that one member of each set is red and all the other members of the set are black. Let's see how this works for dimension 2, but doesn't work for dimension 3.

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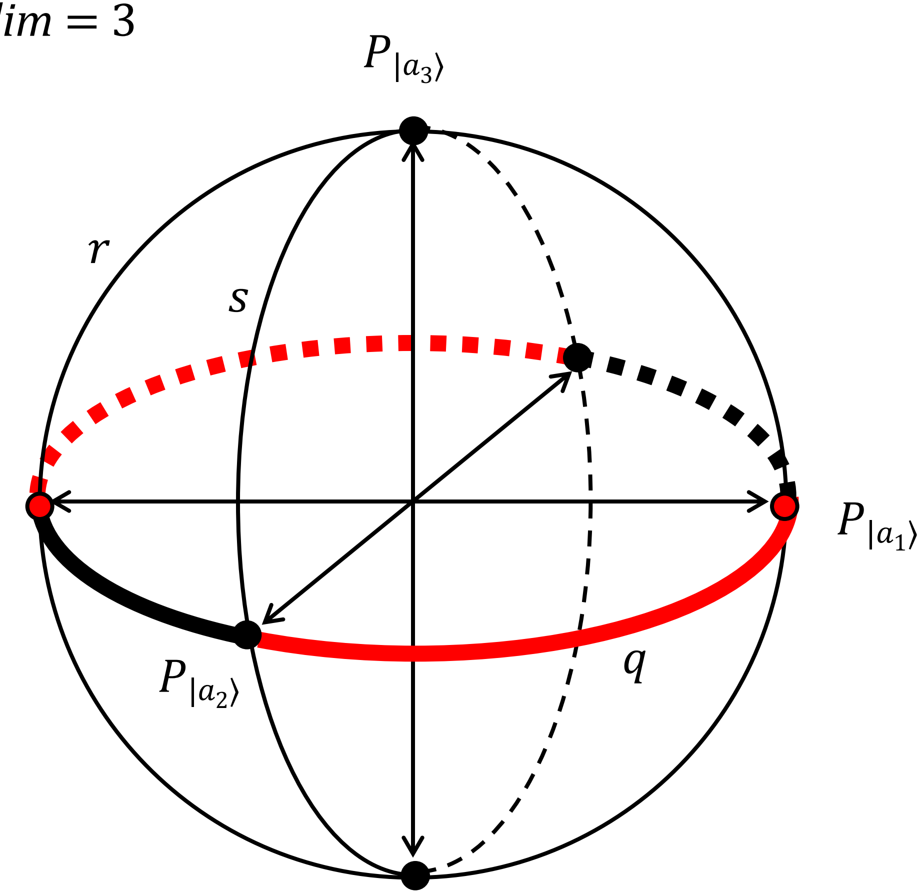

- Rotating about $P_{|a_3\rangle}$ colors the circle q.

Proof Sketch (cont.): Now demonstrate that you can't color all sets of basis vectors of a Hilbert space of dim ≥ 3 such that one member of each set is red and all the other members of the set are black. Let's see how this works for dimension 2, but doesn't work for dimension 3.

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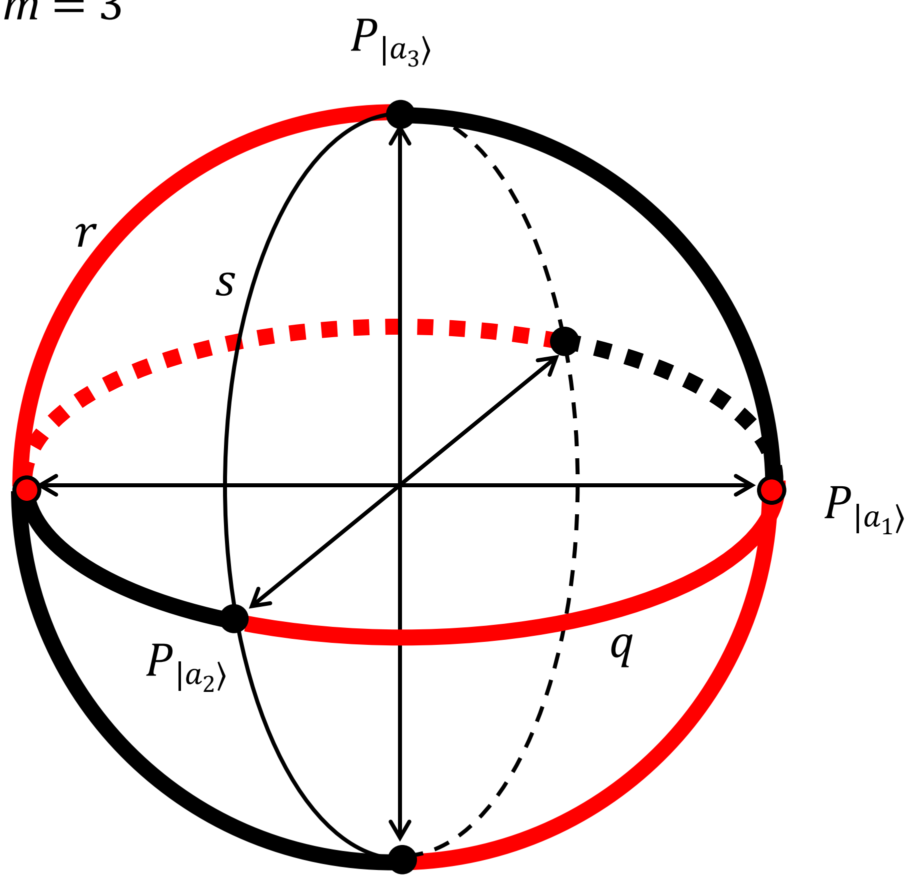

- Rotating about $P_{|a_3\rangle}$ colors the circle q.
- Rotating about $P_{|a_2\rangle}$ colors the circle r.

Proof Sketch (cont.): Now demonstrate that you can't color all sets of basis vectors of a Hilbert space of dim ≥ 3 such that one member of each set is red and all the other members of the set are black. Let's see how this works for dimension 2, but doesn't work for dimension 3.

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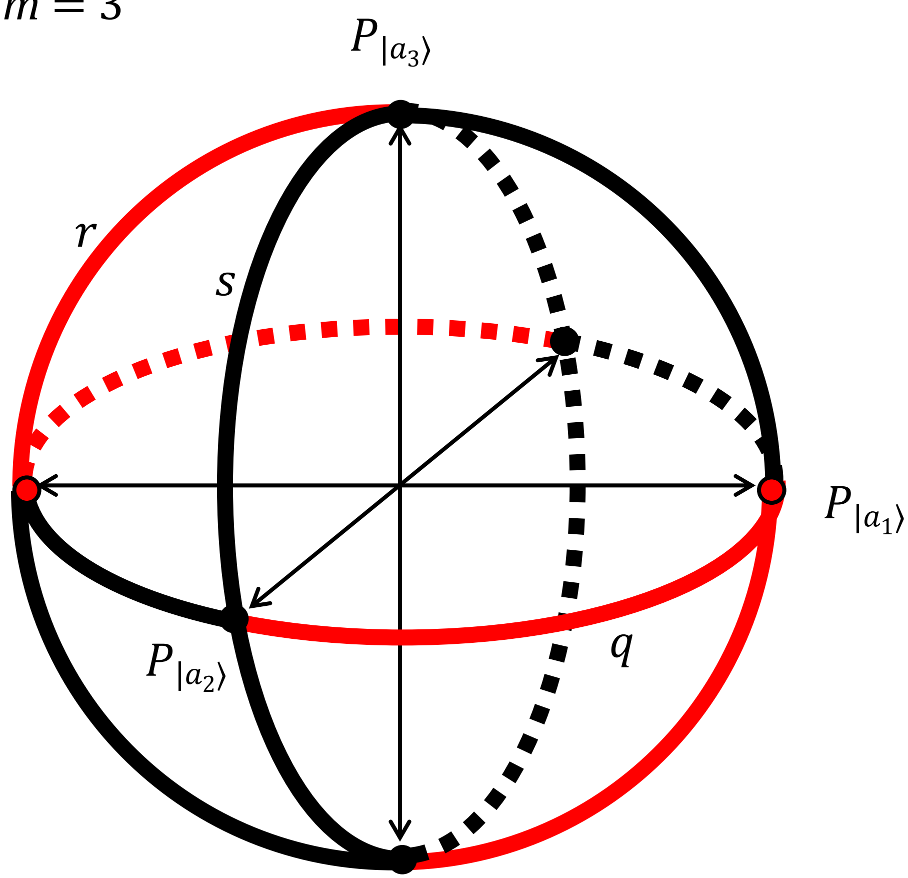


- Rotating about $P_{|a_3\rangle}$ colors the circle q.
- Rotating about $P_{|a_2\rangle}$ colors the circle r.
- Rotating about $P_{|a_1\rangle}$ colors the circle s.
- Can also rotate about any other ray through the origin.
- *Claim*: Can't consistently color the entire surface of the sphere in this manner. At some point, you'll run over a previously colored portion!

How Should Superpositions be Interpreted? Part 3

> *Value Definiteness (VD)*
> The properties of a quantum system are determinate (possess values) at all times, even when the system is in a superposed state.

(A) Literally (QM description is complete):

EPR say: non-local!

Options:

A1. *Standard Claim*: Deny VD. The properties of a quantum system in a superposed state are *indeterminate* (do not possess values).

(B) Non-literally (QM description is incomplete):

Options:

B1. *Local Hidden Variables with VD.*

Bell says: No!

B2. *Non-local Hidden Variables with VD.*

KS Theorem says: No!

Is Option (A1) feasible?

Major Difficulty: The Measurement Problem...

2. The Measurement Problem

There are two ways the state of a quantum system can change:

(a) *In the presence of a measurement*: Indeterministic, instantaneous collapse (Projection Postulate).

- Suppose the state of our system is given by $|Q\rangle = \sqrt{½}\,(|white\rangle + |black\rangle)$.
- Suppose we measure our system for Color and get the value *white*.
- Then the state collpases to $|white\rangle$.

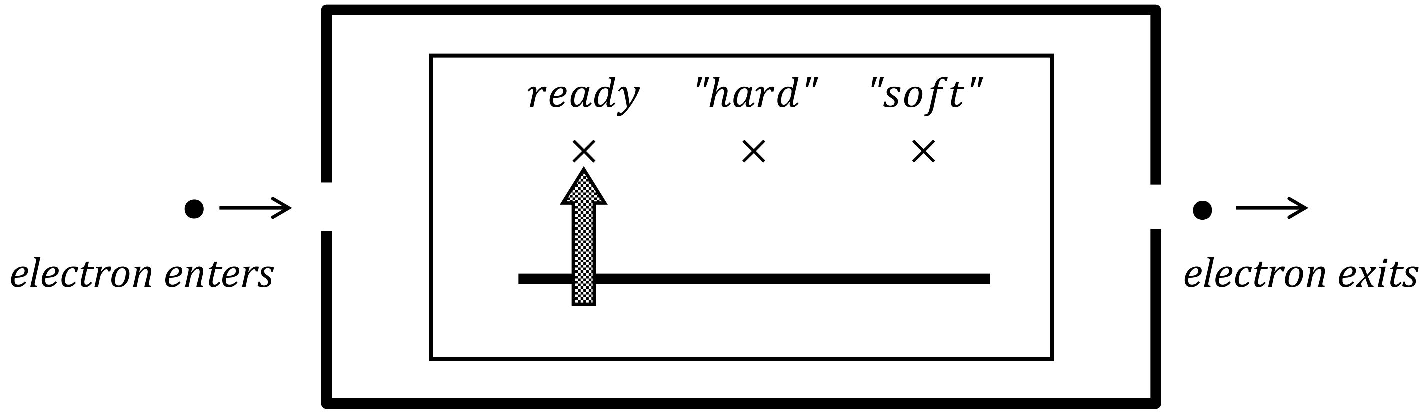
(b) *In the absence of a measurement*: Deterministic, temporal evolution *via* the Schrödinger equation.

- Suppose the state of our system at t_i is given by $|Q\rangle$.
- Then the state of our system at $t_f > t_i$ is given by $S|Q\rangle$, where $S = e^{-iH(t_f - t_i)/\hbar}$ is the linear Schrödinger operator.

- Is (a) inconsistent with (b)?

If we adopt Option (A1), then (a) and (b) make different predictions!

How to Model a Measurement Process:



- Consider composite system of measuring device m and electron e.
- The Schrödinger equation tells us how the state of the m-e system evolves in time.

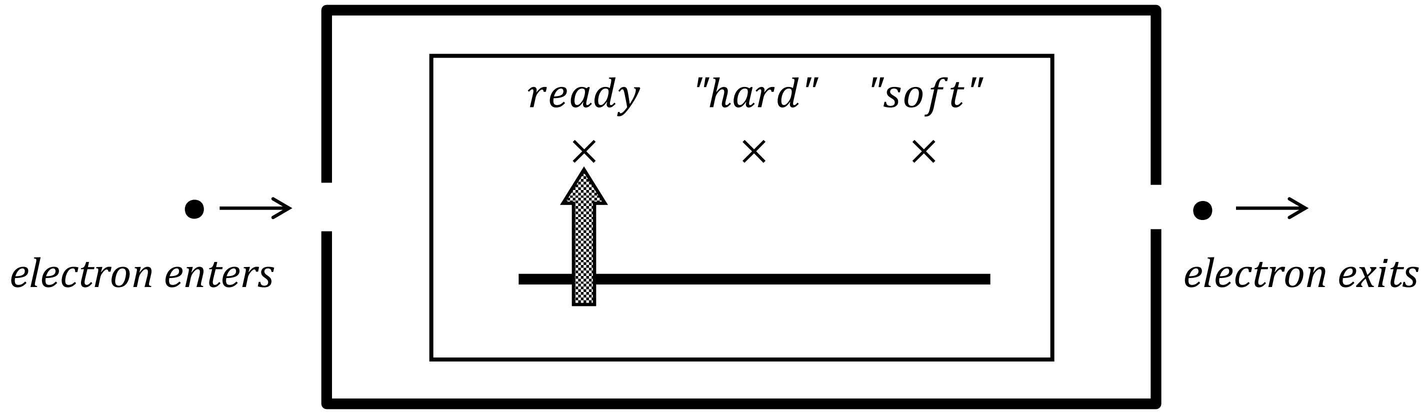
initial state		*final state*
$\|ready\rangle_m \|hard\rangle_e$	$\xrightarrow[\text{Schrödinger evolution}]{}$	$\|\text{"}hard\text{"}\rangle_m \|hard\rangle_e$
pointer set to ready		*pointer set to "hard"*
$\|ready\rangle_m \|soft\rangle_e$	$\xrightarrow[\text{Schrödinger evolution}]{}$	$\|\text{"}soft\text{"}\rangle_m \|soft\rangle_e$

In other words:

$S|ready\rangle_m|hard\rangle_e = |\text{"}hard\text{"}\rangle_m|hard\rangle_e$

$S|ready\rangle_m|soft\rangle_e = |\text{"}soft\text{"}\rangle_m|soft\rangle_e$

How to Model a Measurement Process:



- *Now*: Suppose a *black* electron is measured for *Hardness*.
- According to the Schrödinger equation,

initial state	$\xrightarrow[\text{evolution}]{\text{Schrödinger}}$	*final state*
$\vert ready\rangle_m \vert black\rangle_e$ $= \sqrt{1/2}\,(\vert ready\rangle_m \vert hard\rangle_e + \vert ready\rangle_m \vert soft\rangle_e)$		$\sqrt{1/2}\,(\vert \text{"}hard\text{"}\rangle_m \vert hard\rangle_e + \vert \text{"}soft\text{"}\rangle_m \vert soft\rangle_e)$

- *We know that*:

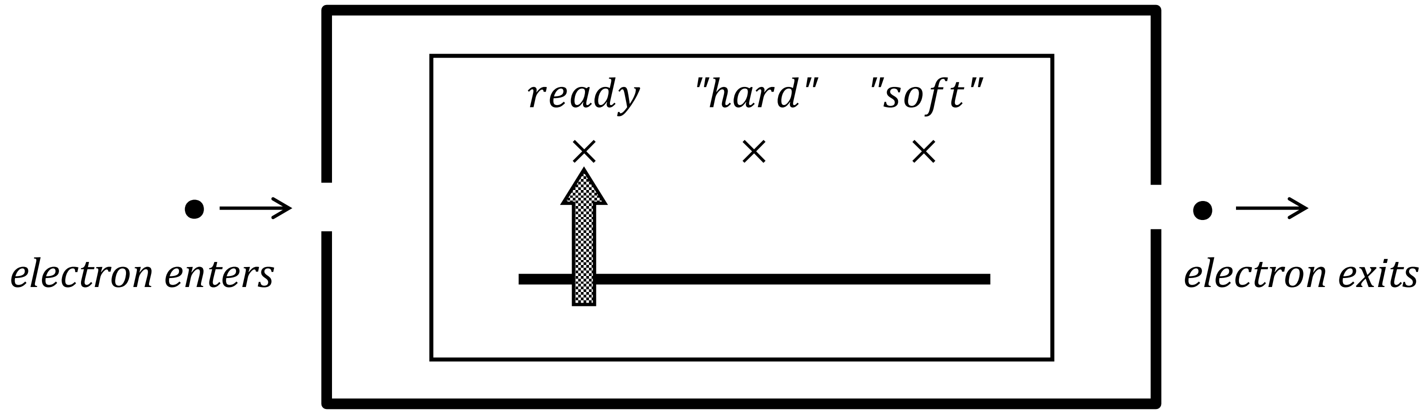
$$S|ready\rangle_m|hard\rangle_e = |\text{"}hard\text{"}\rangle_m|hard\rangle_e$$
$$S|ready\rangle_m|soft\rangle_e = |\text{"}soft\text{"}\rangle_m|soft\rangle_e$$

- *So*:

$$S|ready\rangle_m|black\rangle_e = \sqrt{1/2}\,(S|ready\rangle_m|hard\rangle_e + S|ready\rangle_m|soft\rangle_e)$$
$$= \sqrt{1/2}\,(|\text{"}hard\text{"}\rangle_m|hard\rangle_e + |\text{"}soft\text{"}\rangle_m|soft\rangle_e)$$

The Schrödinger operator S is linear!

How to Model a Measurement Process:



- *Now*: Suppose a *black* electron is measured for *Hardness*.
- *But*: According to the Projection Postulate,

initial state	$\xrightarrow[\text{collapse}]{}$	*final state*	
$\lvert ready\rangle_m \lvert black\rangle_e$		*either*	$\lvert \text{"}hard\text{"}\rangle_m \lvert hard\rangle_e$, prob = ½
$= \sqrt{½}\,(\lvert ready\rangle_m \lvert hard\rangle_e + \lvert ready\rangle_m \lvert soft\rangle_e)$		*or*	$\lvert \text{"}soft\text{"}\rangle_m \lvert soft\rangle_e$, prob = ½

final state

$\sqrt{½}\,(|\text{"}hard\text{"}\rangle_m|hard\rangle_e + |\text{"}soft\text{"}\rangle_m|soft\rangle_e)$ *according to Schrödinger evolution*

either $|\text{"}hard\text{"}\rangle_m|hard\rangle_e$ with prob = ½

or $|\text{"}soft\text{"}\rangle_m|soft\rangle_e$ with prob = ½

according to Projection Postulate

- According to the Eigenvalue/Eigenvector Rule, these represent *different* states!

Initial response:
According the standard formulation, the Projection Postulate is supposed to take over during a measurement. So just ignore what the Schrödinger dynamics predicts when measurements occur.

- *But*: What exactly is a *measurement*? *When* is the Projection Postulate supposed to take over from the Schrödinger dynamics?

Anyone who adopts Option (A1), must answer this question.