07. QIT, Part III.

1. Quantum Error Correction Codes (QECCs)

- *Goal*: To encode information in qubits in such a way that errors due to "noise" can be detected and corrected.
  - *But: Typical quantum algorithms encode information in entangled qubits.*
  - *And: Attempts to detect and correct errors due to noise run the risk of decohering entangled qubits, thus destroying the information.*

Task: To detect and correct errors *without* decohering the relevant entangled qubits.

Set-Up: Suppose information is encoded in a qubit $|Q\rangle = a|0\rangle + b|1\rangle$.

Step 1. Encode $|Q\rangle$ in a **codeword**.

- Do this by performing appropriate transformations on the single-qubit basis states $|0\rangle, |1\rangle$. ← *The type of transformations depends on the type of errors we expect to occur.*
- The new basis states form a space called the **code space** $\mathcal{C}$.
- Complete the set of basis states to form a larger space called the **coding space** $\mathcal{H}$. ← *$\mathcal{C}$ is a subspace of $\mathcal{H}$*

Example: We might transform the single-qubit basis states into three-qubit basis states:

$$|0\rangle \to |000\rangle$$

$$|1\rangle \to |111\rangle$$

- The **codeword** is then $a|000\rangle + b|111\rangle$.
- The **code space** $\mathcal{C}$ is the space spanned by $\{|000\rangle, |111\rangle\}$, which is a 2-dim subspace of the larger 8-dim three-qubit **coding space** space $\mathcal{H}$ spanned by $\{|000\rangle, |001\rangle, |010\rangle, |100\rangle, |110\rangle, |101\rangle, |011\rangle, |111\rangle\}$.

Step 2. Represent **errors** by multi-qubit operators constructed from the single-qubit operators I, X, Y, Z.

- Errors "corrupt" the basis states of $\mathcal{C}$, and hence the codeword, projecting it out of $\mathcal{C}$.

> *Example*: We might transform the single-qubit basis states into three-qubit basis states:
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> $$|0\rangle \rightarrow |000\rangle$$
>
> $$|1\rangle \rightarrow |111\rangle$$
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> - The **codeword** is then $a|000\rangle + b|111\rangle$.
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> - An **error** might be represented by the operator $X \otimes I \otimes I$.
> - This would produce a corrupted codeword $a|100\rangle + b|011\rangle$, which is an element of $\mathcal{H}$ but not of $\mathcal{C}$.

Step 3. Devise an appropriate operation that acts on a corrupted codeword in $\mathcal{H}$ and projects it back into $\mathcal{C}$ (thereby "correcting" it).

Necessary and sufficient condition for error-correction

- Let $\mathcal{C} = \text{span}\{|\psi_1\rangle, \ldots, |\psi_p\rangle\}$, for some number p of basis states.

  Let $\mathcal{E} = \{E_1, \ldots, E_q\}$ be a set of q error operators.

Knill-Laflamme (KL) Condition: A code space $\mathcal{C} = \text{span}\{|\psi_1\rangle, \ldots, |\psi_p\rangle\}$ corrects the error set $\mathcal{E} = \{E_1, \ldots, E_q\}$ *if and only if*

(i) $\langle\psi_i|E^{\dagger}_k E_l|\psi_j\rangle = 0$ ← *Corrupted basis states $E_l|\psi_j\rangle$, $E_k|\psi_i\rangle$ are orthogonal, and hence distinguishable from each other*

(ii) $\langle\psi_i|E^{\dagger}_k E_l|\psi_i\rangle = \langle\psi_j|E^{\dagger}_k E_l|\psi_j\rangle, \quad i \neq j$ ← *Measurements made to determine the error will not give any information about the codeword (and thereby possibly decohere it).*

- Constraints (i) & (ii) together entail:

$$\langle\psi_i|E^{\dagger}_k E_l|\psi_j\rangle = c_{kl}\delta_{ij}$$

← *The projection of the operator $E^{\dagger}_k E_l$ onto the code space is a multiple of the identity*

where c_{kl} are arbitrary constants and δ_{ij} is the identity

Intuition: Errors can be corrected if we can reverse their damage; i.e., if for any error E_l, there is a reverse error $E^{\dagger}_k$.

Example: Single-qubit flip error correction code.

Task: To transmit a qubit $|Q\rangle = a|0\rangle + b|1\rangle$ in the presence of noise that flips single-qubit basis states.

Step 1. Encode $|Q\rangle$ in codeword $|\Phi\rangle = a|000\rangle + b|111\rangle$.

Encoding one qubit in a three-qubit state

- $|\Phi\rangle$ is an element of the 2-dim code space $\mathcal{C} = \text{span}\{|000\rangle, |111\rangle\}$.
- $|\Phi_{corrupt}\rangle = a|\_\rangle_1|\_\rangle_2|\_\rangle_3 + b|\_\rangle_1|\_\rangle_2|\_\rangle_3$ can take one of four forms:

$$a|000\rangle + b|111\rangle$$
$$a|100\rangle + b|011\rangle$$
$$a|010\rangle + b|101\rangle$$
$$a|001\rangle + b|110\rangle$$

$|\Phi_{corrupt}\rangle$ is an element of the 8-dim three-qubit space

$\mathcal{H} = \text{span}\{|000\rangle, |001\rangle, |010\rangle, |100\rangle, |110\rangle, |101\rangle, |011\rangle, |111\rangle\}$

Step 2. Represent single-qubit flip errors by 4 three-qubit operators:

$$\mathcal{E} = \{I \otimes I \otimes I,\ X \otimes I \otimes I,\ I \otimes X \otimes I,\ I \otimes I \otimes X\}$$

does nothing — *flips 1st qubit* — *flips 2nd qubit* — *flips 3rd qubit*

Step 3. Error detection/correction protocol:

(a) Attach two "empty register" qubits $|00\rangle$ to $|\Phi_{corrupt}\rangle$:

$$|\Phi_{corrupt}\rangle|00\rangle = \{a|\_\rangle_1|\_\rangle_2|\_\rangle_3 + b|\_\rangle_1|\_\rangle_2|\_\rangle_3\}\,|0\rangle_4 0\rangle_5$$

(b) *Error detection*:

- Perform *XOR* on qubits 1 and 2 and store result in qubit 4.
- Perform *XOR* on qubits 1 and 3 and store result in qubit 5.

$0\ XOR\ 0 = 0$
$0\ XOR\ 1 = 1$
$1\ XOR\ 0 = 1$
$1\ XOR\ 1 = 0$

(b) *Error correction*: Measure qubits 4 and 5 to determine form of $|\Phi_{corrupt}\rangle$ and what three-qubit operator to use to correct it.

Corrupted codeword/register	*Error detection*	*Error correction*
$\{a\|000\rangle + b\|111\rangle\}\|00\rangle$	$\{a\|000\rangle + b\|111\rangle\}\|00\rangle$	$I \otimes I \otimes I$
$\{a\|100\rangle + b\|011\rangle\}\|00\rangle$	$\{a\|100\rangle + b\|011\rangle\}\|11\rangle$	$X \otimes I \otimes I$
$\{a\|010\rangle + b\|101\rangle\}\|00\rangle$	$\{a\|010\rangle + b\|101\rangle\}\|10\rangle$	$I \otimes X \otimes I$
$\{a\|001\rangle + b\|110\rangle\}\|00\rangle$	$\{a\|001\rangle + b\|110\rangle\}\|01\rangle$	$I \otimes I \otimes X$

Both detection and correction protocols do not decohere $|\Phi_{corrupt}\rangle$!

- *Now*: Check to see if the KL Condition holds for our single-qubit flip error correction code:

  Does $\mathcal{C} = \text{span}\{|000\rangle, |111\rangle\}$ correct the error set $\mathcal{E} = \{III, XII, IXI, IIX\}$?

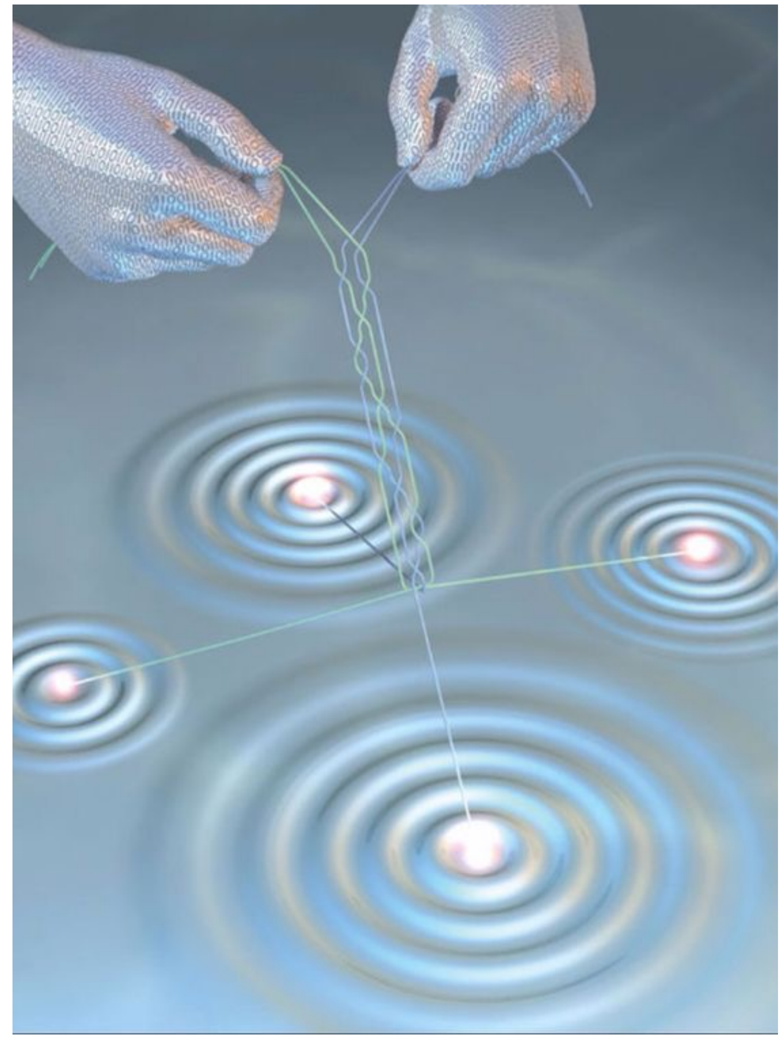
> - Do the following constraints hold, for any $E_k, E_l \in \mathcal{E}$:
>
>   (i) $\langle 000|E^\dagger_k E_l|111\rangle = 0$
>
>   (ii) $\langle 000|E^\dagger_k E_l|000\rangle = \langle 111|E^\dagger_k E_l|111\rangle$
>
> - *Note*: $I^\dagger = I$, $II = I$, $X^\dagger = X$, $XX = I$.
> - *Also*: $(A \otimes B \otimes C)(D \otimes E \otimes F) = (AD) \otimes (BE) \otimes (CF)$
>   - So, *e.g.*, $(XII)^\dagger(IIX) = (XI) \otimes (II) \otimes (IX) = XIX$
> - *In general*: In all combinations of $E^\dagger_k E_l$, there will be at most two X's.
> - *So*: Any combination of $E^\dagger_k E_l$ will fail to convert $|111\rangle$ into $|000\rangle$ or vice-versa.
> - *Thus*: In all cases of both (i) and (ii), the inner products will vanish.

2. Topological Quantum Error Correction Codes

- Is there a way to guarantee the KL Condition for a QECC based on the *topology* of the physical system we use to encode information in qubits?

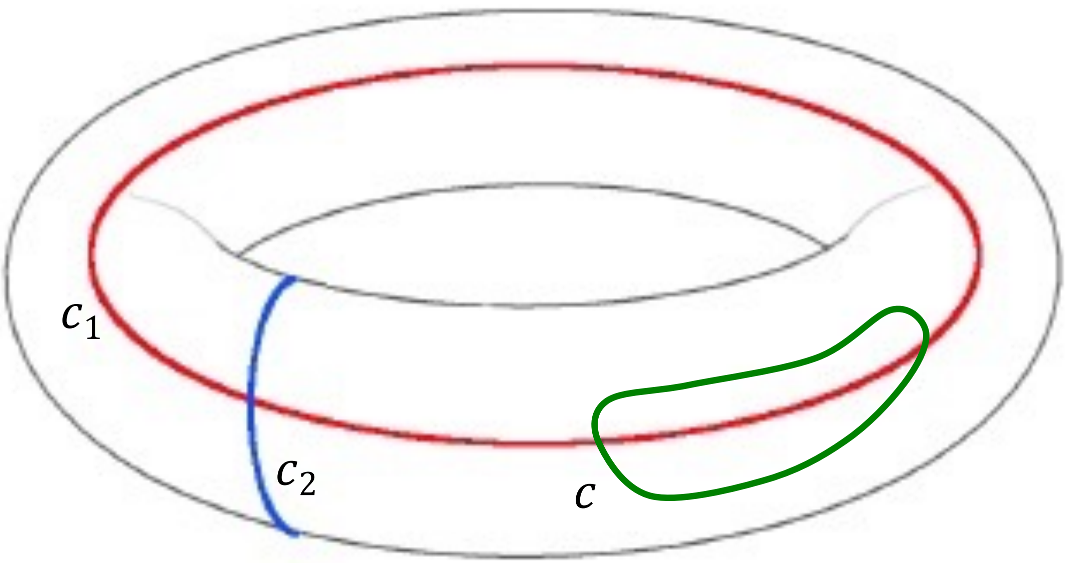
Yes!

- *Immediate goal*: To construct a QECC from a physical system with a non-trivial topology.
- *Ultimate goal*: To build a "topological" quantum computer.



A *topological property* of a surface is a property that remains invariant under continuous deformations of the surface.

Example: Consider 2-dim surface of a torus.

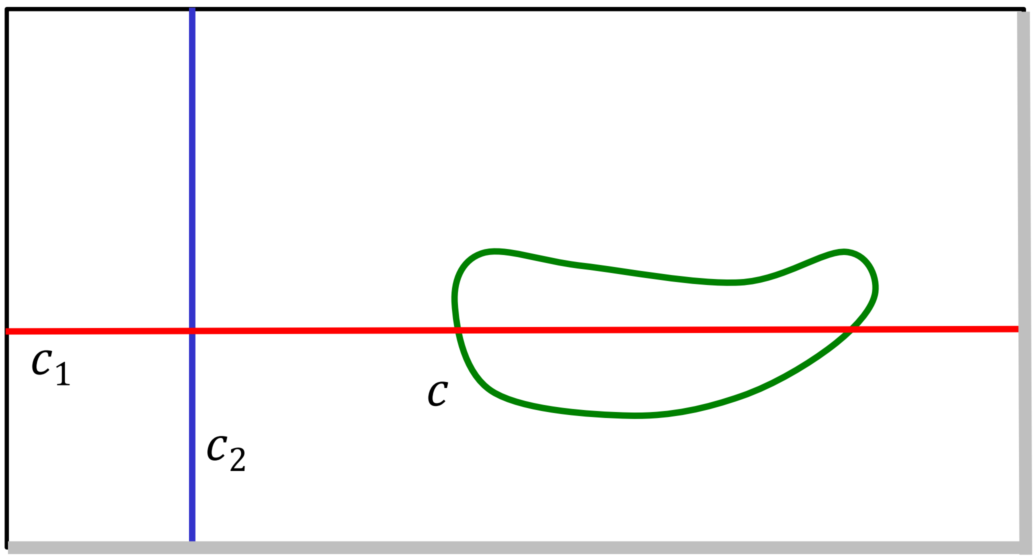


Three types of closed paths:

- Loops c which can be continuously deformed into a point.
- Loops c_1 which cannot be continuously deformed into a point.
- Loops c_2 which cannot be continuously deformed into a point.

- c_1 and c_2 are called "non-contractible" loops.
  - *Neither c, c_1, nor c_2 can be continuously deformed into the others.*
- The surface of a torus is characterized by these three families of loops.
  - *They describe features of the torus that are invariant under continuous deformations of its surface (i.e., they are topological properties).*

Slightly more abstract way to represent a torus: unwind it into a flat surface with periodic boundary conditions.



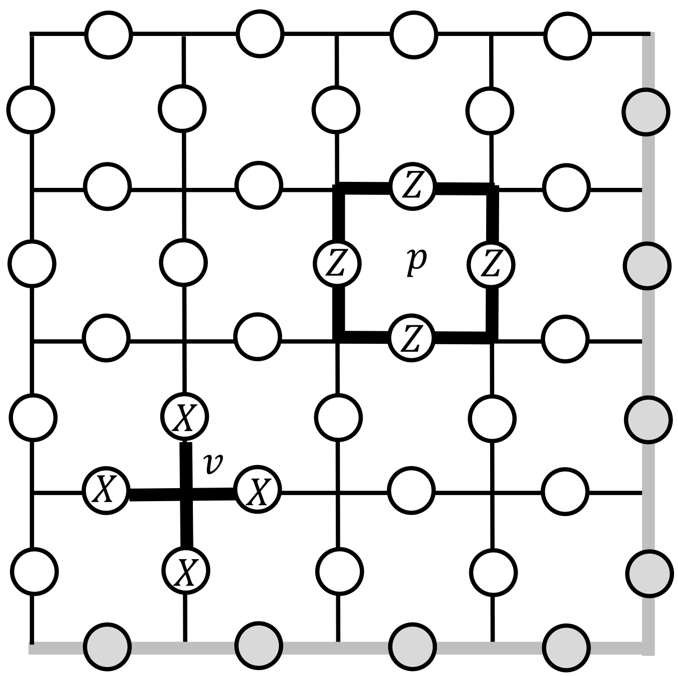
Periodic boundary conditions:

- Identify top and bottom edges.
- Identify left and right edges.

Let's add some (abstract) physics...

The Toric Code

Kitaev (2003)



- Put an $L \times L$ lattice on the torus with L^2 vertices.
- On each lattice edge, place a qubit.

$2L^2$ -qubit Hilbert space $\mathcal{H} = V_1 \otimes \cdots \otimes V_{2L^2}$, where each V is a single-qubit Hilbert space.

- For each vertex v define a vertex operator A_v:

$$A_v = I_1 \otimes \cdots \otimes I_{(i-1)} \otimes (X_{(i)} \otimes X_{(i+1)} \otimes X_{(i+2)} \otimes X_{(i+3)}) \otimes I_{(i+4)} \otimes \cdots I_{(2L^2)}$$

Acts as I on all qubits except those on edges leading to v, on which it acts as X.

- For each plaquette p define a plaquette operator B_p:

$$B_p = I_1 \otimes \cdots \otimes I_{(j-1)} \otimes (Z_{(j)} \otimes Z_{(j+1)} \otimes Z_{(j+2)} \otimes Z_{(j+3)}) \otimes I_{(j+4)} \otimes \cdots I_{(2L^2)}$$

Acts as I on all qubits except those on edges around p, on which it acts as Z.

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue +1.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Claim: $\mathcal{C}$ is a 4-dim (*i.e.*, two-qubit) subspace of $\mathcal{H}$ with "topologically distinct" basis states $\{|\xi\rangle_{ee}, |\xi\rangle_{eo}, |\xi\rangle_{oe}, |\xi\rangle_{oo}\}$ that are entangled with respect to the decomposition $\mathcal{H} = V_1 \otimes \cdots \otimes V_{2L^2}$.

Story to come:

- $\mathcal{C}$ will be our **code space**: use its two entangled qubits to encode information in a topologically non-local way.
- Operators that act like the identity on $\mathcal{C}$ will be "local" operators associated with contractible loops.
- Operators that transform codewords to other codewords (that are not the identity) will be "non-local" operators associated with non-contractible loops on the torus. They preserve the non-local aspects of $\mathcal{C}$.
- **Error** operators will be "local" operators associated with contractible open paths.

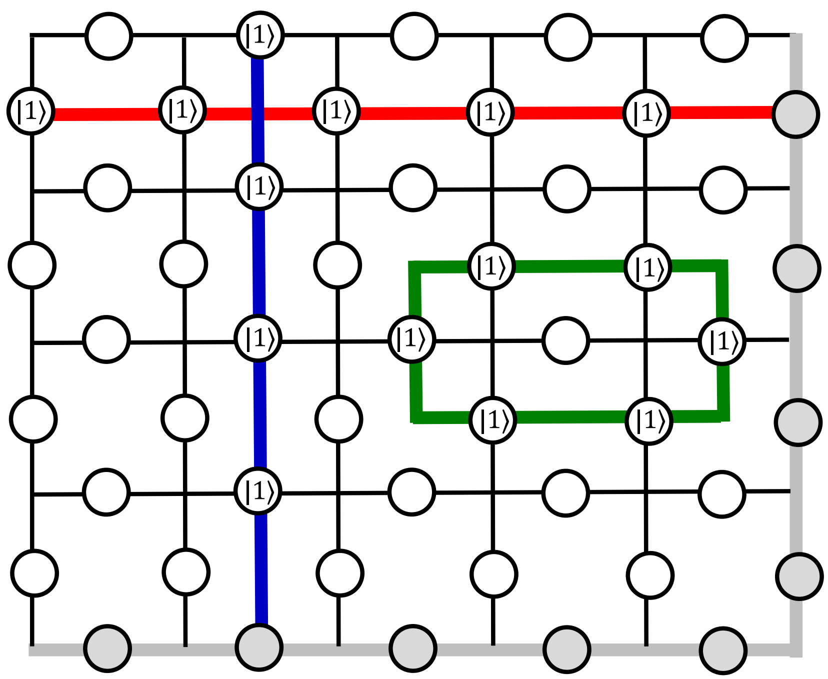
Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue $+1$.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Constraints:

(a) $B_p|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must either be the $|0\rangle$ $2L^2$–qubit state, or have an *even number* of $|1\rangle$ qubits per plaquette, since $Z|1\rangle = -|1\rangle$.

Claim: Constraint (a) entails $|\xi\rangle$ is either the $|0\rangle$ $2L^2$–qubit state or a *loop state*.



Def: A *loop state* is a $2L^2$–qubit state that has $|1\rangle$'s along one or more closed loops that do not intersect vertices, and $|0\rangle$'s everywhere else.

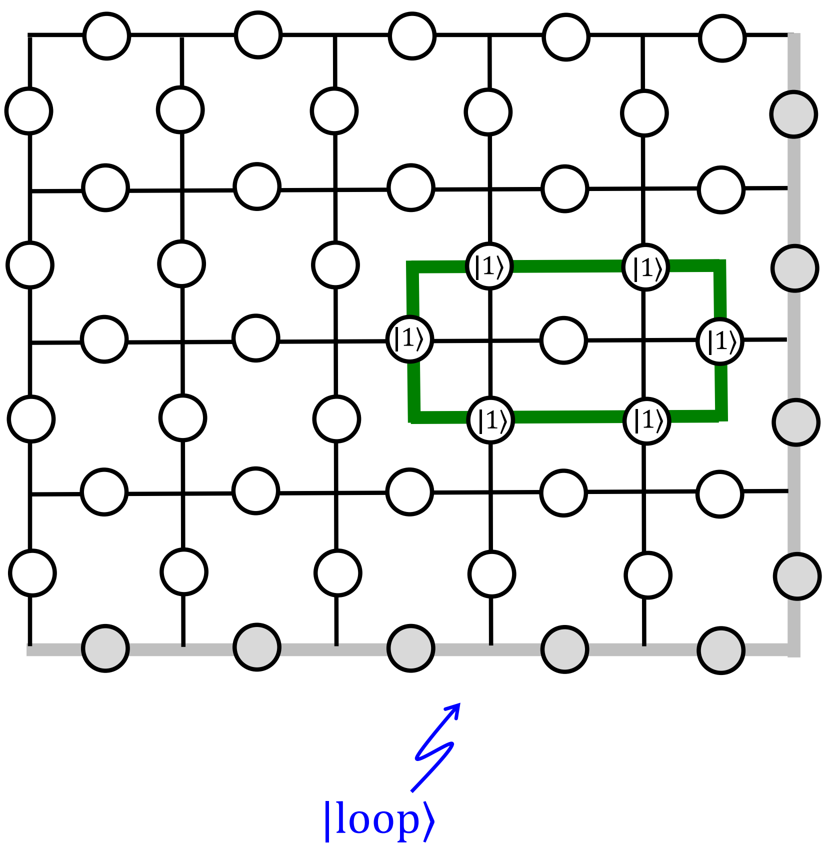
A loop state consisting of three closed loops of $|1\rangle$'s. (The "empty" qubits are $|0\rangle$'s.)

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue $+1$.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Constraints:

(b) $A_v|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must be a superposition of a vector in $\mathcal{H}$ and its A_v-flipped counterpart, since X flips qubits.



Example:

- *This loop state is not a $+1$ eigenvector of any A_v, since they flip $|0\rangle$'s to $|1\rangle$'s and $|1\rangle$'s to $|0\rangle$'s.*

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue $+1$.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{for all } v, p\}$$

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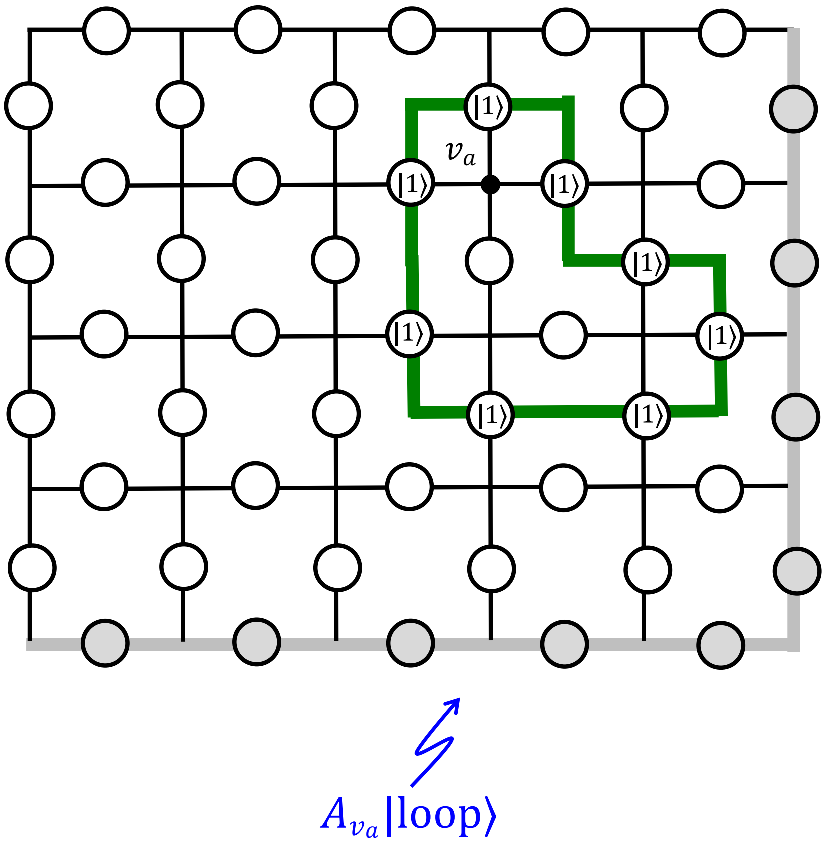

Example:

- *But: The sum of the loop state and any of its A_v-flipped counterparts is a $+1$ eigenvector of that vertex operator!*
- $A_{v_a}\{|\text{loop}\rangle + A_{v_a}|\text{loop}\rangle\} = \{A_{v_a}|\text{loop}\rangle + |\text{loop}\rangle\}$
- *Also note: Any A_v that acts on all $|0\rangle$'s creates a loop. And any A_v that "touches" a loop deforms it into another loop.*
- *So: If $|\xi\rangle$ is a loop state, then so is $A_v|\xi\rangle$ for all v.*

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue $+1$.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Constraints:

(a) $B_p|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must either be the $|0\rangle$ $2L^2$–qubit state or a loop state.

(b) $A_v|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must be a superposition of a vector in $\mathcal{H}$ and its A_v-flipped counterpart.

Claim: A vector that satisfies (a) and (b) is given by:

$$|\xi\rangle_{ee} = \prod_{i=1}^{L^2} 2^{-½}(I + A_{v_i})|0\rangle_1 \cdots |0\rangle_{2L^2}$$

- *Start with $|0\rangle$ $2L^2$–qubit state.*
- *Then add all other states that can be obtained via A_v-flips, and their unflipped counterparts.*

Proof: Let $j = 1, \ldots, L^2$. Then

$$B_{p_j}|\xi\rangle_{ee} = 2^{-L^2/2} B_{p_j}(I + A_{v_1})\cdots(I + A_{v_{L^2}})|0\cdots 0\rangle$$

$$= 2^{-L^2/2}(I + A_{v_1})\cdots(I + A_{v_{L^2}})B_{p_j}|0\cdots 0\rangle \qquad B_{p_j} \textit{ commutes with } (I + A_{v_i}) \textit{ for all } i, j$$

$$= 2^{-L^2/2}(I + A_{v_1})\cdots(I + A_{v_{L^2}})|0\cdots 0\rangle \qquad B_{p_j}|0\cdots 0\rangle = |0\cdots 0\rangle, \textit{ for all } j$$

$$= |\xi\rangle_{ee}$$

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue $+1$.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

Constraints:

(a) $B_p|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must either be the $|0\rangle$ $2L^2$–qubit state or a loop state.

(b) $A_v|\xi\rangle = |\xi\rangle$ requires that any $|\xi\rangle$ must be a superposition of a vector in $\mathcal{H}$ and its A_v-flipped counterpart.

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$$|\xi\rangle_{ee} = \prod_{i=1}^{L^2} 2^{-½}\,(I + A_{v_i})|0\rangle_1 \cdots |0\rangle_{2L^2}$$

- *Start with $|0\rangle$ $2L^2$–qubit state.*
- *Then add all other states that can be obtained via A_v-flips, and their unflipped counterparts.*

Proof: Let $j = 1, \ldots, L^2$. Then

$$\begin{aligned}
A_{v_j}|\xi\rangle_{ee} &= 2^{-L^2/2} A_{v_j}(I + A_{v_1})\cdots(I + A_{v_{L^2}})|0\ldots0\rangle \\
&= 2^{-L^2/2}(I + A_{v_1})\cdots A_{v_j}(I + A_{v_j})\cdots(I + A_{v_{L^2}})|0\ldots0\rangle && A_{v_j} \text{ commutes with } (I + A_{v_i}) \\
&= 2^{-L^2/2}(I + A_{v_1})\cdots(A_{v_j} + A_{v_j}A_{v_j})\cdots(I + A_{v_{L^2}})|0\ldots0\rangle \\
&= 2^{-L^2/2}(I + A_{v_1})\cdots(I + A_{v_j})\cdots(I + A_{v_{L^2}})|0\ldots0\rangle && A_{v_j}A_{v_j} = I \\
&= |\xi\rangle_{ee}
\end{aligned}$$

Exercise: Find the space $\mathcal{C}$ of eigenvectors of all A_v and B_p operators with eigenvalue $+1$.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

$$|\xi\rangle_{ee} = \prod_{i=1}^{L^2} 2^{-1/2} (I + A_{v_i})|0\rangle_1 \cdots |0\rangle_{2L^2}$$

- Start with $|0\rangle$ $2L^2$–qubit state.
- Then add all other states that can be obtained via A_v-flips, and their unflipped counterparts.

- *Note*: The $|0\rangle$ $2L^2$–qubit state has an even number (zero!) of c_1 and c_2 loops.
- *And*: The $(I + A_{v_i})$ operators do not change the parity of the number of c_1 and c_2 loops (so $|\xi\rangle_{ee}$ also has an even number of c_1 and c_2 loops).
- *Which means*: There are four *topologically distinct* types of elements of $\mathcal{C}$:

  $|\xi\rangle_{ee}$: loop state with even # c_1 loops and even # c_2 loops.

  $|\xi\rangle_{eo}$: loop state with even # c_1 loops and odd # c_2 loops.

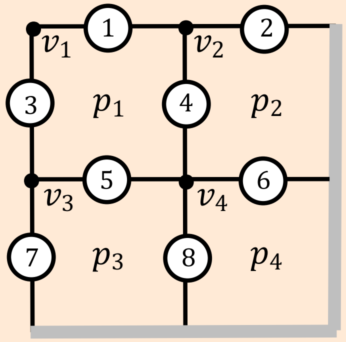
  $|\xi\rangle_{oe}$: loop state with odd # c_1 loops and even # c_2 loops.

  $|\xi\rangle_{oo}$: loop state with odd # c_1 loops and odd # c_2 loops.

- *So*: $\mathcal{C} = \text{span}\{|\xi\rangle_{ee}, |\xi\rangle_{eo}, |\xi\rangle_{oe}, |\xi\rangle_{oo}\}$

- Topologically distinct bases vectors.
- Entangled with respect to $\mathcal{H} = V_1 \otimes \cdots \otimes V_{2L^2}$.

Example: $L = 2\ |\xi\rangle_{ee}$ state.



8 qubits (so $\mathcal{H}$ has $2^8 = 256$ dimensions!)

4 plaquettes: $p_1 = \{1, 3, 4, 5\}$ $p_2 = \{2, 3, 4, 6\}$ $p_3 = \{1, 5, 7, 8\}$ $p_4 = \{2, 6, 7, 8\}$

4 vertices: $v_1 = \{1, 2, 3, 7\}$ $v_2 = \{1, 2, 4, 8\}$ $v_3 = \{3, 5, 6, 7\}$ $v_4 = \{4, 5, 6, 8\}$

$$|\xi\rangle_{ee} = \prod_{i=1}^{4} 2^{-1/2}\,(I + A_{v_i})|00000000\rangle$$

$$= \tfrac{1}{4}(I + A_{v_1})(I + A_{v_2})(I + A_{v_3})(I + A_{v_4})|00000000\rangle$$

$$= \tfrac{1}{4}(I + A_{v_1})(I + A_{v_2})(I + A_{v_3})\{|00000000\rangle + |00011101\rangle\}$$

$$= \tfrac{1}{4}(I + A_{v_1})(I + A_{v_2})\{|00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle\}$$

$$= \tfrac{1}{4}(I + A_{v_1})\{|00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle + |11010001\rangle + |11001100\rangle + |11111111\rangle + |11100010\rangle\}$$

$$= \tfrac{1}{4}\{|00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle + |11010001\rangle + |11001100\rangle + |11111111\rangle + |11100010\rangle + |11100010\rangle + |11111111\rangle + |11001100\rangle + |11010001\rangle + |00110011\rangle + |00101110\rangle + |00011101\rangle + |00000000\rangle\}$$

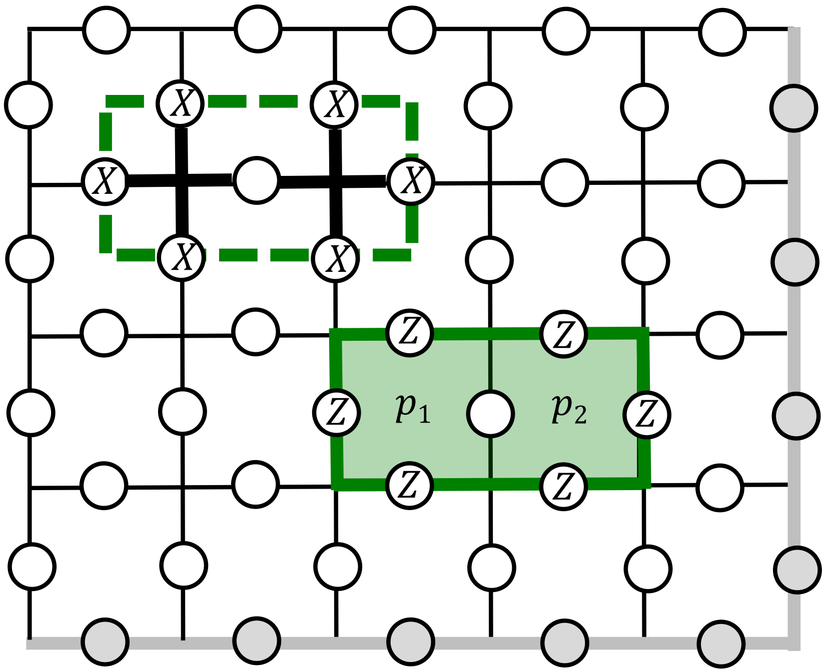
$$= \tfrac{1}{2}\{|00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle + |11010001\rangle + |11001100\rangle + |11111111\rangle + |11100010\rangle\}$$

- *entangled state!*
- *each term has even #of 1's*
- *each term has an A_v-flipped counterpart*

Three types of operators that act on $\mathcal{C}$

First type: "Stabilizer" operators.

- Composing adjacent plaquette operators B_{p_1}, B_{p_2} to form $B_{p_1}B_{p_2}$ results in a *loop* of Z operators:



- B_{p_1} and B_{p_2} share an edge.
- $B_{p_1}B_{p_2}$ includes the square of the Z operator of the shared edge, and $Z^2 = I$.
- *So*: The Z's that appear in $B_{p_1}B_{p_2}$ will act on the qubits that form the *boundary* of the two plaquettes!
- The same holds for any number of adjacent plaquette operators.
- The same holds for vertex operators A_v.

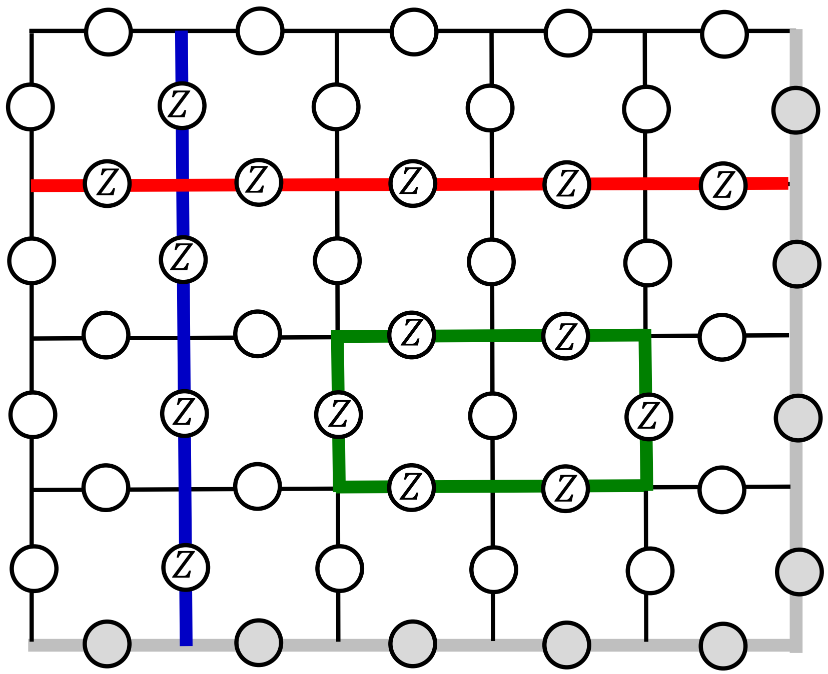
- *Note*: These loops are of type c on the torus.

> Type c loop operators are called "stabilizer" operators:
>
> - *They act like the identity on $\mathcal{C}$ (since they are compositions of A_v and B_p operators).*
> - *They are "local" (in the sense that they are associated with contractible loops).*

Three types of operators that act on $\mathcal{C}$

Second type: "Encoded logical" operators.

- There are two other types of loops on a torus: *non-contractible* loops c_1 and c_2.



- Let $\bar{Z}_1$ and $\bar{Z}_2$ refer to the two types of products of Z operators along loops of type c_1 and c_2.
- Let $\bar{X}_1$ and $\bar{X}_2$ refer to the two types of products of X operators along loops of type c_1 and c_2.

Types c_1 and c_2 loop operators are called "encoded logical" operators:

- *They act on codewords in $\mathcal{C}$ and transform them into other codewords (they are not the identity on $\mathcal{C}$).*
- *They are not "local" operators (in the sense that they are associated with non-contractible loops).*

Aside: Why do the non-contractible loop operators map vectors in $\mathcal{C}$ to other vectors in $\mathcal{C}$?

Claim 1. Any operator D that maps $\mathcal{C}$ to $\mathcal{C}$ must commute with all A_v and B_p operators.

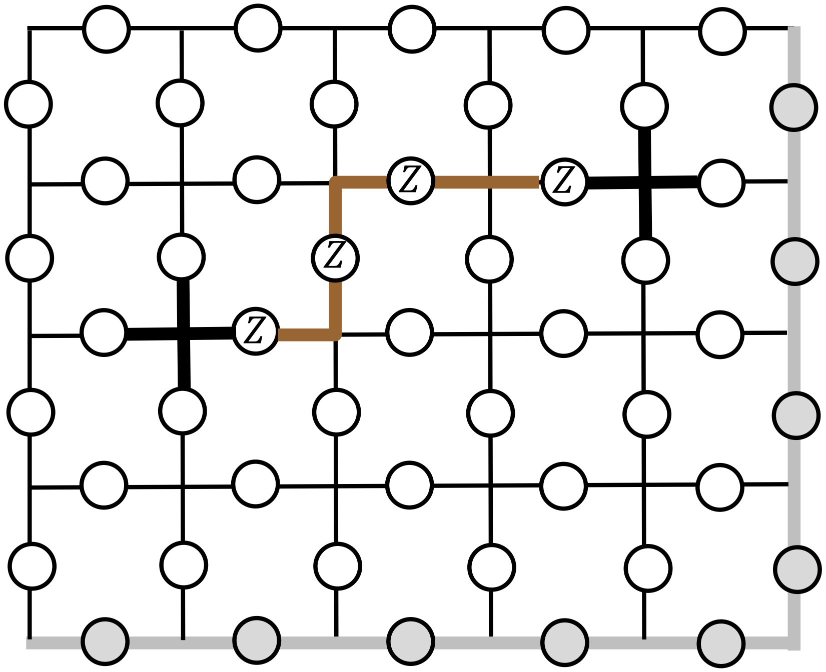
Proof. Recall $\mathcal{C} = \{|\phi\rangle : \mathcal{O}|\phi\rangle = |\phi\rangle$, for $\mathcal{O} = A_v$ or $B_p\}$.

- Let D be an operator such that $D|\psi\rangle \in \mathcal{C}$, for any $|\psi\rangle \in \mathcal{C}$.
- *Suppose* $D\mathcal{O} = -\mathcal{O}D$ (D anticommutes with $\mathcal{O}$).
  - Then for any $|\psi\rangle \in \mathcal{C}$, $D|\psi\rangle = D\mathcal{O}|\psi\rangle = -\mathcal{O}D|\psi\rangle$.
  - *Or*: $\mathcal{O}D|\psi\rangle = -D\mathcal{O}|\psi\rangle = -D|\psi\rangle$.
  - *So*: $\mathcal{O}(D|\psi\rangle) = -(D|\psi\rangle) \neq D|\psi\rangle$.
  - *But*: $\mathcal{O}$ is the identity on $\mathcal{C}$!
  - *So*: $D|\psi\rangle \notin \mathcal{C}$ (contradiction!)
- *Hence*: D must commute with $\mathcal{O}$.

Claim 1 *(reworded)*.
(maps $\mathcal{C}$ to $\mathcal{C}$) $\Rightarrow$ (commutes with all A_v, B_p)

<u>*Aside: Why do the non-contractible loop operators map vectors in $\mathcal{C}$ to other vectors in $\mathcal{C}$?*</u>

Claim 2. Any operator formed from an open path of X's or Z's will anticommute with two A_v's or two B_p's.



- Consider an operator $S^Z(t)$ which is the identity on all qubits except for an open path t of Z's.
- $S^Z(t)$ commutes with all B_p's.
  - We only need to consider B_p's with Z's that overlap the Z's in $S^Z(t)$.
  - Those B_p's commute with $S^Z(t)$, since Z commutes with itself.
- $S^Z(t)$ commutes with all A_v's, except for the two at the endpoints of t.
  - We only need to consider A_v's with X's that overlap the Z's in $S^Z(t)$.
  - Those A_v's that are not at the endpoints of t commute with $S^Z(t)$, since each one has 2 X's that overlap 2 Z's in $S^Z(t)$, and Z anticommutes with X.
- $S^Z(t)$ anticommutes with the two A_v's at the endpoints of t.
  - Each of these A_v's has 1 X that overlaps 1 Z in $S^Z(t)$, and Z anticommutes with X.

Claim 2 ***(reworded)*****.**
(open path operator) $\Rightarrow$
$\neg$(commutes with all A_v, B_p)

Aside: Why do the non-contractible loop operators map vectors in $\mathcal{C}$ to other vectors in $\mathcal{C}$?

Claim 1 *(reworded)*.
(maps $\mathcal{C}$ to $\mathcal{C}$) $\Rightarrow$ (commutes with all A_v, B_p)

Claim 2 *(reworded)*.
(open path operator) $\Rightarrow$ $\neg$(commutes with all A_v, B_p)

- *Claims 1 & 2 entail*:

  (maps $\mathcal{C}$ to $\mathcal{C}$) $\Rightarrow$ $\neg$(open path operator)

- *So*: A code space operator is either an individual A_v or B_p, or a loop operator.

- *And*: The identity on $\mathcal{C}$ is either an individual A_v or B_p, or a contractible loop operator.

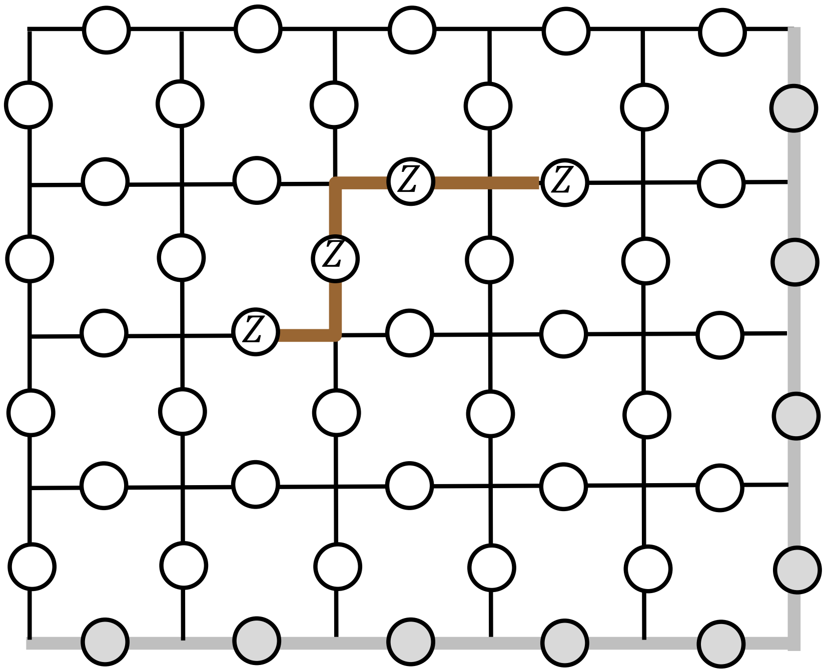
  - *Moreover: A non-contractible loop operator cannot be constructed by a product of A_v's or B_p's (such a product is the boundary of a set of adjacent plaquettes or vertices).*

So: A non-contractible loop operator is a code space operator that is not the identity on $\mathcal{C}$.

Three types of operators that act on $\mathcal{C}$

Third type: Error operators.

- By definition, error operators act on codewords and corrupt them (transform them into states not in $\mathcal{C}$).



- Error operators can't be associated with products of Z's or X's on loops: There are only three types, and each type transforms codewords to codewords.
- What about "open path" products of Z's or X's?

Claim: Open path products of Z's or X's transform codewords in $\mathcal{C}$ out of $\mathcal{C}$.

Proof: We've just seen that open path products of Z's or X's anticommute with two A_v's or tow B_p's, and hence transform codewords out of $\mathcal{C}$.

- "Open path" operators are "local" (in the sense that they are associated with *contractible* line segments).

Summary: Three types of operators that act on $\mathcal{C}$

1. Stabilizer operators (*local*).

$S^Z(c) = \bigotimes_{j\in c} Z_j$

$S^X(c') = \bigotimes_{j\in c'} X_j$

c, c' = contractible loops

2. Encoded logical operators (*non-local*).

$\bar{Z}_1 = \bigotimes_{j\in \gamma_1} Z_j$ $\quad$ $\bar{Z}_2 = \bigotimes_{j\in \gamma_2} Z_j$

$\bar{X}_1 = \bigotimes_{j\in \gamma'_1} X_j$ $\quad$ $\bar{X}_2 = \bigotimes_{j\in \gamma'_2} X_j$

γ_1, γ'_1 = non-contractible loops of type c_1

γ_2, γ'_2 = non-contractible loops of type c_2

3. Error operators (*local*).

$S^Z(t) = \bigotimes_{j\in t} Z_j$

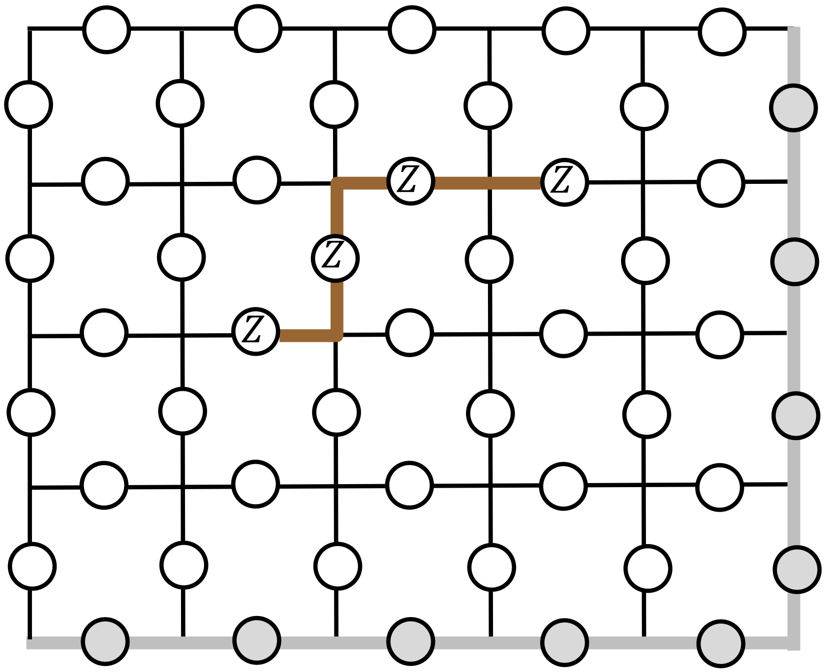
$S^X(t') = \bigotimes_{j\in t'} X_j$

t, t' = contractible open paths

Now: Check to see if the KL Condition holds for the toric code.

- Does $\mathcal{C}$ correct the error set $\mathcal{E} = \{S^Z(t), S^X(t') : \text{for all } t, t'\}$?
  - *Is it the case that* $\langle\psi_i|E^\dagger_k E_l|\psi_j\rangle = c_{kl}\delta_{ij}$, *for any* $E_k,\ E_l \in \mathcal{E}$, *and* $\psi_i,\ \psi_j \in \mathcal{C}$?

Yes!



- For any open-path operator E_l between two endpoints...

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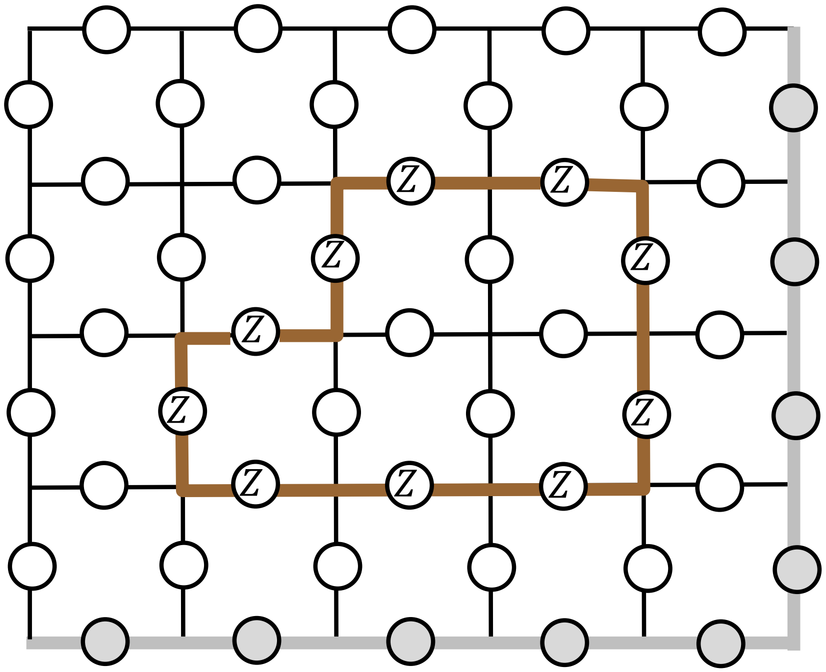

- For any open-path operator E_l between two endpoints...
- ... there is always another E_k with the same endpoints such that $E^\dagger_k E_l$ is a "type-c" loop operator; *i.e.*, a stabilizer operator.

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- Does $\mathcal{C}$ correct the error set $\mathcal{E} = \{S^Z(t), S^X(t') : \text{for all } t, t'\}$?
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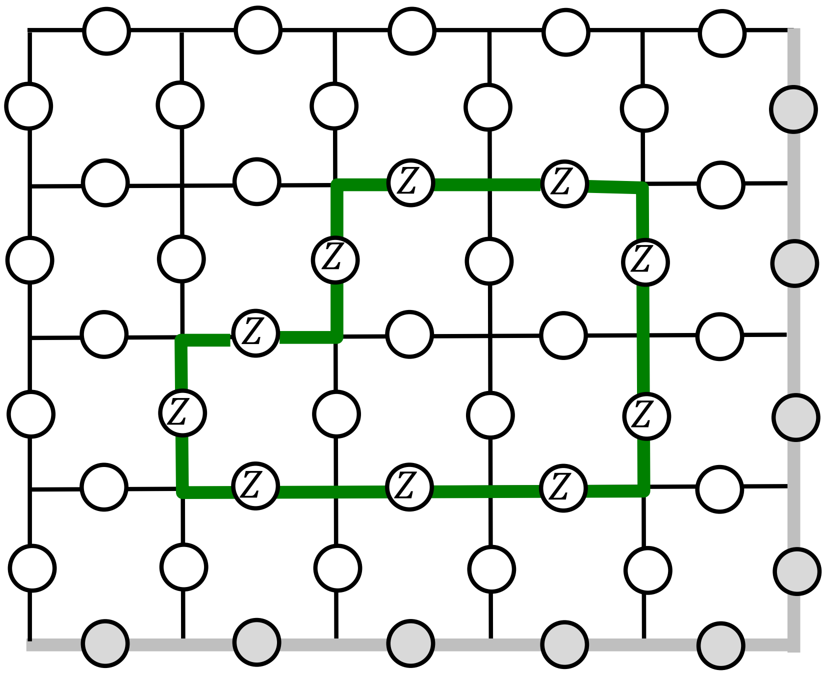


- For any open-path operator E_l between two endpoints...
- ... there is always another E_k with the same endpoints such that $E^\dagger_k E_l$ is a "type-c" loop operator; *i.e.*, a stabilizer operator.
- *And*: Stabilizer operators act as the identity on $\mathcal{C}$.

Upshot: We've encoded information "non-locally" in $\mathcal{C}$ in such a way that local errors can be detected and corrected.

Two senses of "non-locality" in the Toric Code

- *Entanglement non-locality*: The codewords (elements of $\mathcal{C}$) are entangled states.
  - *Entanglement non-locality = Einstein non-locality + Bell non-locality*

> *Recall*:
> - Einstein non-locality occurs when two systems are correlated and the correlation cannot be explained by a direct cause that travels from one system to the other.
> - Bell non-locality occurs when two systems are correlated and the correlation cannot be explained by a common cause

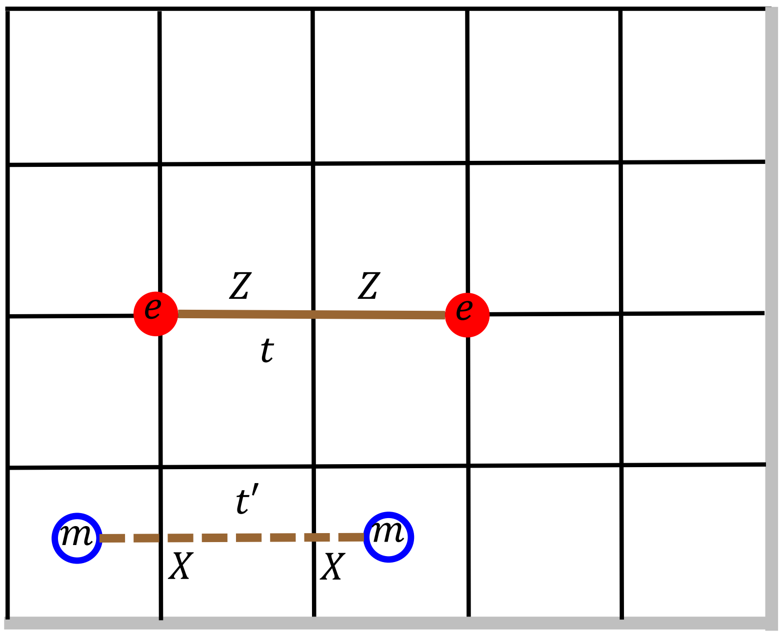
- *Topological non-locality*: The operators that act on codewords are non-contractible loop operators.

> *Suppose*: Topological non-locality occurs when a quantity is not localized to a contractible region of space.

Open Question: Under what conditions does entanglement non-locality entail topological non-locality and/or vice-versa?

Let's add some (slightly more concrete) physics...

- Interpret the code space $\mathcal{C}$ as the space of ground-states $|q\rangle$ (states of lowest energy) of a physical system.



- Interpret a Z (or X) error operator as acting on a ground-state to produce a pair of "e" (or "m") "quasiparticle" excitations at the ends of the open path.
- What happens when we move an m around an e?
- $|\Psi_{initial}\rangle = S^Z(t)S^X(t')|q\rangle$

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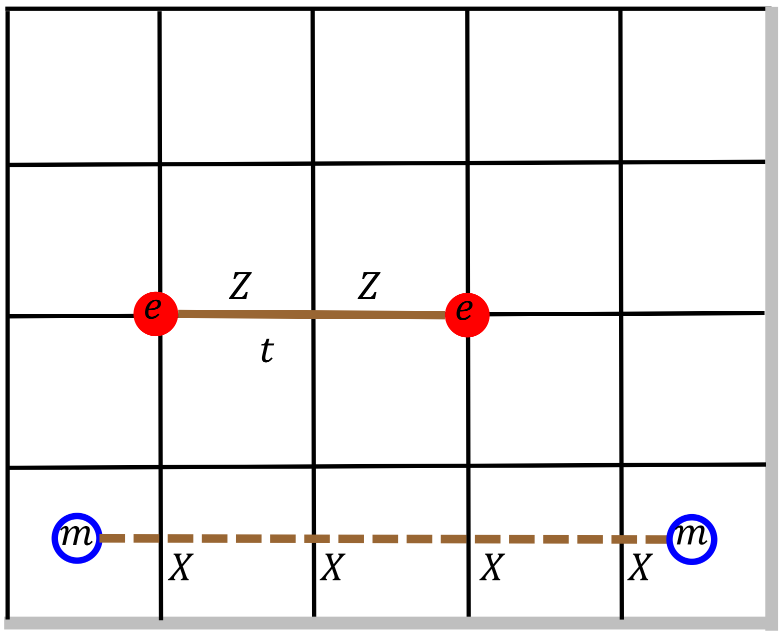

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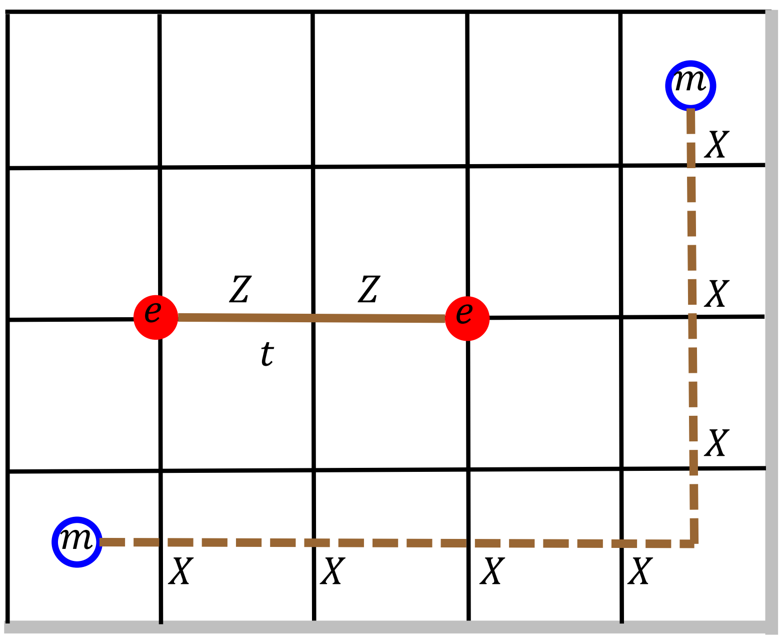

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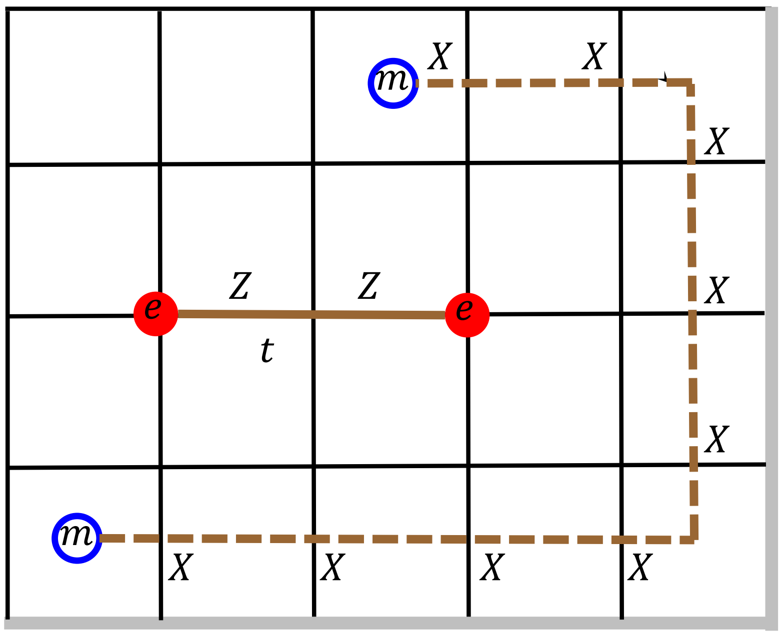

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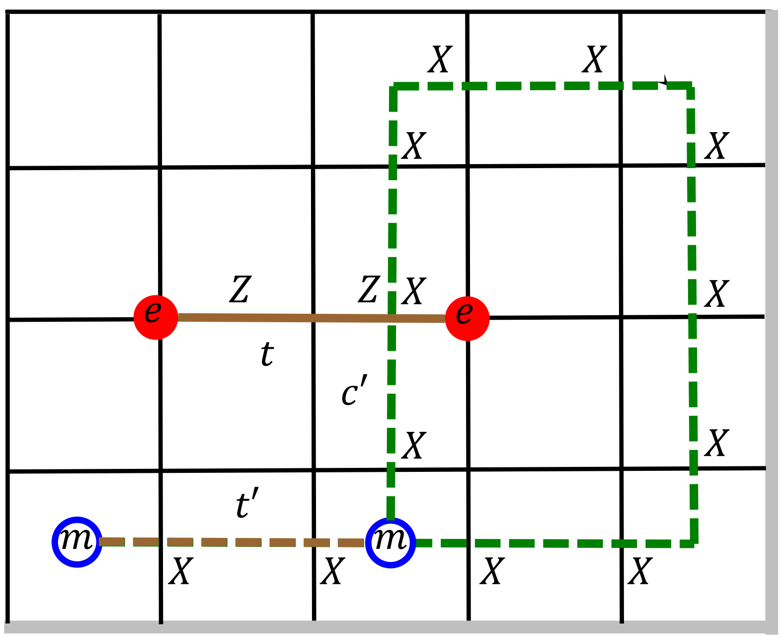


- Interpret a Z (or X) error operator as acting on a ground-state to produce a pair of "e" (or "m") "quasiparticle" excitations at the ends of the open path.
- What happens when we move an m around an e?
- $|\Psi_{initial}\rangle = S^Z(t)S^X(t')|q\rangle$

- $|\Psi_{final}\rangle = S^X(c')S^Z(t)S^X(t')|q\rangle$

$= -S^Z(t)S^X(c')S^X(t')|q\rangle$ — *$S^Z(t)$ and $S^X(c')$ anticommute*

$= -S^Z(t)S^X(t')S^X(c')|q\rangle$ — *$S^X(c')$ and $S^X(t')$ commute*

$= -|\Psi_{initial}\rangle$ — *$S^X(c')$ acts like the identity on $\mathcal{C}$*

- <u>So</u>: Moving an m quasiparticle completely around an e quasiparticle changes the phase of the initial 4-particle state by -1.

In general: When two particles are exchanged in a multiparticle system, the multiparticle state $|\Psi\rangle$ picks up a phase $|\Psi\rangle \rightarrow e^{i\theta}|\Psi\rangle$.

- Taking one particle around another is equivalent to two exchanges; so $|\Psi\rangle \rightarrow e^{2i\theta}|\Psi\rangle$.

- *So*: Taking an m quasiparticle around an e quasi-particle produces the phase $e^{2i\theta} = -1$, or $\theta = \pi/2$.

$e^{2i\theta} = \cos 2\theta + i\sin 2\theta$

- *So*: One exchange of an m quasiparticle and an e quasiparticle produces the phase $|\Psi\rangle \rightarrow e^{i\pi/2}|\Psi\rangle$.

Bosons:	Particle exchange phase $\theta = 0$.
Fermions:	Particle exchange phase $\theta = \pi$.
Anyons:	Particle exchange phase $\theta \in (0, \pi)$.

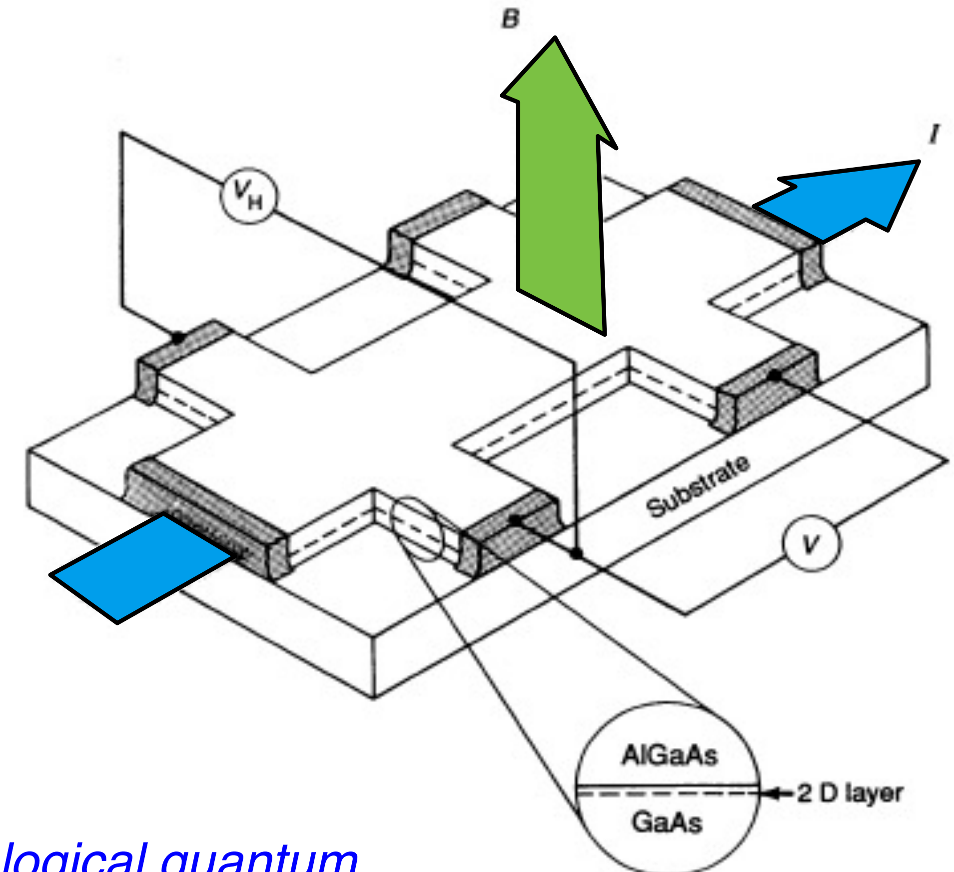
Upshot: m and e quasiparticles are anyons!
(They obey "fractional statistics".)

Physical significance: There are physical systems that exhibit characteristics of the toric code!

1998 Nobel Prize in Physics

- Fractional quantum Hall system:

- 2-dim conductor in external magnetic field *B*.
- At low temps, longitudinal resistance vanishes, and transverse (Hall) resistance becomes quantized.
- *Prediction*: Low-energy anyonic excitations.



Open Question: Can we build a topological quantum computer out of a fractional quantum Hall system?