07. QIT, Part III.

1. Quantum Error Correction Codes (QECCs)

- *Goal*: To encode information in qubits in such a way that errors due to "noise" can be detected and corrected.
  - *But: Typical quantum algorithms encode information in entangled qubits.*
  - *And: Attempts to detect and correct errors due to noise run the risk of decohering entangled qubits, thus destroying the information.*

Task: To detect and correct errors *without* decohering the relevant entangled qubits.

Set-Up: Suppose information is encoded in a qubit $|Q\rangle = a|0\rangle + b|1\rangle$.

Step 1. Encode $|Q\rangle$ in a **codeword**.

- Do this by performing appropriate transformations on the single-qubit basis states $|0\rangle$, $|1\rangle$.

The type of transformations depends on the type of errors we expect to occur.

- The new basis states form a space called the **code space** $\mathcal{C}$.
- Complete the set of basis states to form a larger space called the **coding space** $\mathcal{H}$.

$\mathcal{C}$ is a subspace of $\mathcal{H}$

Example: We might transform the single-qubit basis states into three-qubit basis states:

$$|0\rangle \to |000\rangle$$

$$|1\rangle \to |111\rangle$$

- The **codeword** is then $a|000\rangle + b|111\rangle$.
- The **code space** $\mathcal{C}$ is the space spanned by $\{|000\rangle, |111\rangle\}$, which is a 2-dim subspace of the larger 8-dim three-qubit **coding space** space $\mathcal{H}$ spanned by $\{|000\rangle, |001\rangle, |010\rangle, |100\rangle, |110\rangle, |101\rangle, |011\rangle, |111\rangle\}$.

Step 2. Represent **errors** by multi-qubit operators constructed from the single-qubit operators I, X, Y, Z.

- Errors "corrupt" the basis states of $\mathcal{C}$, and hence the codeword, projecting it out of $\mathcal{C}$.

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- An **error** might be represented by the operator $X \otimes I \otimes I$.
- This would produce a corrupted codeword $a|100\rangle + b|011\rangle$, which is an element of $\mathcal{H}$ but not of $\mathcal{C}$.

Step 3. Devise an appropriate operation that acts on a corrupted codeword in $\mathcal{H}$ and projects it back into $\mathcal{C}$ (thereby "correcting" it).

Necessary and sufficient condition for error-correction

- Let $\mathcal{C} = \text{span}\{|\psi_1\rangle, ..., |\psi_p\rangle\}$, for some number p of basis states.

  Let $\mathcal{E} = \{E_1, ..., E_q\}$ be a set of q error operators.

Knill-Laflamme (KL) Condition: A code space $\mathcal{C} = \text{span}\{|\psi_1\rangle, ..., |\psi_p\rangle\}$ corrects the error set $\mathcal{E} = \{E_1, ..., E_q\}$ *if and only if*

(i) $\langle\psi_i|E^\dagger{}_k E_l|\psi_j\rangle = 0$

(ii) $\langle\psi_i|E^\dagger{}_k E_l|\psi_i\rangle = \langle\psi_j|E^\dagger{}_k E_l|\psi_j\rangle, \quad i \neq j$

- *Condition (i) says*: Corrupted basis states $E_l|\psi_j\rangle$, $E_k|\psi_i\rangle$ are orthogonal, and hence distinguishable from each other.
- *Condition (ii) says*: Measurements made to determine the error will not give any information about the codeword itself (and thereby possibly decohere it).
- *Conditions (i) & (ii) together say*: The projection of the operator $E^\dagger{}_k E_l$ onto the code space is a multiple of the identity: $\langle\psi_i|E^\dagger{}_k E_l|\psi_j\rangle = c_{kl}\delta_{ij}$, for arbitrary constants c_{kl}.

Example: Single-qubit flip error correction code.

Task: To transmit a qubit $|Q\rangle = a|0\rangle + b|1\rangle$ in the presence of noise that flips single-qubit basis states.

Step 1. Encode $|Q\rangle$ in codeword $|\Phi\rangle = a|000\rangle + b|111\rangle$. A three-qubit state

- $|\Phi\rangle$ is an element of the 2-dim code space $\mathcal{C} = \text{span}\{|000\rangle, |111\rangle\}$.
- $|\Phi_{corrupt}\rangle = a|\_\_\rangle_1|\_\_\rangle_2|\_\_\rangle_3 + b|\_\_\rangle_1|\_\_\rangle_2|\_\_\rangle_3$ can take one of four forms:

$$\boxed{\begin{array}{l} a|000\rangle + b|111\rangle \\ a|100\rangle + b|011\rangle \\ a|010\rangle + b|101\rangle \\ a|001\rangle + b|110\rangle \end{array}}$$

$|\Phi_{corrupt}\rangle$ is an element of the 8-dim three-qubit space
$\mathcal{H} = \text{span}\{|000\rangle, |001\rangle, |010\rangle, |100\rangle, |110\rangle, |101\rangle, |011\rangle, |111\rangle\}$

Step 2. Represent single-qubit flip errors by 4 three-qubit operators:

$$\mathcal{E} = \{I \otimes I \otimes I, X \otimes I \otimes I, I \otimes X \otimes I, I \otimes I \otimes X\}$$

does nothing — flips 1st qubit — flips 2nd qubit — flips 3rd qubit

Step 3. Error detection/correction protocol:

(a) Attach two "empty register" qubits $|00\rangle$ to $|\Phi_{corrupt}\rangle$:

$$|\Phi_{corrupt}\rangle|00\rangle = \{a|\_\_\rangle_1|\_\_\rangle_2|\_\_\rangle_3 + b|\_\_\rangle_1|\_\_\rangle_2|\_\_\rangle_3\}|0\rangle_4 0\rangle_5$$

(b) *Error detection*:

- Perform *XOR* on qubits 1 and 2 and store result in qubit 4.
- Perform *XOR* on qubits 1 and 3 and store result in qubit 5.

$0\ XOR\ 0 = 0$
$0\ XOR\ 1 = 1$
$1\ XOR\ 0 = 1$
$1\ XOR\ 1 = 0$

(b) *Error correction*: Measure qubits 4 and 5 to determine form of $|\Phi_{corrupt}\rangle$ and what three-qubit operator to use to correct it.

Corrupted codeword/register	*Error detection*	*Error correction*
$\{a\|000\rangle + b\|111\rangle\}\|00\rangle$	$\{a\|000\rangle + b\|111\rangle\}\|00\rangle$	$I \otimes I \otimes I$
$\{a\|100\rangle + b\|011\rangle\}\|00\rangle$	$\{a\|100\rangle + b\|011\rangle\}\|11\rangle$	$X \otimes I \otimes I$
$\{a\|010\rangle + b\|101\rangle\}\|00\rangle$	$\{a\|010\rangle + b\|101\rangle\}\|10\rangle$	$I \otimes X \otimes I$
$\{a\|001\rangle + b\|110\rangle\}\|00\rangle$	$\{a\|001\rangle + b\|110\rangle\}\|01\rangle$	$I \otimes I \otimes X$

Both detection and correction protocols do not decohere $|\Phi_{corrupt}\rangle$!

- *Now*: Check to see if the KL Condition holds for our single-qubit flip error correction code.
  - Does $\mathcal{C}$ = span{$|000\rangle$, $|111\rangle$} correct the error set $\mathcal{E} = \{III, XII, IXI, IIX\}$?

- Do the following conditions hold, for any $E_k, E_l \in \mathcal{E}$:

  (i) $\langle 000|E^\dagger{}_k E_l|111\rangle = 0$

  (ii) $\langle 000|E^\dagger{}_k E_l|000\rangle = \langle 111|E^\dagger{}_k E_l|111\rangle$

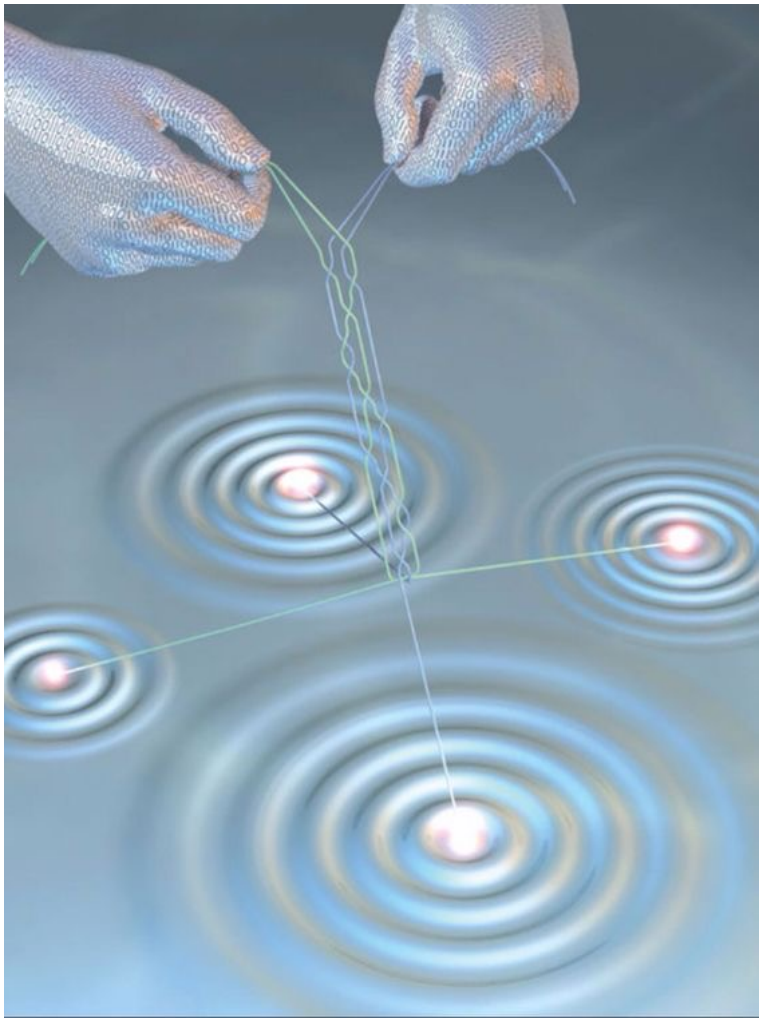
- *Note*: $I^\dagger = I, II = I, X^\dagger = X, XX = I$.
- *Also*: $(A \otimes B \otimes C)(D \otimes E \otimes F) = (AD) \otimes (BE) \otimes (CF)$
  - So, *e.g.*, $(XII)^\dagger(IIX) = (XI) \otimes (II) \otimes (IX) = XIX$
- *In general*: In all combinations of $E^\dagger{}_k E_l$, there will be at most two X's.
- *So*: Any combination of $E^\dagger{}_k E_l$ will fail to convert $|111\rangle$ into $|000\rangle$ or vice-versa.
- *Thus*: In all cases of both (i) and (ii), the inner products will vanish.

2. Topological Quantum Error Correction Codes

- Is there a way to guarantee the KL Condition for a QECC based on the *topology* of the physical system we use to encode information in qubits?

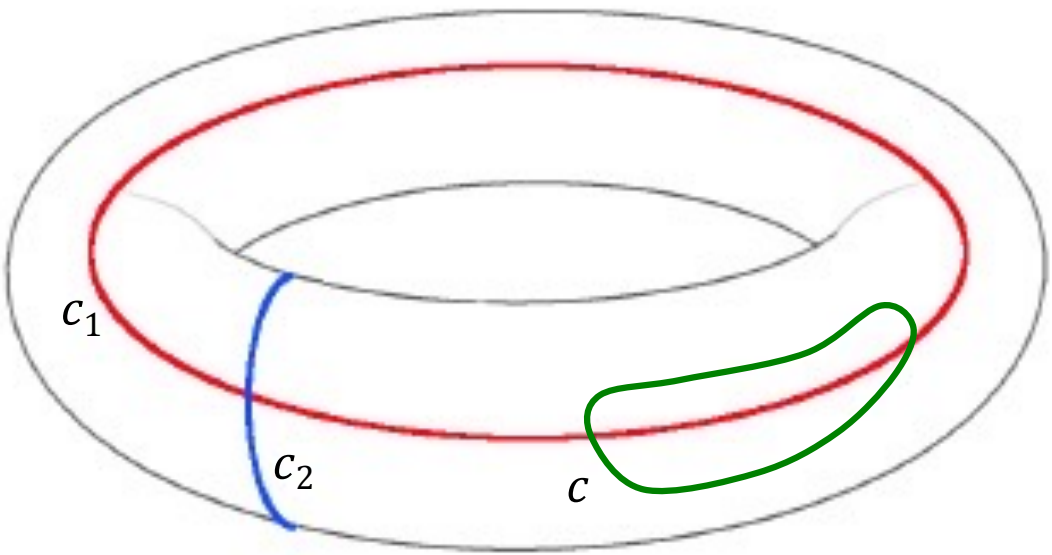
Yes!

- *Immediate goal*: To construct a QECC from a physical system with a non-trivial topology.
- *Ultimate goal*: To build a "topological" quantum computer.



A *topological property* of a surface is a property that remains invariant under continuous deformations of the surface.

<u>*Example*</u>: Consider 2-dim surface of a torus.

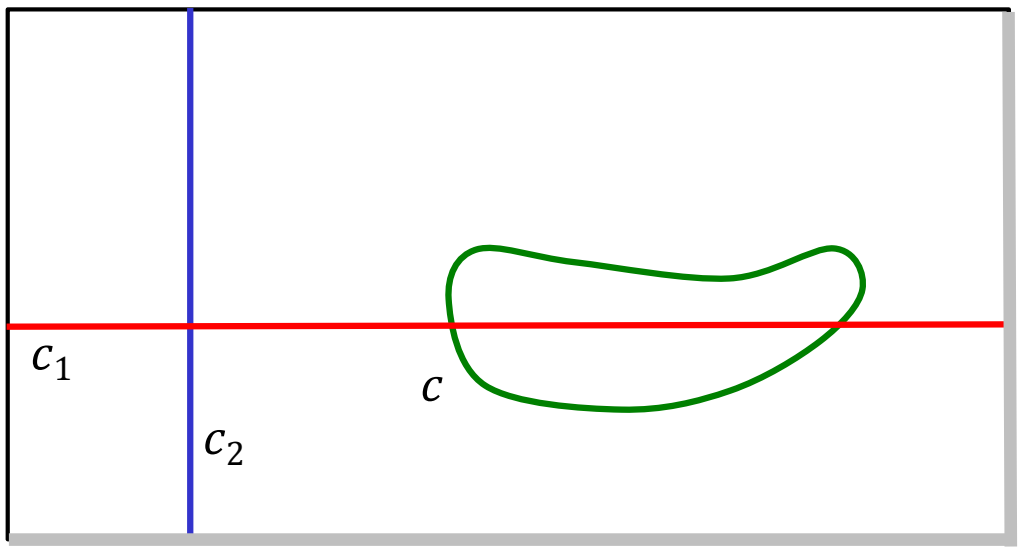


<u>*Three types of closed paths*</u>:

- Closed loops c which can be continuously deformed into a point.
- Closed loops c_1 which cannot be continuously deformed into a point.
- Closed loops c_2 which cannot be continuously deformed into a point.

- c_1 and c_2 are called "non-contractible" closed loops.
  - *Neither c, c_1, nor c_2 can be continuously deformed into the others.*
- The surface of a torus is characterized by these three families of closed loops.
  - *They describe features of the torus that are invariant under continuous deformations of its surface (i.e., they are topological properties).*

Slightly more abstract way to represent a torus: unwind it into a flat surface with periodic boundary conditions.



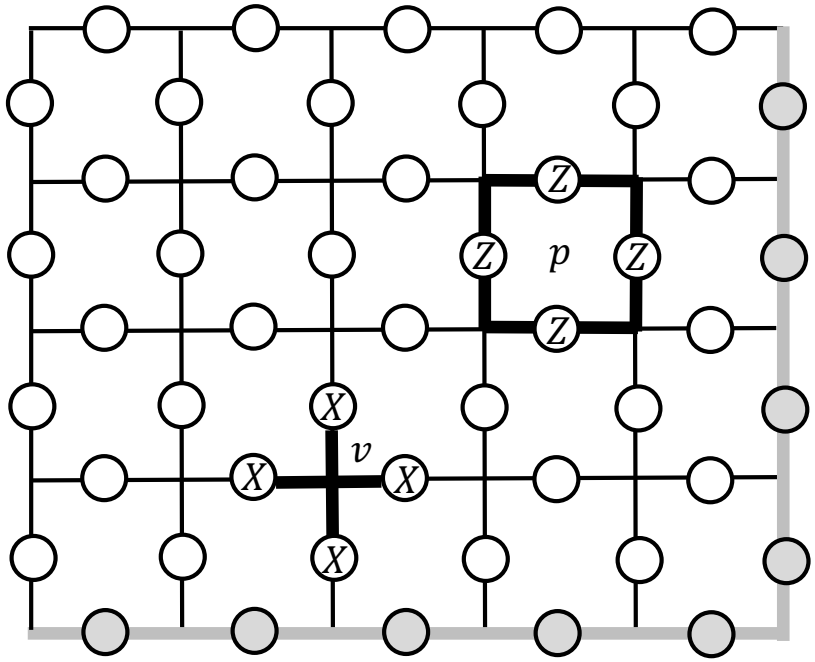
Periodic boundary conditions:

- Identify top and bottom edges.
- Identify left and right edges.

Let's add some (abstract) physics...

The Toric Code

Kitaev (2003)



- Put a square $L \times L$ lattice on the torus with L^2 vertices.
- On each lattice edge, place a qubit.

$2L^2$ -qubit Hilbert space $\mathcal{H} = V_1 \otimes \cdots \otimes V_{2L^2}$, where each V is a single-qubit Hilbert space.

- For each vertex v define a vertex operator A_v:

$$A_v = I_1 \otimes \cdots \otimes I_{(i-1)} \otimes (X_{(i)} \otimes X_{(i+1)} \otimes X_{(i+2)} \otimes X_{(i+3)}) \otimes I_{(i+4)} \otimes \cdots I_{(n)}$$

A_v acts as the identity on all qubits except those on the 4 edges leading to v. On each of these, it acts as the X operator.

- For each plaquette p define a plaquette operator B_p:

$$B_p = I_1 \otimes \cdots \otimes I_{(j-1)} \otimes (Z_{(j)} \otimes Z_{(j+1)} \otimes Z_{(j+2)} \otimes Z_{(j+3)}) \otimes I_{(j+4)} \otimes \cdots I_{(n)}$$

B_p acts as the identity on all qubits except those on the 4 edges around p. On each of these, it acts as the Z operator.

Exercise in linear algebra: Find the space $\mathcal{C}$ of eigenvectors of A_v and B_p with eigenvalue $+1$, for all vertices v and plaquettes p.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H}, \text{ such that } A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \text{ for all } v, p\}$$

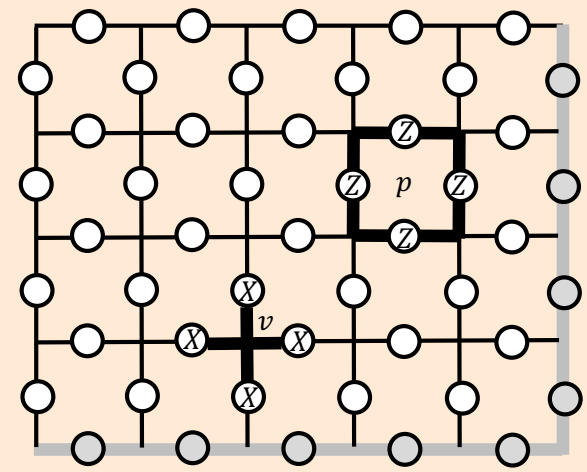
- *Result*: $\mathcal{C}$ is a 4-dim (*i.e.*, two-qubit) subspace of $\mathcal{H}$ whose elements are entangled with respect to the decomposition $\mathcal{H} = V_1 \otimes \cdots \otimes V_{2L^2}$.

Story to come:

- $\mathcal{C}$ will be our **code space**: use its two entangled qubits to encode our information.
- Given $\mathcal{C}$, we now need to identify **error** operators. These will be "local" operators that act on codewords in $\mathcal{C}$ and transform them out of $\mathcal{C}$.
- The only operators that transform codewords to other codewords, and that are not the identity, are "non-local" in the sense of being associated with the non-contractible closed loops c_1, c_2 on the torus.

Aside: Find the space $\mathcal{C}$ of eigenvectors of A_v and B_p with eigenvalue $+1$, for all vertices v and plaquettes p.

$$\mathcal{C} = \{|\xi\rangle \in \mathcal{H} : A_v|\xi\rangle = |\xi\rangle, B_p|\xi\rangle = |\xi\rangle, \forall v, p\}$$



Constraints:

(a) $B_p|\xi\rangle = |\xi\rangle$ requires that there must be an *even number* of $|1\rangle$ qubits per plaquette, since $Z|1\rangle = -|1\rangle$.

(b) $A_v|\xi\rangle = |\xi\rangle$ requires that $|\xi\rangle$ must be a superposition of an element of $\mathcal{H}$ and its single-qubit-flipped counterpart.

Claim: A vector $|\xi\rangle$ that satisfies (a) and (b) is given by:

$$|\xi\rangle = \prod_{i=1}^{L^2} 2^{-1/2}(I + A_{v_i})|0\rangle_1 \cdots |0\rangle_{2L^2}$$

entangled state!

Proof: Let $j = 1, \ldots, L^2$. Then

$$\begin{aligned} B_{p_j}|\xi\rangle &= 2^{-L^2/2} B_{p_j}(I + A_{v_1})\cdots(I + A_{v_{L^2}})|0\ldots0\rangle \\ &= 2^{-L^2/2}(I + A_{v_1})\cdots(I + A_{v_{L^2}})B_{p_j}|0\ldots0\rangle && B_{p_j} \textit{ commutes with } (I + A_{v_i}) \textit{ for all } i, j \\ &= 2^{-L^2/2}(I + A_{v_1})\cdots(I + A_{v_{L^2}})|0\ldots0\rangle && B_{p_j}|0\ldots0\rangle = |0\ldots0\rangle, \textit{for all } j \\ &= |\xi\rangle \end{aligned}$$

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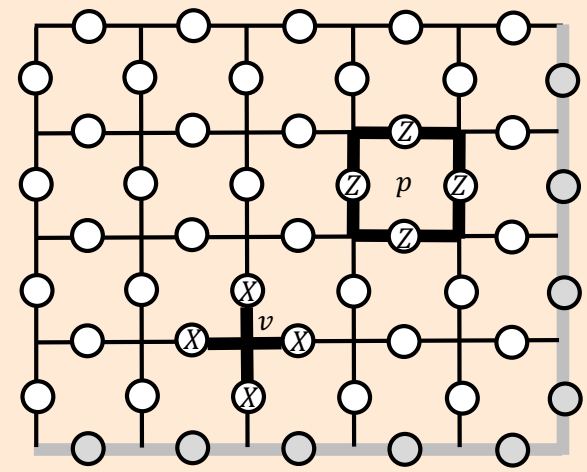

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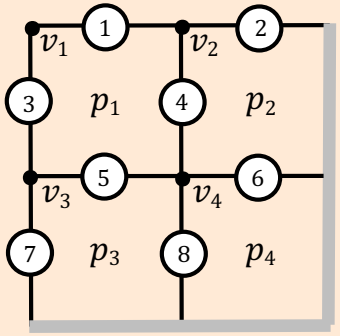
Proof: Let $j = 1, \ldots, L^2$. Then

$$\begin{aligned}
A_{v_j}|\xi\rangle &= 2^{-L^2/2} A_{v_j}(I + A_{v_1}) \cdots (I + A_{v_{L^2}})|0\ldots0\rangle \\
&= 2^{-L^2/2}(I + A_{v_1}) \cdots A_{v_j}(I + A_{v_j}) \cdots (I + A_{v_{L^2}})|0\ldots0\rangle && A_{v_j} \text{ commutes with } (I + A_{v_i}) \\
&= 2^{-L^2/2}(I + A_{v_1}) \cdots (A_{v_j} + A_{v_j}A_{v_j}) \cdots (I + A_{v_{L^2}})|0\ldots0\rangle \\
&= 2^{-L^2/2}(I + A_{v_1}) \cdots (I + A_{v_j}) \cdots (I + A_{v_{L^2}})|0\ldots0\rangle && A_{v_j}A_{v_j} = I \\
&= |\xi\rangle
\end{aligned}$$

Example: Let $L = 2$.



8 qubits (so $\mathcal{H}$ has $2^8 = 256$ dimensions!)

4 plaquettes: $p_1 = \{1, 3, 4, 5\}$ $p_2 = \{2, 3, 4, 6\}$

$p_3 = \{1, 5, 7, 8\}$ $p_4 = \{2, 6, 7, 8\}$

4 vertices: $v_1 = \{1, 2, 3, 7\}$ $v_2 = \{1, 2, 4, 8\}$

$v_3 = \{3, 5, 6, 7\}$ $v_4 = \{4, 5, 6, 8\}$

$$|\xi\rangle = \prod_{i=1}^{4} 2^{-1/2}\,(I + A_{v_i})|00000000\rangle$$

$$= \tfrac{1}{4}(I + A_{v_1})(I + A_{v_2})(I + A_{v_3})(I + A_{v_4})|00000000\rangle$$

$$= \tfrac{1}{4}(I + A_{v_1})(I + A_{v_2})(I + A_{v_3})\{|00000000\rangle + |00011101\rangle\}$$

$$= \tfrac{1}{4}(I + A_{v_1})(I + A_{v_2})\{|00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle\}$$

$$= \tfrac{1}{4}(I + A_{v_1})\{|00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle + |11010001\rangle + |11001100\rangle + |11111111\rangle + |11100010\rangle\}$$

$$= \tfrac{1}{4}\{|00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle + |11010001\rangle + |11001100\rangle + |11111111\rangle + |11100010\rangle + |11100010\rangle + |11111111\rangle + |11001100\rangle + |11010001\rangle + |00110011\rangle + |00101110\rangle + |00011101\rangle + |00000000\rangle\}$$

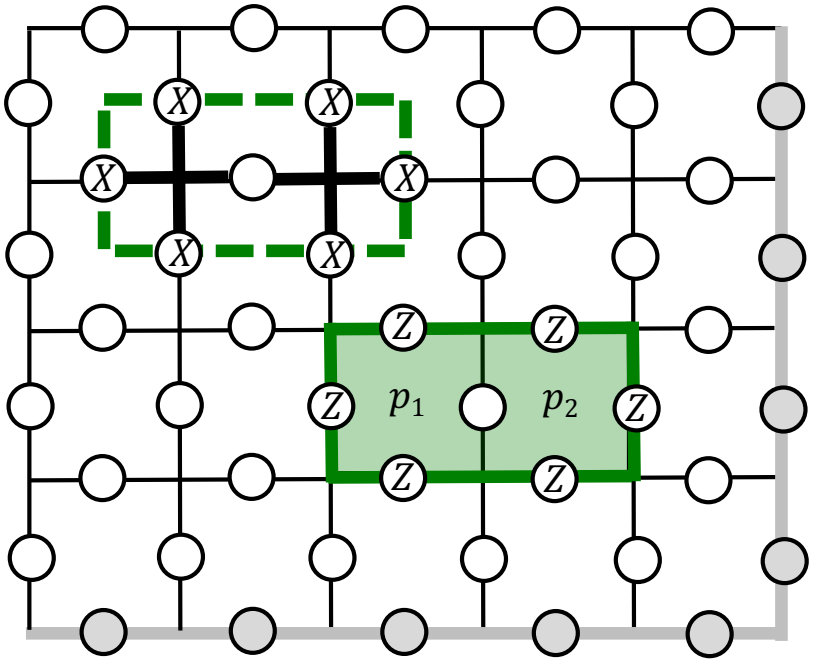
$$= \tfrac{1}{2}\{|00000000\rangle + |00011101\rangle + |00101110\rangle + |00110011\rangle + |11010001\rangle + |11001100\rangle + |11111111\rangle + |11100010\rangle\}$$

entangled state!

Three types of operators that act on $\mathcal{C}$

First type: "Stabilizer" operators.

- Composing adjacent plaquette operators B_{p_1}, B_{p_2} to form $B_{p_1}B_{p_2}$ results in a *closed loop* of Z operators:



- B_{p_1} and B_{p_2} share an edge.
- $B_{p_1}B_{p_2}$ includes the square of the Z operator of the shared edge, and $Z^2 = I$.
- *So*: The Z's that appear in $B_{p_1}B_{p_2}$ will act on the qubits that form the *boundary* of the two plaquettes!
- The same holds for any number of adjacent plaquette operators.
- The same holds for vertex operators A_v.

- *Note*: These closed loops are of type c on the torus.

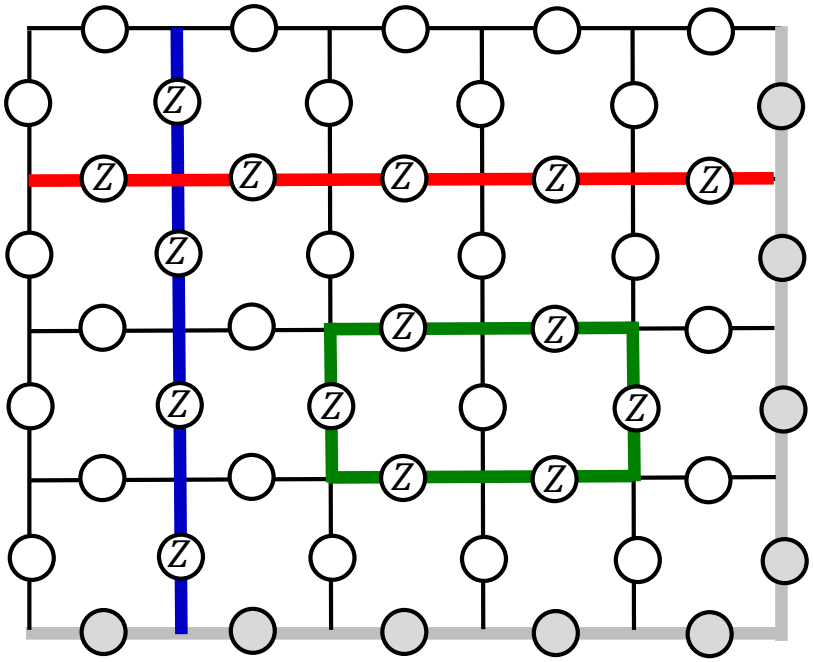
Type c closed loop operators are called "stabilizer" operators:

- *They act like the identity on $\mathcal{C}$ (since they are compositions of A_v and B_p operators).*
- *They are "local" (in the sense that they are associated with contractible closed loops).*

Three types of operators that act on $\mathcal{C}$

Second type: "Encoded logical" operators.

- There are two other types of closed loops on a torus: *non-contractible* closed loops c_1 and c_2.



- Let $\bar{Z}_1$ and $\bar{Z}_2$ refer to the two types of products of Z operators along closed loops of type c_1 and c_2.
- Let $\bar{X}_1$ and $\bar{X}_2$ refer to the two types of products of X operators along closed loops of type c_1 and c_2.

Types c_1 and c_2 closed loop operators are called "encoded logical" operators:

- *They act on codewords in $\mathcal{C}$ and transform them into other codewords (they are not the identity on $\mathcal{C}$).*
- *They are not "local" operators (in the sense that they are associated with non-contractible closed loops).*

Why do the encoded logical operators map vectors in $\mathcal{C}$ to other vectors in $\mathcal{C}$?

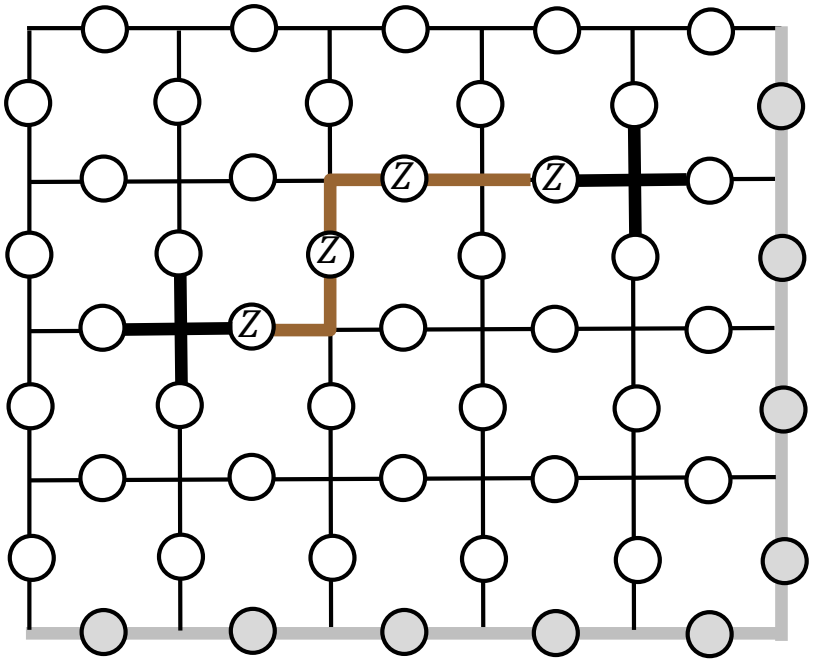
Claim 1. Any operator D that maps $\mathcal{C}$ to $\mathcal{C}$ must commute with all A_v and B_p operators.

Proof: Recall $\mathcal{C} = \{|\phi\rangle : \mathcal{O}|\phi\rangle = |\phi\rangle, \text{ for } \mathcal{O} = A_v \text{ or } B_p\}$.

- Let D be an operator such that $D|\psi\rangle \in \mathcal{C}$, for any $|\psi\rangle \in \mathcal{C}$.
- Suppose $D\mathcal{O} = -\mathcal{O}D$ (D anticommutes with $\mathcal{O}$).
- Then for any $|\psi\rangle \in \mathcal{C}$, $D|\psi\rangle = D\mathcal{O}|\psi\rangle = -\mathcal{O}D|\psi\rangle$.
- *So*: $\mathcal{O}(D|\psi\rangle) = -(D|\psi\rangle) \neq D|\psi\rangle$.
- *So*: $D|\psi\rangle \notin \mathcal{C}$ (contradiction!)
- *Hence*: D must commute with $\mathcal{O}$.

Why do the encoded logical operators map vectors in $\mathcal{C}$ to other vectors in $\mathcal{C}$?

Claim 2. Any operator formed from an open path of X's or Z's will anticommute with some A_v or B_p.



- Consider an operator formed from a product of I's on all qubits except for an open path of Z's.
- This operator commutes with all B_p's (since Z commutes with itself).
- It commutes with all A_v's, except for the two that contain the endpoint Z's.
- It anticommutes with these two A_v's (since Z anticommutes with X).

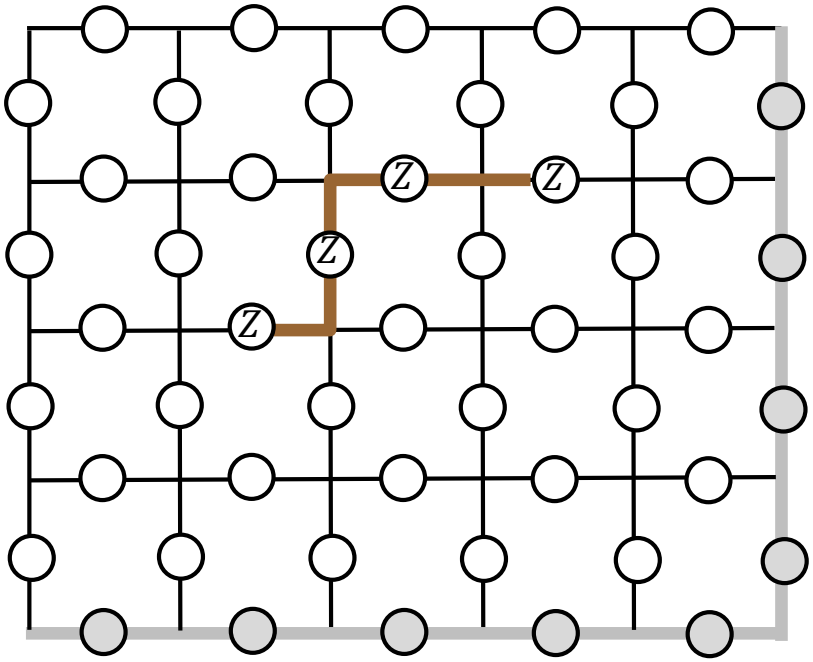
- To avoid "hanging" Z's at endpoints, form an operator from a closed loop of Z's (or X's). (Closed loop operators commute with all B_p's and A_v's.)

  - *But: A type-c closed loop operator is a stabilizer operator that acts like the identity on $\mathcal{C}$.*
  - *Solution: Form an operator from a non-contractible closed loop that has no endpoints!*

Three types of operators that act on $\mathcal{C}$

Third type: Error operators.

- By definition, error operators act on codewords and corrupt them (transform them into states not in $\mathcal{C}$).



- Error operators can't be associated with products of Z's or X's on closed loops: There are only three types, and each type transforms codewords to codewords.
- What about "open path" products of Z's or X's?

Claim: Open path products of Z's or X's transform codewords in $\mathcal{C}$ out of $\mathcal{C}$.

Proof: We've just seen that open path products of Z's or X's anticommute with some A_v or B_p, and hence transform codewords out of $\mathcal{C}$.

- "Open path" operators are "local" (in the sense that they are associated with *contractible* line segments).

Summary: Three types of operators that act on $\mathcal{C}$

1. Stabilizer operators (*local*).

$S^Z(c) = \bigotimes_{j \in c} Z_j$

$S^X(c') = \bigotimes_{j \in c'} X_j$

c, c' = contractible closed loops

2. Encoded logical operators (*non-local*).

$\bar{Z}_1 = \bigotimes_{j \in \gamma_1} Z_j \qquad \bar{Z}_2 = \bigotimes_{j \in \gamma_2} Z_j$

$\bar{X}_1 = \bigotimes_{j \in \gamma'_1} X_j \qquad \bar{X}_2 = \bigotimes_{j \in \gamma'_2} X_j$

γ_1, γ'_1 = non-contractible closed loops of type c_1

γ_2, γ'_2 = non-contractible closed loops of type c_2

3. Error operators (*local*).

$S^Z(t) = \bigotimes_{j \in t} Z_j$

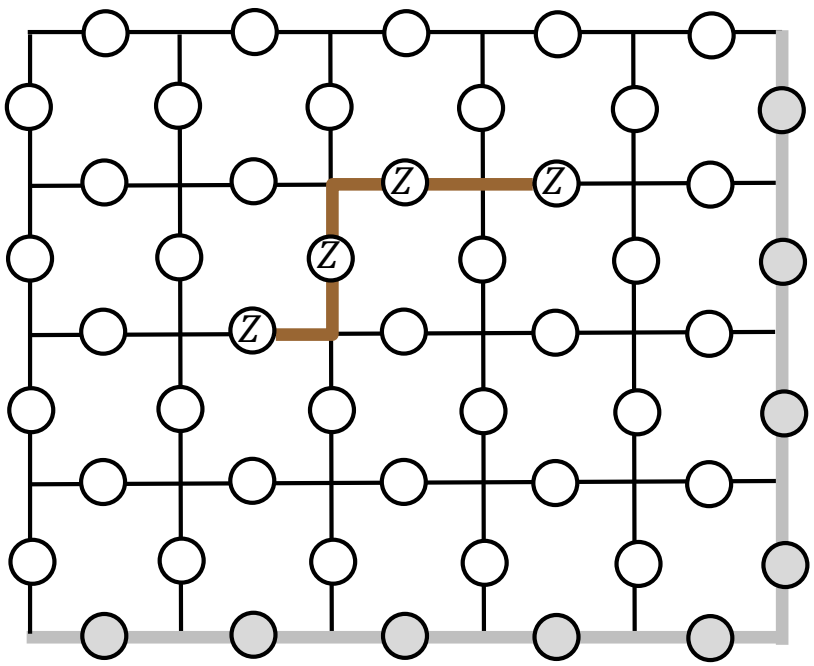
$S^X(t') = \bigotimes_{j \in t'} X_j$

t, t' = contractible open paths

Now: Check to see if the KL Condition holds for the toric code.

- Does $\mathcal{C}$ correct the error set $\mathcal{E} = \{S^Z(t), S^X(t') : \text{for all } t, t'\}$?
  - *Is it the case that* $\langle\psi_i|E_k^\dagger E_l|\psi_j\rangle = c_{kl}\delta_{ij}$, *for any* E_k, $E_l \in \mathcal{E}$, *and* ψ_i, $\psi_j \in \mathcal{C}$?

Yes!



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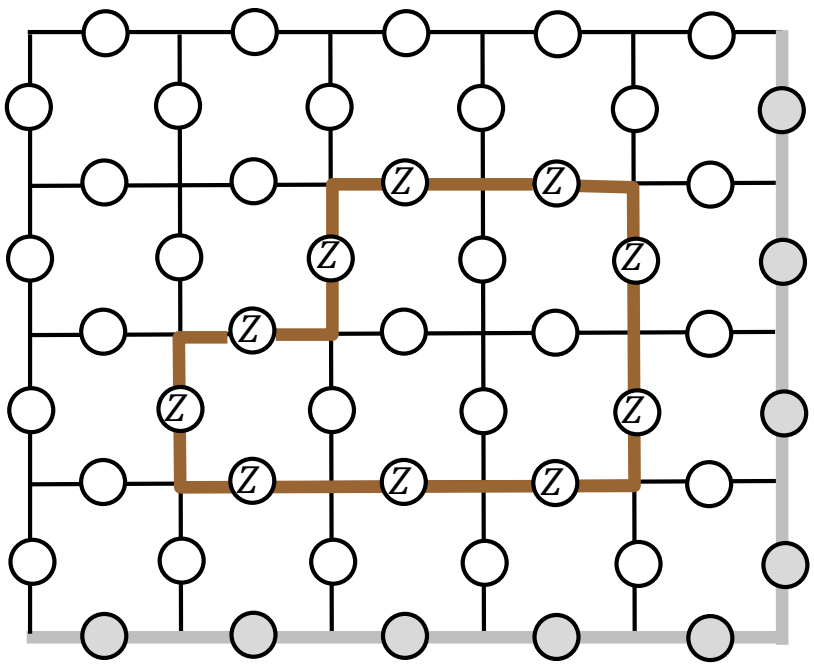

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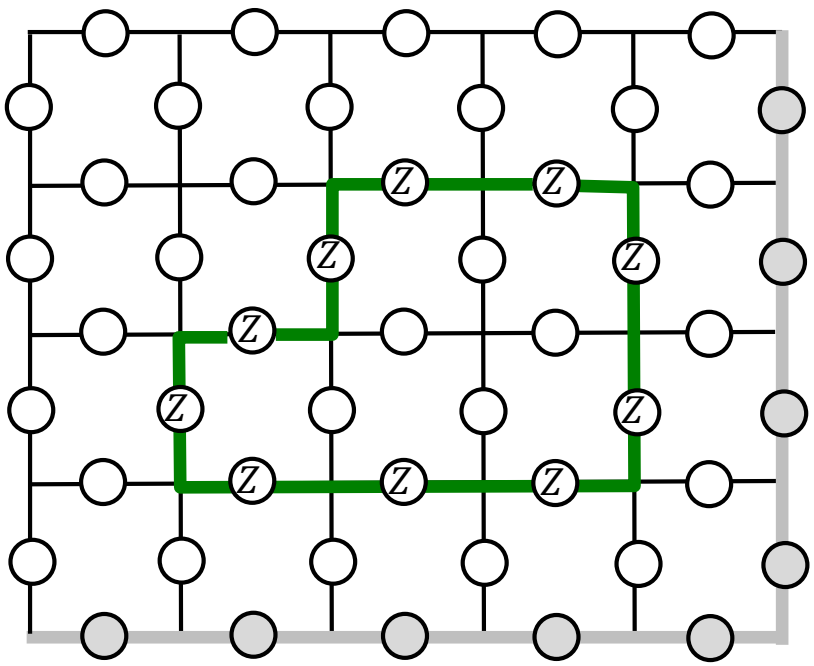

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- *And*: Stabilizer operators act as the identity on $\mathcal{C}$.

Upshot: We've encoded information "non-locally" in $\mathcal{C}$ in such a way that local errors can be detected and corrected.

Two senses of "non-locality" in the Toric Code

- *Entanglement non-locality*: The codewords (elements of $\mathcal{C}$) are entangled states.
  - *Entanglement non-locality* = *Einstein non-locality* + *Bell non-locality*

> *Recall*:
> - Einstein non-locality occurs when two systems are correlated and the correlation cannot be explained by a direct cause that travels from one system to the other.
> - Bell non-locality occurs when two systems are correlated and the correlation cannot be explained by a common cause

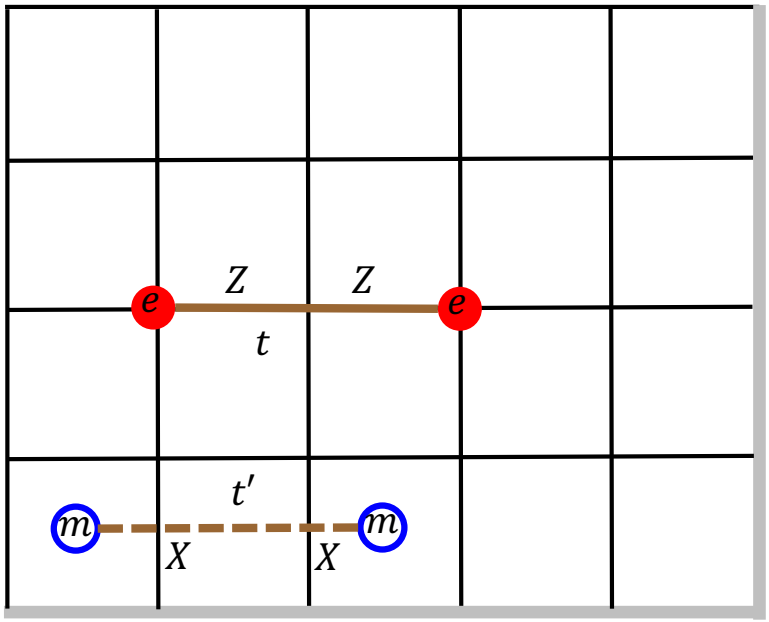
- *Topological non-locality*: The operators that act on codewords are non-contractible loop operators.

> *Suppose*: Topological non-locality occurs when a quantity is not localized to a contractible region of space.

Open Question: Under what conditions does entanglement non-locality entail topological non-locality and/or vice-versa?

Let's add some (slightly more concrete) physics...

- Interpret the code space $\mathcal{C}$ as the space of ground-states $|q\rangle$ (states of lowest energy) of a physical system.



- Interpret a Z (or X) error operator as acting on a ground-state to produce a pair of "e" (or "m") "quasiparticle" excitations at the ends of the open path.
- What happens when we move an m around an e?
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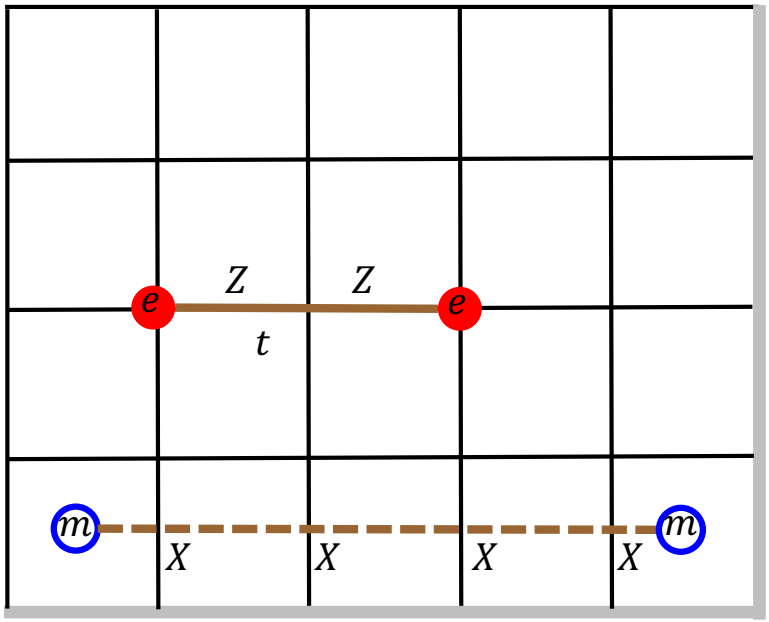

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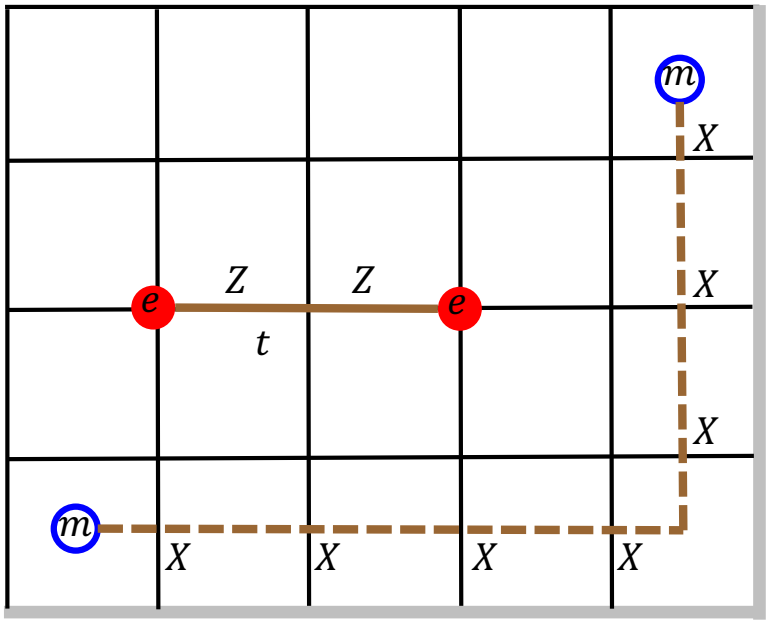

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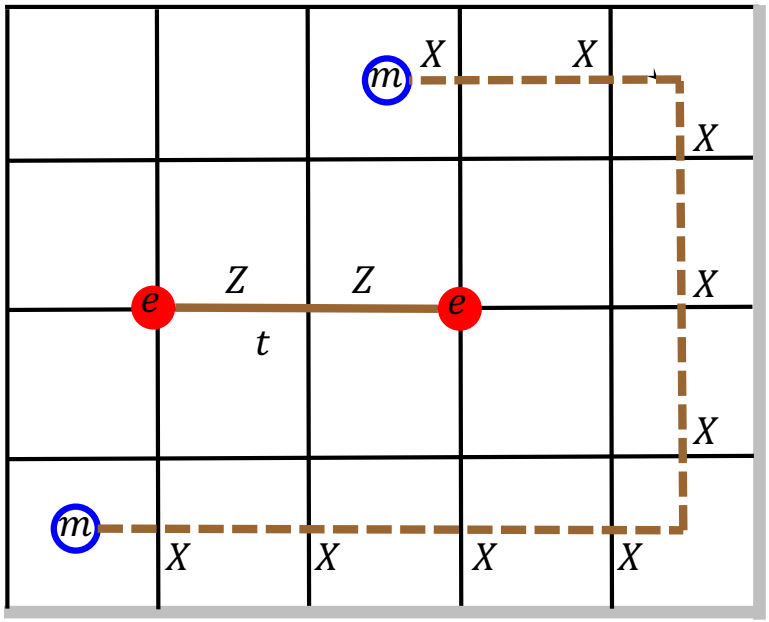

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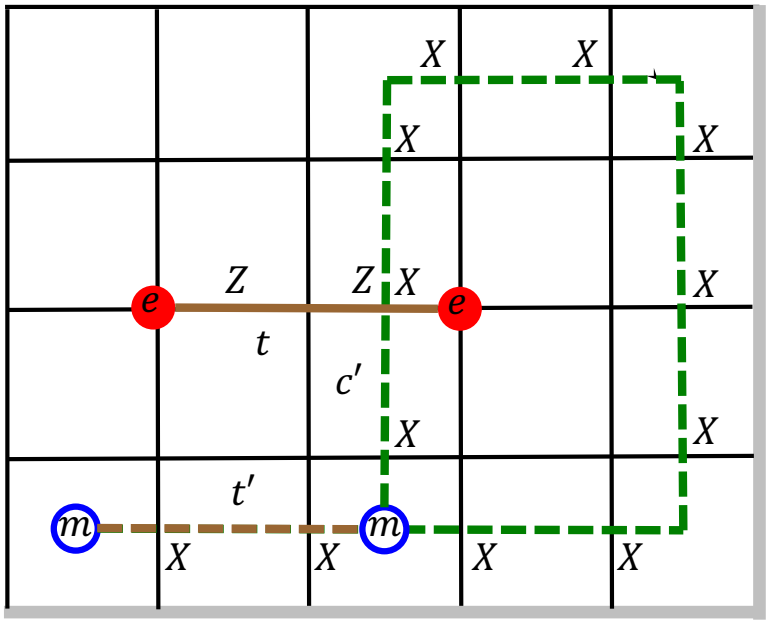

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- $|\Psi_{final}\rangle = S^X(c')S^Z(t)S^X(t')|q\rangle$

$$= -S^Z(t)S^X(c')S^X(t')|q\rangle \qquad S^Z(t) \text{ and } S^X(c') \text{ anticommute}$$

$$= -S^Z(t)S^X(t')S^X(c')|q\rangle \qquad S^X(c') \text{ and } S^X(t') \text{ commute}$$

$$= -|\Psi_{initial}\rangle \qquad S^X(c') \text{ acts like the identity on } \mathcal{C}$$

- <u>*So*</u>: Moving an m quasiparticle completely around an e quasiparticle changes the phase of the initial 4-particle state by -1.

In general: When two particles are exchanged in a multiparticle system, the multiparticle state $|\Psi\rangle$ picks up a phase $|\Psi\rangle \to e^{i\theta}|\Psi\rangle$.

- Taking one particle around another is equivalent to two exchanges; so $|\Psi\rangle \to e^{2i\theta}|\Psi\rangle$.

- *So*: Taking an m quasiparticle around an e quasiparticle produces the phase $e^{2i\theta} = -1$, or $\theta = \pi/2$.

$e^{2i\theta} = \cos 2\theta + i\sin 2\theta$

- *So*: One exchange of an m quasiparticle and an e quasiparticle produces the phase $|\Psi\rangle \to e^{i\pi/2}|\Psi\rangle$.

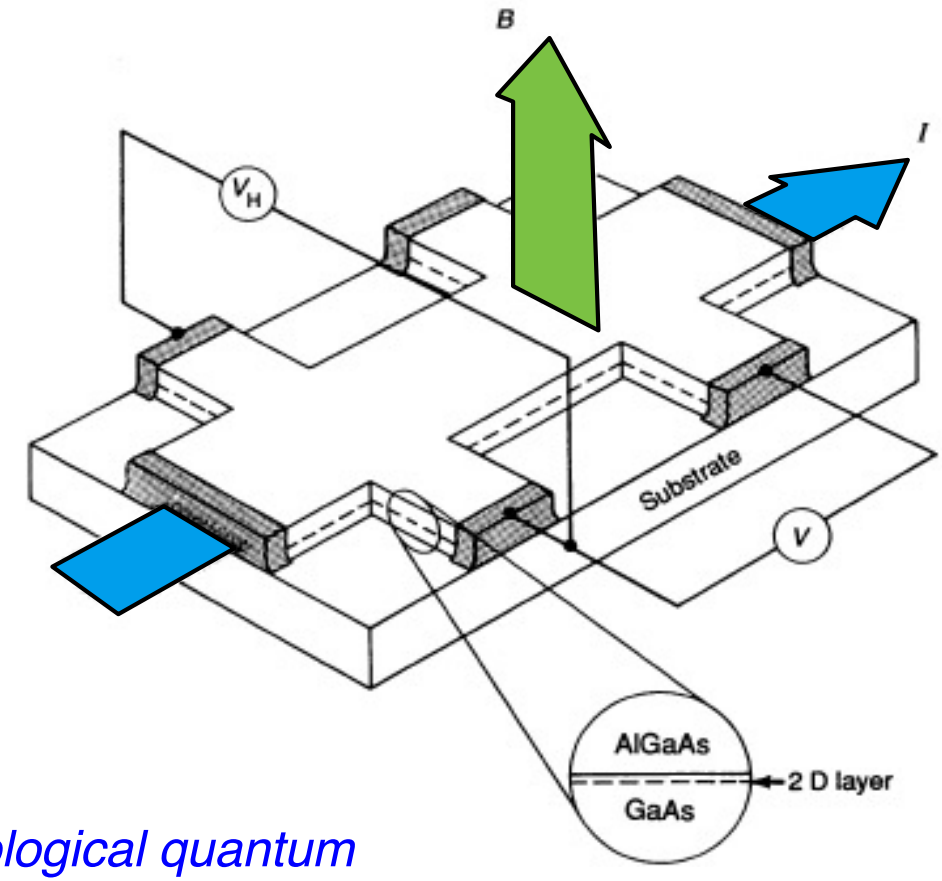
Bosons:	Particle exchange phase $\theta = 0$.
Fermions:	Particle exchange phase $\theta = \pi$.
Anyons:	Particle exchange phase $\theta \in (0, \pi)$.

Upshot: m and e quasiparticles are anyons!
(They obey "fractional statistics".)

Physical significance: There are physical systems that exhibit characteristics of the toric code!

- Fractional quantum Hall system:

  - 2-dim conductor in external magnetic field B.
  - At low temps, longitudinal resistance vanishes, and transverse (Hall) resistance becomes quantized.
  - *Prediction*: Low-energy anyonic excitations.



Open Question: Can we build a topological quantum computer out of a fractional quantum Hall system?