05. Multiparticle Systems

1. 2-Particle Product Spaces

- *Suppose*: Particle$_1$ and particle$_2$ are represented by vector spaces V and W.
- *Then*: The composite 2-particle system is represented by a *product vector space* $V \otimes W$.

Let V and W be n-dim and m-dim vector spaces. The **product vector space** $V \otimes W$ is an $(n \times m)$-dim vector space with the following property:

For any $|v\rangle \in V$, $|w\rangle \in W$, one can form a vector $|\psi\rangle \in V \otimes W$ *via* the "tensor product" $|v\rangle \otimes |w\rangle$, which satisfies:

(i) $(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$

(ii) $|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$

(iii) $\alpha(|v\rangle \otimes |w\rangle) = \alpha|v\rangle \otimes |w\rangle = |v\rangle \otimes \alpha|w\rangle$, for any scalar α.

Instead of "$|v\rangle \otimes |w\rangle$", we can alternatively write "$|v\rangle|w\rangle$" or "$|vw\rangle$".

Further characteristics of the 2-particle product space $V \otimes W$

$V \otimes W$ inherits an inner-product, bases, and operators from V and W:

1. An **inner-product** on $V \otimes W$ is defined by the following: For any $|\psi\rangle = |vw\rangle$, $|\phi\rangle = |tu\rangle \in V \otimes W$, with $|v\rangle, |t\rangle \in V$ and $|w\rangle, |u\rangle \in W$,

   $$\langle\psi|\phi\rangle \equiv \langle v|t\rangle\langle w|u\rangle$$

2. If $\{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\}$ and $\{|w_1\rangle, |w_2\rangle, \ldots, |w_m\rangle\}$ are bases for V and W, then a **basis** for $V \otimes W$ is given by

   $$\{|v_1w_1\rangle, |v_1w_2\rangle, \ldots, |v_1w_m\rangle, |v_2w_1\rangle, \ldots, |v_nw_m\rangle\}$$

3. Any vector $|\psi\rangle$ in $V \otimes W$ can be *expanded* in this basis:

   $$|\psi\rangle = a_{11}|v_1w_1\rangle + a_{12}|v_1w_2\rangle + \cdots + a_{21}|v_2w_1\rangle + \cdots + a_{nm}|v_nw_m\rangle$$

4. Let A and B be operators on V and W such that

   $$A|v\rangle = a|v\rangle, B|w\rangle = b|w\rangle, \quad \text{where } |v\rangle \in V, |w\rangle \in W.$$

   Then there is an **operator** $A \otimes B$ on $V \otimes W$ such that

   $$(A \otimes B)|vw\rangle = ab|vw\rangle$$

Extension to multiparticle (multi-partite) systems

- A product vector space $\mathcal{H}$ may be formed from the tensor product of more than two lower-dim vector spaces.
- A product vector space $\mathcal{H}$ may admit more than one decomposition into lower-dim vector spaces.

Ex. Let $\mathcal{H}$ be a 16-dim vector space.

Then: One can always find 2-dim vector spaces V_1, V_2, V_3, V_4 such that

$$\mathcal{H} = V_1 \otimes V_2 \otimes V_3 \otimes V_4$$

And: One can always find 4-dim vector spaces W_1, W_2 such that

$$\mathcal{H} = W_1 \otimes W_2$$

- *Note*: A "factor" vector space must have dim > 1.
- *So*: As long as the dimension n of a vector space isn't a *prime number*, it will admit at least one decomposition into the tensor product of lower-dim vector spaces.
- *And*: How many it will admit depends on the *prime factorization* of n.

Two-particle example

- *Let*: V, W be the 2-dim spin state spaces for two electrons.
  - *Combined 2-particle spin space is given by 4-dim $V \otimes W$.*

- *Suppose*:

  $\{|hard\rangle_1, |soft\rangle_1\}$ is a basis for V.

  $\{|hard\rangle_2, |soft\rangle_2\}$ is a basis for W.

- *Then*: A basis for $V \otimes W$ is given by

  $\{|hard\rangle_1|hard\rangle_2, |hard\rangle_1|soft\rangle_2, |soft\rangle_1|hard\rangle_2, |soft\rangle_1|soft\rangle_2\}$

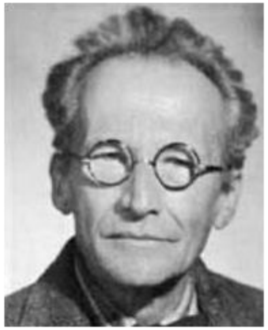
- *And*: Any 2-particle state $|A\rangle$ in $V \otimes W$ can be expanded in this basis:

$$|A\rangle = a_{11}|hard\rangle_1|hard\rangle_2 + a_{12}|hard\rangle_1|soft\rangle_2$$
$$+ a_{21}|soft\rangle_1|hard\rangle_2 + a_{22}|soft\rangle_1|soft\rangle_2$$

2. Entangled States

An **entangled state** in a product vector space $\mathcal{H}$ with respect to a decomposition $\mathcal{H} = V_1 \otimes \cdots \otimes V_n$ is a vector $|\psi\rangle$ that *cannot* be written as a product of n terms,

$$|\psi\rangle = |v_1\rangle \otimes \cdots \otimes |v_n\rangle, \qquad \text{where } |v_i\rangle \in V_i$$



Erwin Schrödinger
(1887-1961)

- *What this means*: An entangled n-particle state cannot be written as a product of a particle_1 state, and a particle_2 state, and a particle_3 state, *etc*.
  - *In an entangled n-particle state, the states of all n particles are "entangled with each other": they cannot be separated out.*

Two initial observations

1. Nothing about this mathematical definition tells us what the notion of "entangled with each other" means physically.
2. Entanglement is a *relative* property!
   - *A vector in $\mathcal{H}$ can be entangled with respect to one decomposition of $\mathcal{H}$, but not entangled with respect to another decomposition of $\mathcal{H}$.*

Examples:

- *Entangled*: $|\Psi^{+}\rangle = \sqrt{1/2}\,\{|hard\rangle_1|hard\rangle_2 + |soft\rangle_1|soft\rangle_2\}$

- *Nonentangled (Separable)*:

$$\begin{aligned} |A\rangle &= \sqrt{1/4}\{|hard\rangle_1|hard\rangle_2 + |hard\rangle_1|soft\rangle_2 + |soft\rangle_1|hard\rangle_2 + |soft\rangle_1|soft\rangle_2\} \\ &= \sqrt{1/4}\{|hard\rangle_1 + |soft\rangle_1\}\{|hard\rangle_2 + |soft\rangle_2\} \end{aligned}$$

$$|B\rangle = \sqrt{1/2}\,\{|hard\rangle_1|hard\rangle_2 + |soft\rangle_1|hard\rangle_2\} = \sqrt{1/2}\,\{|hard\rangle_1 + |soft\rangle_1\}|hard\rangle_2$$

$$|C\rangle = |hard\rangle_1|hard\rangle_2$$

According to the Eigenvalue-Eigenvector Rule:

- In states $|\Psi^{+}\rangle$ and $|A\rangle$, both electrons have no determinate Hardness value, but the combined system *as a whole* does have a determinate value of some other property.*
- In state $|B\rangle$, electron_1 has no determinate Hardness value, but electron_2 *does* (*i.e.*, *hard*), and the combined system as a whole has a determinate value of some other property.*
- In state $|C\rangle$, both electrons have determinate Hardness values, and the combined system as a whole has a determinate value of some other property.*

*<u>*General fact*</u>: Any vector is the eigenvector of some operator.

- *So:* According to the EE Rule, in any 2-particle state, either particle may or may not have well-defined properties.

- *But:* According to EE, *the combined 2-particle system as a whole will always have well-defined properties!*

- *Why?* Because, again, any vector in a vector space (including $V \otimes W$) is an eigenvector of some (Hermitian) operator on that space.
  - So there exist 2-particle operators with eigenvectors $|\Psi^+\rangle$, $|A\rangle$, $|B\rangle$ and $|C\rangle$ that represent properties of the 2-particle system *as a whole*.

2-Particle "Holistic" Properties

Suppose: $|Q\rangle = \sqrt{½}\{|5\rangle_1|7\rangle_2 + |9\rangle_1|11\rangle_2\}$

$|5\rangle_1$ *is an eigenvector of the position operator* $X^{(1)}$

$|11\rangle_2$ *is an eigenvector of the position operator* $X^{(2)}$

- If we only want to measure P1's position (and not P2's), we must use the 2-particle operator $X^{(1)} \otimes I^{(2)}$, where $I^{(2)}$ = identity operator on W.
- If we only want to measure P2's position (and not P1's), we must use the 2-particle operator $I^{(1)} \otimes X^{(2)}$, where $I^{(1)}$ = identity operator on V.
- The *difference in the positions* of P1 and P2 is a property of the 2-particle system represented by the 2-particle operator $(I^{(1)} \otimes X^{(2)}) - (X^{(1)} \otimes I^{(2)})$.

> *Claim*: $|Q\rangle$ is an eigenstate of $(I^{(1)} \otimes X^{(2)}) - (X^{(1)} \otimes I^{(2)})$ but *not* of $X^{(1)} \otimes I^{(2)}$ or $I^{(1)} \otimes X^{(2)}$!

- *So*: According to the EE Rule, P1 and P2 have no definite position in the 2-particle state $|Q\rangle$, but the difference in their positions *is* a definite property of the 2-particle state as a whole!

Claim: $|Q\rangle$ is an eigenstate of $(I^{(1)} \otimes X^{(2)}) - (X^{(1)} \otimes I^{(2)})$ but *not* of $X^{(1)} \otimes I^{(2)}$ or $I^{(1)} \otimes X^{(2)}$!

Check:

(a) $|Q\rangle$ is an eigenstate of $(I^{(1)} \otimes X^{(2)}) - (X^{(1)} \otimes I^{(2)})$:

$$\{(I^{(1)} \otimes X^{(2)}) - (X^{(1)} \otimes I^{(2)})\}|Q\rangle = (I^{(1)} \otimes X^{(2)})|Q\rangle - (X^{(1)} \otimes I^{(2)})|Q\rangle$$

$$= (I^{(1)} \otimes X^{(2)})\sqrt{½}\{|5\rangle_1|7\rangle_2 + |9\rangle_1|11\rangle_2\} - (X^{(1)} \otimes I^{(2)})\sqrt{½}\{|5\rangle_1|7\rangle_2 + |9\rangle_1|11\rangle_2\}$$

$$= \sqrt{½}\{7|5\rangle_1|7\rangle_2 + 11|9\rangle_1|11\rangle_2\} - \sqrt{½}\{5|5\rangle_1|7\rangle_2 + 9|9\rangle_1|11\rangle_2\}$$

$$= \sqrt{½}\{2|5\rangle_1|7\rangle_2 + 2|9\rangle_1|11\rangle_2\} = 2|Q\rangle$$

In the state represented by $|Q\rangle$, the value of the difference-in-position operator is 2; i.e., P1 *and* P2 *differ in position by 2.*

(b) $|Q\rangle$ is not an eigenstate of $X^{(1)} \otimes I^{(2)}$ or $I^{(1)} \otimes X^{(2)}$:

$$X^{(1)} \otimes I^{(2)}|Q\rangle = (X^{(1)} \otimes I^{(2)})\sqrt{½}\{|5\rangle_1|7\rangle_2 + |9\rangle_1|11\rangle_2\}$$

$$= \sqrt{½}\{5|5\rangle_1|7\rangle_2 + 9|9\rangle_1|11\rangle_2\}$$

$$\neq \lambda|Q\rangle, \quad \text{for any value of } \lambda.$$

- Similarly for $I^{(1)} \otimes X^{(2)}$.

3. Born Rule for 2-Particle States

1. Suppose a 2-particle system is in a state represented by $|k\rangle$, and suppose we measure properties of *both* P1 and P2 represented by operators $A^{(1)}$ and $B^{(2)}$.

   Then: The probability that the value of $A^{(1)}$ is a_i and the value of $B^{(2)}$ is b_i is:

   $$\boxed{\Pr(\textit{value of } A^{(1)} \textit{ is } a_i \textit{ and value of } B^{(2)} \textit{ is } b_i \textit{ in state } |k\rangle) \equiv |\langle a_i b_i | k\rangle|^2}$$

   where $|a_i b_i\rangle$ *is an eigenvector of the 2-particle operator* $A^{(1)} \otimes B^{(2)}$

2. Suppose a 2-particle system is in a state represented by $|k\rangle$, and *only* the property of P1, represented by $A^{(1)}$, is measured.

   Then: The probability that the value of $A^{(1)}$ is a_i is:

   $$\boxed{\Pr(\textit{value of } A^{(1)} \textit{ is } a_i \textit{ in state } |k\rangle) \equiv |\langle a_i \ell_1 | k\rangle|^2 + \cdots + |\langle a_i \ell_N | k\rangle|^2}$$

   where $|a_i \ell_j\rangle$, $j = 1, \ldots, N$, *are eigenvectors of the 2-particle operator* $A^{(1)} \otimes L^{(2)}$, *for any* P2 *property represented by* $L^{(2)}$

> *Motivation* (*Law of Total Probability*): The probability that the value of $A^{(1)}$ is a_i is equal to the sum of the probabilities of *all* the different ways in which the value of $A^{(1)}$ *could* be a_i.

4. 2-Particle Projection Postulate

- *Suppose*: A 2-particle system is in a state represented by $|D\rangle$, and a property of P1 represented by $A^{(1)}$ is measured with the resulting value a_i.

- *Then*: $|D\rangle$ collapses to the state given by the following:

  (a) Expand $|D\rangle$ in eigenvectors of the 2-particle operator $A^{(1)} \otimes L^{(2)}$, for *any* arbitrary operator $L^{(2)}$:

$$|D\rangle = d_{11}|a_1 \ell_1\rangle + \cdots + d_{1N}|a_1 \ell_N\rangle + d_{21}|a_2 \ell_1\rangle + \cdots + d_{NN}|a_N \ell_N\rangle$$

  (b) Throw out all terms other than ones with a_i. Then divide by an appropriate normalization term Λ to make sure the result is a vector with unit length:

$$|D\rangle \xrightarrow[\textit{collapse}]{} \frac{d_{i1}|a_i \ell_1\rangle + d_{i2}|a_i \ell_2\rangle + \cdots}{\Lambda}$$

Example 1 (collapse of separable state)

- *Suppose*: $|D\rangle = |q_3\rangle|m_4\rangle$ is an eigenvector of $Q^{(1)} \otimes M^{(2)}$.
- *Now*: Suppose the property represented by $A^{(1)}$ is measured with the resulting value a_5.
- What happens to $|D\rangle$?

- *First*: Expand $|D\rangle$ in the eigenvectors of $A^{(1)} \otimes L^{(2)}$, for *any* arbitrary $L^{(2)}$.
- *Note*: The P2 part of $|D\rangle$ already is an eigenvector of $M^{(2)}$.
- *So*: Use eigenvectors of $A^{(1)} \otimes M^{(2)}$ for simplicity.

$$\begin{aligned} |D\rangle &= |q_3 m_4\rangle \\ &= d_1|a_1 m_4\rangle + d_2|a_2 m_4\rangle + \cdots + d_N|a_N m_4\rangle \end{aligned}$$

- *Next*: Throw out all terms other than ones with a_5, and normalize the result.
- This just leaves $d_5|a_5 m_4\rangle$.
- To normalize it, divide by its length, which is just d_5.

- *So*: $\boxed{|D\rangle \xrightarrow[\text{collapse}]{} |a_5\rangle|m_4\rangle}$

No change to state of P2.

Example 2 (collapse of entangled state)

- *Suppose*: $|D\rangle = \sqrt{½}\,\{|a_4\ell_7\rangle + |a_5\ell_{24}\rangle\}$
- *Now*: Suppose the property represented by $A^{(1)}$ is measured with the resulting value a_5.
- What happens to $|D\rangle$?

Note: The P1 part of $|D\rangle$ is *already* in an eigenvector basis of $A^{(1)}$.

So: Simply throw out all terms in $|D\rangle$ that don't contain a_5.

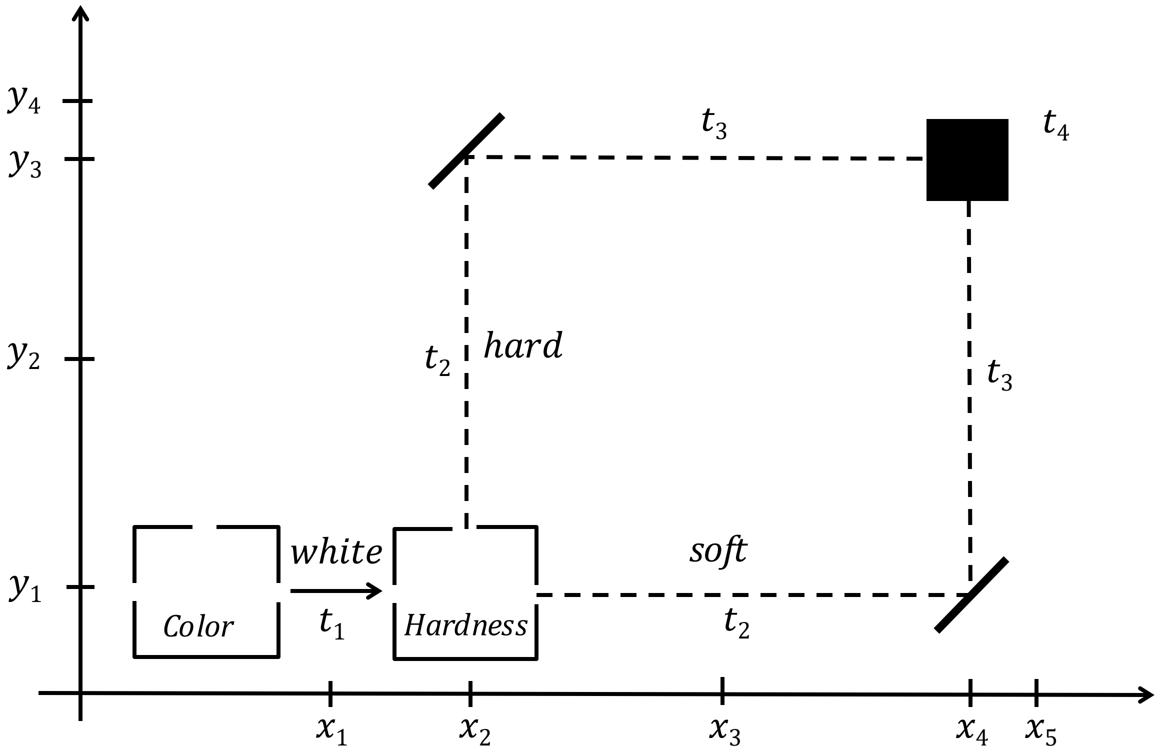
Result: $|D\rangle = |a_5\ell_{24}\rangle$

- *So*:

$$|D\rangle \xrightarrow[\textit{collapse}]{} |a_5\rangle|\ell_{24}\rangle$$

The state of the unmeasured P2 changes!

5. 2-Path Experiment Again



Without Barrier:
100% of exiting electrons are white.

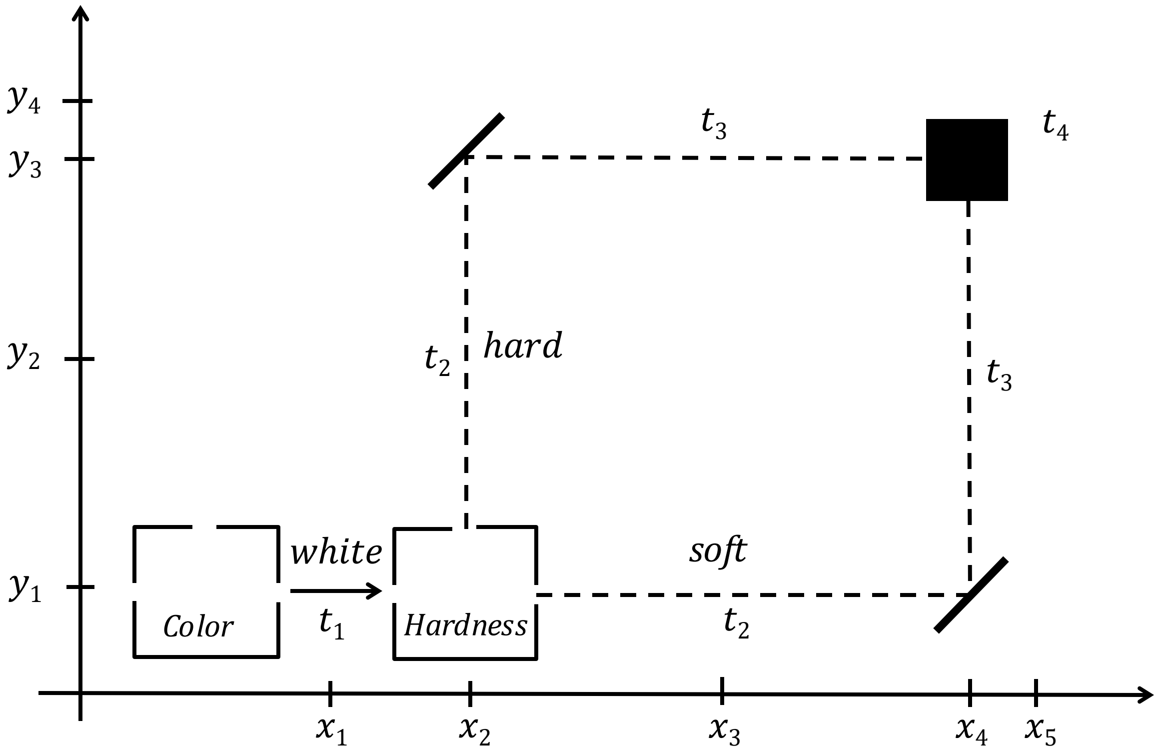
- At t_1, the electron's state is: $|white\rangle|x_1, y_1\rangle = \sqrt{½}\{|hard\rangle|x_1, y_1\rangle - |soft\rangle|x_1, y_1\rangle\}$

Definite color state

Definite position state

- One particle with two properties in a "two-property" state.
- Represented by product vector in a product vector space.
- Just like a "2-particle" product vector for two particles, each with a single property.

5. 2-Path Experiment Again



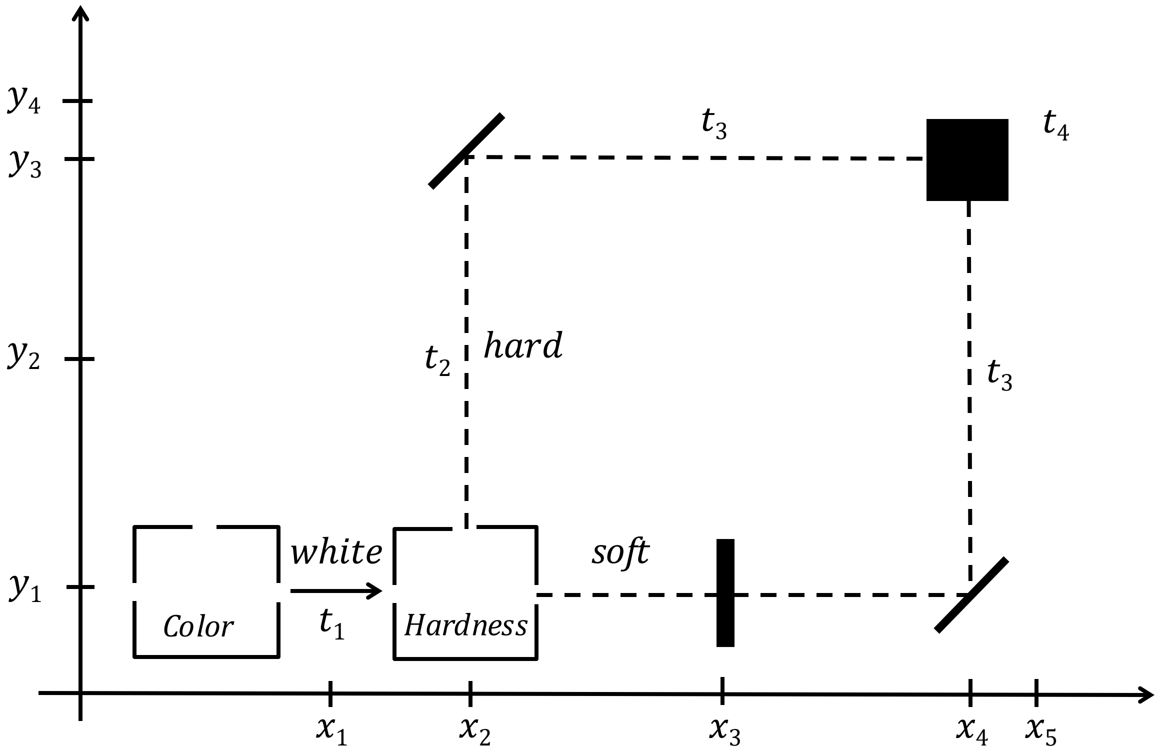
Without Barrier:
100% of exiting electrons are white.

- At t_1, the electron's state is: $|white\rangle|x_1, y_1\rangle = \sqrt{1/2}\{|hard\rangle|x_1, y_1\rangle - |soft\rangle|x_1, y_1\rangle\}$
- At t_2, the electron's state is: $\sqrt{1/2}\{|hard\rangle|x_2, y_2\rangle - |soft\rangle|x_3, y_1\rangle\}$

entangled states!

- At t_3, the electron's state is: $\sqrt{1/2}\{|hard\rangle|x_3, y_3\rangle - |soft\rangle|x_4, y_2\rangle\}$
- At t_4, the electron's state is: $\sqrt{1/2}\{|hard\rangle|x_5, y_4\rangle - |soft\rangle|x_5, y_4\rangle\} = |white\rangle|x_5, y_4\rangle$
- $\Pr(\textit{value of C is white in state at } t_4) = |\langle white, x_5, y_4|white, x_5, y_4\rangle|^2 \overset{\checkmark}{=} 1$

5. 2-Path Experiment Again



With Barrier:
50% of exiting electrons are white, 50% are black.

- At t_4, the electron's state is: $|k\rangle = \sqrt{½}\,\{|hard\rangle|x_5, y_4\rangle - |soft\rangle|x_3, y_1\rangle\}$
- To measure Color at t_4, expand $|k\rangle$ in Color basis:

$$|k\rangle = \left(\sqrt{½}\right)\left(\sqrt{½}\right)\{|black\rangle + |white\rangle\}|x_5, y_4\rangle - \left(\sqrt{½}\right)\left(\sqrt{½}\right)\{|black\rangle - |white\rangle\}|x_3, y_1\rangle$$

$$= ½|black\rangle|x_5, y_4\rangle - ½|black\rangle|x_3, y_1\rangle + ½|white\rangle|x_5, y_4\rangle + ½|white\rangle\}|x_3, y_1\rangle$$

- $\Pr(\textit{value of C is white in state } |k\rangle) = |\langle white, x_5, y_4|k\rangle|^2 + |\langle white, x_3, y_1|k\rangle|^2$

$$= |½|^2 + |½|^2 \stackrel{\checkmark}{=} ½$$