04. Five Principles of Quantum Mechanics

1. States are represented by vectors of length 1.

- The state space of a *physical system* is represented by a *linear vector space* (the space of all its possible states).

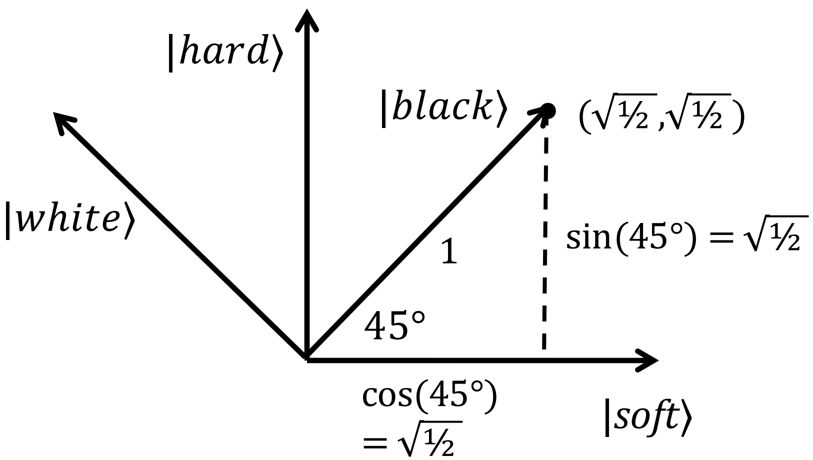
2. Properties are represented by operators.

- An operator O represents a *property*.
- Its eigenvectors $|\lambda\rangle$ represent the *value states* ("eigenstates") associated with the property.
- Its eigenvalues λ represent the (numerical) *values* of the property.

> *Eigenvector/Eigenvalue (EE) Rule*:
>
> A quantum system possesses the value λ of a property represented by operator O *if and only if* it is in a state represented by an eigenvector of O with eigenvalue λ.

Why is this helpful?

- *Recall:* Black electrons appear to have no determinate value of Hardness.
- Let's represent the *value states* of Hardness and Color as *orthonormal basis vectors*.
- Let's suppose the Hardness basis $\{|hard\rangle, |soft\rangle\}$ is rotated by 45° with respect to the Color basis $\{|white\rangle, |black\rangle\}$:



Then: $|black\rangle = \sqrt{½}\,|hard\rangle + \sqrt{½}\,|soft\rangle$

A black vector state of an electron...

... is in a "superposition" of hard and soft vector states.

> *So*: Since an electron in the vector state $|black\rangle$ cannot be in either of the vector states $|hard\rangle$, $|soft\rangle$, the EE Rule says it cannot be said to possess a value of Hardness.

Let's be a bit more precise...

- Represent the Hardness basis vectors by column vectors:

$$|hard\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |soft\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Define the Hardness operator by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- *Then*: $|hard\rangle$ and $|soft\rangle$ are eigenvectors of H:

$$H|hard\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +1|hard\rangle$$

$$H|soft\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1|soft\rangle$$

Orthonormality check:

$$\langle hard|soft\rangle = (1{,}0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\langle hard|hard\rangle = (1{,}0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$\langle soft|soft\rangle = (0{,}1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

Stipulate: $+1$ is the number corresponding to the Hardness value *hard*.
-1 is the number corresponding to the Hardness value *soft*.

Thus: An electron in the vector state $|hard\rangle$ has a Hardness value of *hard*.
An electron in the vector state $|soft\rangle$ has a Hardness value of *soft*.

- Represent the Color basis vectors by column vectors:

$$|black\rangle = \begin{pmatrix} \sqrt{½} \\ \sqrt{½} \end{pmatrix} \quad |white\rangle = \begin{pmatrix} \sqrt{½} \\ -\sqrt{½} \end{pmatrix}$$

Orthonormality check:

$$\langle black|white\rangle = \left(\sqrt{½}, \sqrt{½}\right) \begin{pmatrix} \sqrt{½} \\ -\sqrt{½} \end{pmatrix} = 0$$

$$\langle black|black\rangle = \left(\sqrt{½}, \sqrt{½}\right) \begin{pmatrix} \sqrt{½} \\ \sqrt{½} \end{pmatrix} = 1$$

$$\langle white|white\rangle = \left(\sqrt{½}, -\sqrt{½}\right) \begin{pmatrix} \sqrt{½} \\ -\sqrt{½} \end{pmatrix} = 1$$

Check: Angle between $|black\rangle$ and $|soft\rangle$ is 45°:

$$\langle black|soft\rangle = \left(\sqrt{½}, \sqrt{½}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \sqrt{½} = 1 \times 1 \times \cos 45°$$

- Define the Color operator by $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- *Then*: $|black\rangle$ and $|white\rangle$ are eigenvectors of C:

$$C|black\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{½} \\ \sqrt{½} \end{pmatrix} = \begin{pmatrix} \sqrt{½} \\ \sqrt{½} \end{pmatrix} = +1|black\rangle$$

$$C|white\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{½} \\ -\sqrt{½} \end{pmatrix} = -\begin{pmatrix} \sqrt{½} \\ -\sqrt{½} \end{pmatrix} = -1|white\rangle$$

Stipulate: $+1$ is the number corresponding to the Color value *black*.
-1 is the number corresponding to the Color value *white*.

- Can now expand Color states in Hardness basis:

$$|black\rangle = \begin{pmatrix} \sqrt{½} \\ \sqrt{½} \end{pmatrix} = \sqrt{½}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{½}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{½}\,|hard\rangle + \sqrt{½}|soft\rangle$$

$$|white\rangle = \begin{pmatrix} \sqrt{½} \\ -\sqrt{½} \end{pmatrix} = \sqrt{½}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sqrt{½}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{½}\,|hard\rangle - \sqrt{½}|soft\rangle$$

- *The EE Rule says*: To say a *white* electron has a Hardness value (*hard* or *soft*), it must be in an eigenstate of the Hardness operator.

- *But*: The state represented by $|white\rangle$ is *not* an eigenstate of the operator H representing the Hardness property:

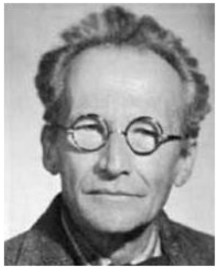
$$H|white\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{½} \\ -\sqrt{½} \end{pmatrix} = \begin{pmatrix} \sqrt{½} \\ \sqrt{½} \end{pmatrix} \neq \lambda|white\rangle, \qquad \text{for any value of } \lambda.$$

So: According to the Eigenvector/Eigenvalue Rule...

- A *white* electron has no definite value of Hardness.
- A *black* electron has no definite value of Hardness.
- A *hard*/*soft* electron has no definite value of Color.

3. Dynamics: States evolve in time via the Schrödinger equation

- Plug an initial state $|\psi(t_1)\rangle$ into the Schrödinger equation, and it produces a unique final state $|\psi(t_2)\rangle$.



Erwin Schrödinger
(1887-1961)

$$|\psi(t_1)\rangle \xrightarrow[\text{Schrödinger evolution}]{} |\psi(t_2)\rangle$$

state at time t_1 — state at later time t_2

- The Schrödinger equation can be encoded in an operator $S \equiv e^{-iH(t_2-t_1)/\hbar}$ (where H is the Hamiltonian operator that encodes the energy).

$$|A\rangle \longrightarrow |A'\rangle = S|A\rangle$$

state at t_1 — state at t_2

> *Important property*: S is a *linear operator*.
>
> $S(\alpha|A\rangle + \beta|B\rangle) = \alpha S|A\rangle + \beta S|B\rangle$, where α, β are numbers.

- *Recall*: Experimental Result #1: There is no correlation between Hardness measurements and Color measurements.
  - *If the Hardness of a batch of white electrons is measured,* 50% *will be soft and* 50% *will be hard.*

- Let's assume:

> *"Born Rule"*:
> The probability that a quantum system in a state $|\psi\rangle$ possesses the value b of a property B is given by the square of the expansion coefficient of the basis state $|b\rangle$ in the expansion of $|\psi\rangle$ in the basis corresponding to all values of the property.

Max Born
(1882-1970)

- *So*: The probability that a *black* electron has the value *hard* when measured for Hardness is ½!

$$|black\rangle = \sqrt{1/2}\,|hard\rangle + \sqrt{1/2}\,|soft\rangle$$

An electron in a black vector state...

... has a probability of ½ of being in a hard vector state upon measurement for Hardness.

More precisely...

4. Born Rule

The *probability* that a state $|\psi\rangle$ possesses the value b_i of the property represented by B is given by

$$\Pr(\textit{value of } B \textit{ is } b_i \textit{ in state } |\psi\rangle) \equiv |\langle\psi|b_i\rangle|^2 = |a_i|^2$$

where $|b_i\rangle$ is the eigenvector of B with eigenvalue b_i, and a_i is the expansion coefficient corresponding to $|b_i\rangle$ in the expansion of $|\psi\rangle$ in the eigenvector basis of B.

- Suppose a physical system is in a state represented by $|\psi\rangle$.
- To measure the value of a property represented by an operator B:

(1) First expand $|\psi\rangle$ in a basis given by a set of eigenvectors of B:

$$|\psi\rangle = a_1|b_1\rangle + a_2|b_2\rangle + \cdots + a_N|b_N\rangle$$

expansion coefficients of $|\psi\rangle$ in basis $|b_1\rangle, \ldots, |b_N\rangle$

eigenvectors of B: $B|b_1\rangle = b_1|b_1\rangle$, $B|b_2\rangle = b_2|b_2\rangle$, etc...

(2) The *probability* that $|\psi\rangle$ possesses the value b_1, say, of the property represented by B is then $|a_1|^2$, according to the Born Rule.

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where $|b_i\rangle$ is the eigenvector of B with eigenvalue b_i, and a_i is the expansion coefficient corresponding to $|b_i\rangle$ in the expansion of $|\psi\rangle$ in the eigenvector basis of B.

- When $|\psi\rangle$ is itself an eigenvector $|b_i\rangle$ of B, then the probability that it possesses the value b_i is equal to 1.
  - *Suppose*: $|\psi\rangle = |b_i\rangle$.
  - *Then*: $|\langle\psi|b_i\rangle|^2 = |\langle b_i|b_i\rangle|^2 = 1$.
  - This is consistent with the EE Rule!

5. Projection Postulate

When a measurement of a property B is made on a system in the state $|\psi\rangle = a_1|b_1\rangle + \cdots + a_N|b_N\rangle$ expanded in the eigenvector basis of B, and the result is the value b_i, then $|\psi\rangle$ *collapses* to the state $|b_i\rangle$: $|\psi\rangle \xrightarrow[collapse]{} |b_i\rangle$.

John von Neumann
(1903-1957)

Example: Suppose we measure a *black* electron for Hardness.

- The pre-measurement state is given by:

$$|black\rangle = \sqrt{½}\,|hard\rangle + \sqrt{½}|soft\rangle$$

- *Suppose*: The outcome of the measurement is the value *hard*.
- *Then*: The post-measurement state is given by $|hard\rangle$.

5. Projection Postulate

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John von Neumann
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- *Motivation*: Guarantees that if we obtain the value b_i once, then we should get the same value b_i on a second measurement (provided the system is not interferred with in-between).

Check:

- Suppose we conduct a B-measurement on the state $|\psi\rangle = a_1|b_1\rangle + \cdots + a_N|b_N\rangle$.
  - *Born Rule says*: The probability of getting b_i is $a_i^2 < 1$.
- Suppose we get b_i upon initial measurement.
  - *Projection Postulate says*: $|\psi\rangle$ collapses to $|b_i\rangle$.
- *Born Rule says*: The probability of getting b_i upon a second measurement is 1!
- *So*: If we measure the property represented by B again, we should get b_i with certainty.

Aside: The Wave Function

- Have been considering Color and Hardness (*i.e.*, spin) properties for electrons:
  - *Only 2 values.*
  - *State space is a 2-dimensional vector space.*
  - *Many orthonormal bases; each associated with a spin property: Color, Hardness, Gelb, Scrad, etc.; all of which are mutually incompatible.*

- Now consider Position and Momentum properties:
  - *Infinite continuum of values.*
  - *State space is an infinite-dimensional vector space (!).*
  - *Two distinct orthonormal bases; one associated with Position, the other with Momentum; both of which are mutually incompatible.*

  - Let X be the operator that represents the property of Position.
  - Let the eigenvectors and eigenvalues of X be given by $|x\rangle$ and x:

$$X|x\rangle = x|x\rangle$$

  - There are an infinite number of values for x, and thus an infinite number of eigenvectors $|x\rangle$!

- Suppose $|\psi\rangle$ represents the state of an electron located at some position.

- Can expand $|\psi\rangle$ in eigenvectors $|x\rangle$ of position operator X:

$$|\psi\rangle = a_1|1\rangle + a_{1.00001}|1.00001\rangle + \cdots + a_{72.93}|72.93\rangle + \cdots$$

  - All of the infinite number of position eigenvectors $|x\rangle$ are orthogonal to each other and form a basis for the ∞-dim position state space.

  - Any expansion coefficient a_x is given by $a_x = \langle\psi|x\rangle$, where x can be any number from $-\infty$ to $+\infty$!

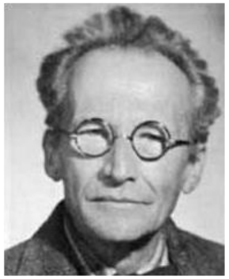
  - $\langle\psi|x\rangle$ is a continuous function, call it $\psi(x)$, of x called the *wave function*.

Just a functional way of representing a vector that has a continuum of eigenvalues!

- *<u>According to the Born Rule</u>*:

$$\Pr(\textit{electron is located at position } x \textit{ in state } |\psi\rangle) = |\langle\psi|x\rangle|^2 = |\psi(x)|^2$$

The square of the amplitude of the wavefunction $\psi(x)$!



Erwin Schrödinger
(1887-1961)

Schrodinger's (1926)
"wave mechanics"
(mathematics of waves)

Werner Heisenberg
(1901-1976)

Heisenberg's (1925)
"matrix mechanics"
(mathematics of matrices)

Paul Dirac
(1902-1984)

Dirac's (1926)
"transformation theory"
(linear algebra;
functional analysis)