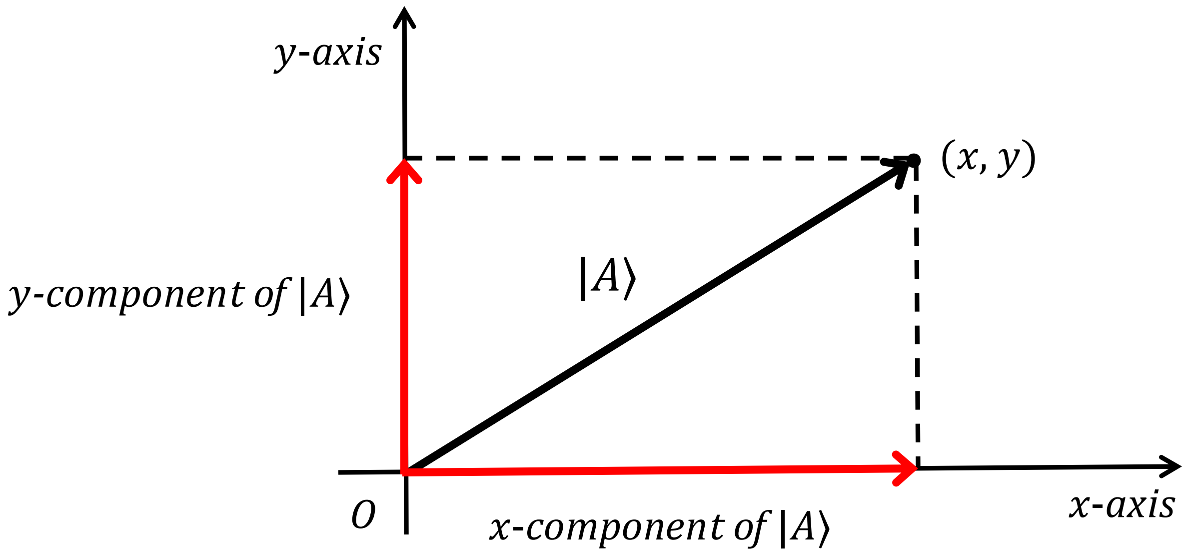
I. Vectors and Vector Spaces: 9 Easy Steps

1. Vectors

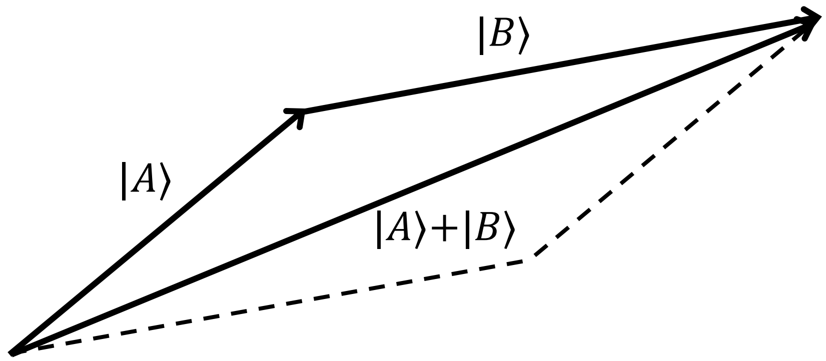
- A vector = a magnitude ("length") *and* a direction
- One way to represent vectors:



- Every point in x-y plane has a vector $|A\rangle$ associated with it.
- In this example, the collection of all vectors $|A\rangle$, $|B\rangle$, $|C\rangle$, ... forms a 2-dimensional *vector space*, call it V.

2. Vector addition

- To add vectors $|A\rangle$ and $|B\rangle$ in V, place tail of $|B\rangle$ to head of $|A\rangle$ to form a third vector in V, $|A\rangle+|B\rangle$, whose tail is the tail of $|A\rangle$ and whose head is the head of $|B\rangle$:



3. Scalar (number) multiplication

- *Numbers* can be multiplied to vectors. The result is another vector.

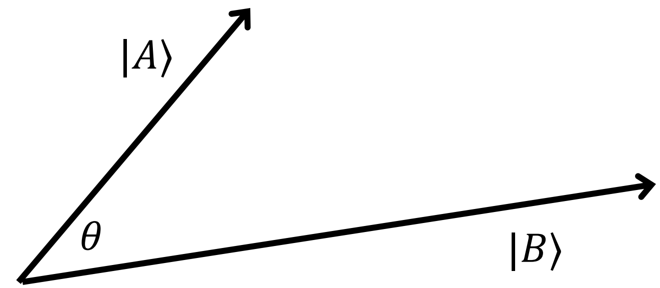
> *Ex*: $5|A\rangle$ is a vector in V that you get by multiplying the vector $|A\rangle$ by the number 5.

4. Inner-product (or dot-product)

- Vectors can be multiplied to each other.
- One type of vector multiplication is called the *inner-product*.
- The inner-product of two vectors $|A\rangle$, $|B\rangle$ is written as $\langle A|B\rangle$ and is a *number* defined by:

$$\langle A|B\rangle \equiv |A||B|\cos\theta$$

length of $|A\rangle$ — *length of* $|B\rangle$ — *angle between* $|A\rangle$ *and* $|B\rangle$

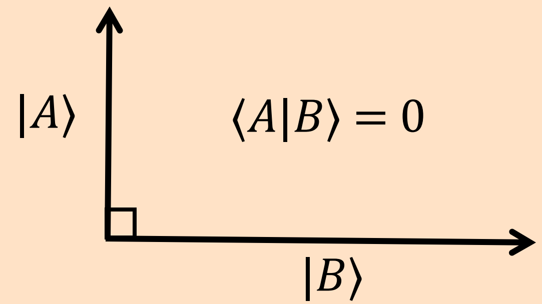


> *Note*: This means $\langle A|A\rangle = |A||A|\cos(0) = |A|^2$.
>
> So the *length* of a vector $|A\rangle$ is given by $|A| = \sqrt{\langle A|A\rangle}$

5. Orthogonality

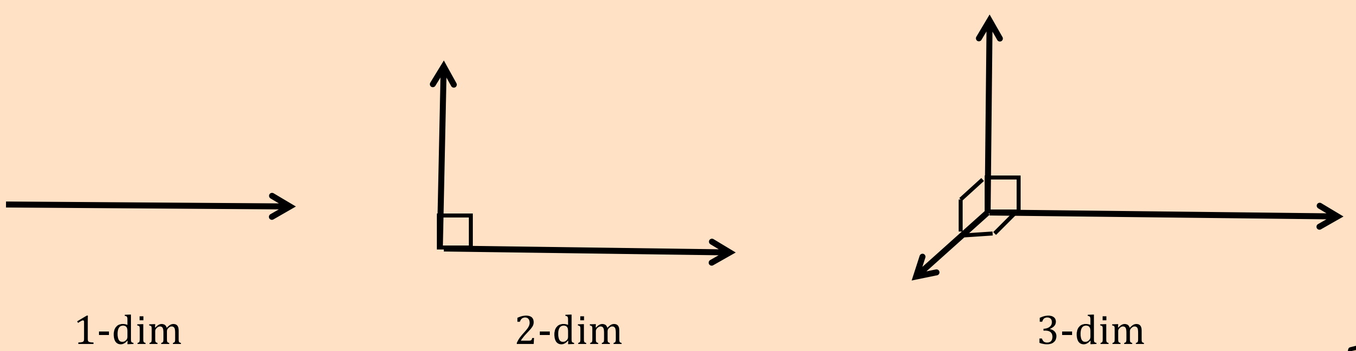
- Two non-zero-length vectors $|A\rangle$ and $|B\rangle$ are *orthogonal* (perpendicular) just when their inner-product is zero: $\langle A|B\rangle = 0$.

Check: $\langle A|B\rangle = |A||B|\cos\theta = 0$, just when $\theta = 90°$, given $|A| \neq 0 \neq |B|$.



6. Dimensionality

- The *dimension* of a linear vector space is equal to the maximum number N of mutually orthogonal vectors.



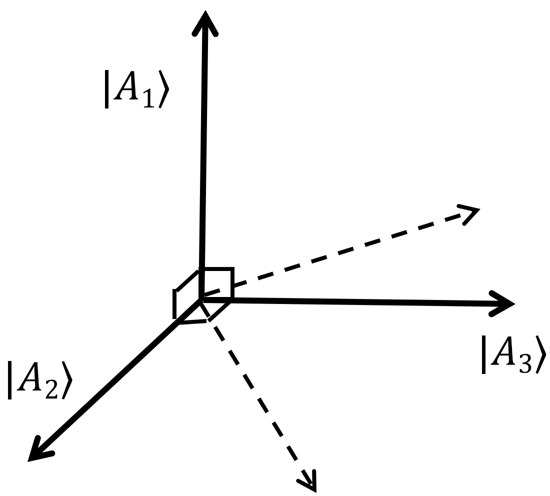
7. Orthonormal bases

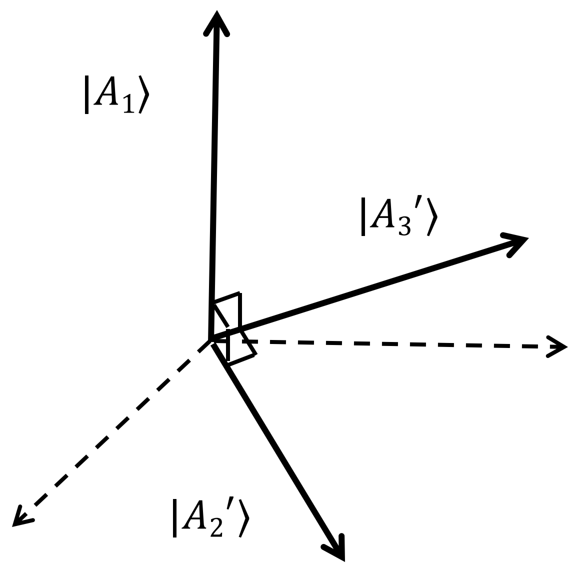
- An *orthonormal basis* of an N-dimensional vector space is a set of N mutually orthogonal vectors, each with unit length (or *norm*).

> *Note*: An N-dim vector space can have many different orthonormal bases!

Ex: Let $N = 3$.

Then $|A_1\rangle$, $|A_2\rangle$, $|A_3\rangle$ and $|A_1\rangle$, $|A_2'\rangle$, $|A_3'\rangle$ are two different orthonormal bases:



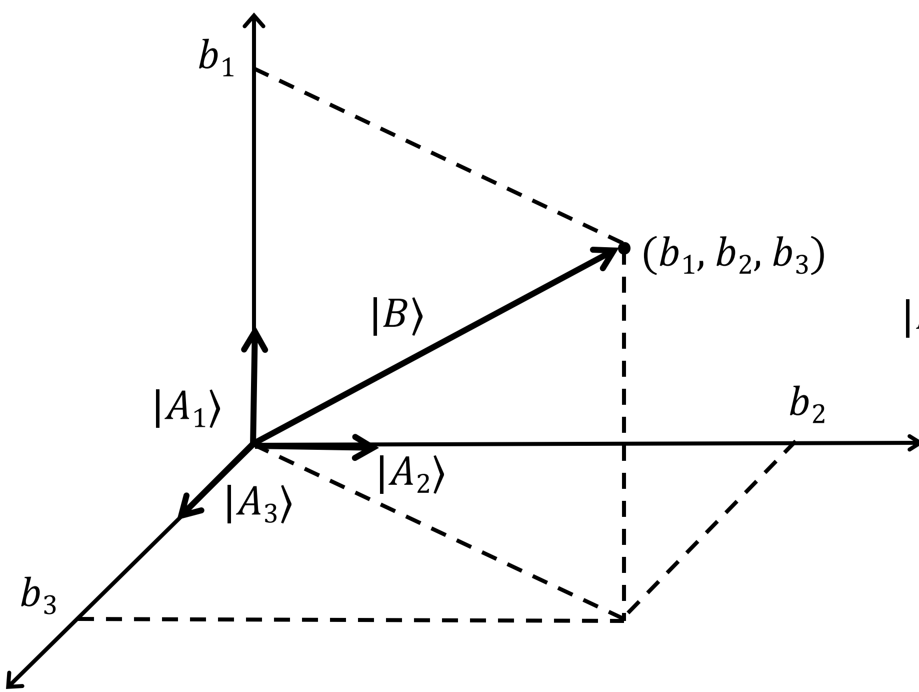


8. Expansion in an orthonormal basis

- In an N-dimensional vector space, any vector $|B\rangle$ can be *expanded* in terms of any orthonormal basis.

> - *Suppose*: $|A_1\rangle$, $|A_2\rangle$, ... $|A_N\rangle$ are N orthonormal basis vectors.
> - *Suppose*: $|B\rangle$ is an N-dim vector.
> - *Then*: $|B\rangle = b_1|A_1\rangle + b_2|A_2\rangle + ... + b_N|A_N\rangle$.
> - *And*: The numbers $b_i = \langle A_i|B\rangle$, $i = 1, ..., N$, are called *expansion coefficients*.

Ex: Let $N = 3$.



$$|B\rangle = b_1|A_1\rangle + b_2|A_2\rangle + b_3|A_3\rangle$$

9. Column vectors and row vectors

- One way to represent vectors is in terms of columns or rows of their expansion coefficients in a given basis:

Ex: $|B\rangle = b_1|A_1\rangle + b_2|A_2\rangle + b_3|A_3\rangle$

$$|B\rangle = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

column vector

Rule: To turn a column vector into a row vector, take the complex conjugate of its expansion coefficients:

$$\langle B| = b_1{}^*\langle A_1| + b_2{}^*\langle A_2| + b_3{}^*\langle A_3|$$

$$\langle B| = (b_1{}^*, b_2{}^*, b_3{}^*) = b_1{}^*(1, 0, 0) + b_2{}^*(0, 1, 0) + b_3{}^*(0, 0, 1)$$

row vector

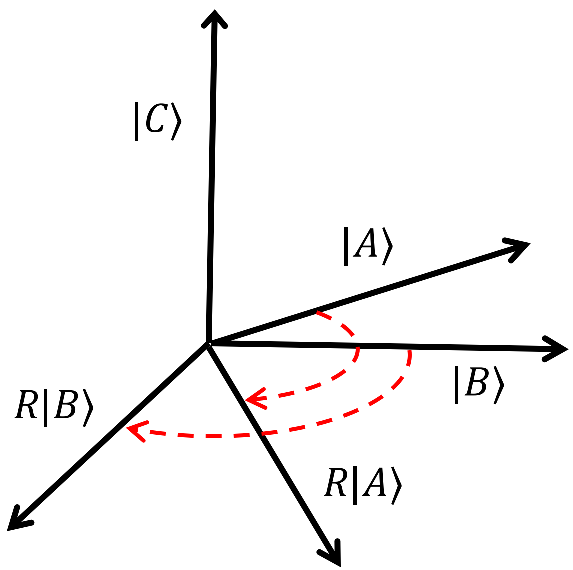
II. Operators: 3 Easy Steps

1. Linear Operators

- A *linear operator* O is a map that assigns to any vector $|A\rangle$, another vector $O|A\rangle$, such that, for any other vector $|B\rangle$ and numbers n, m,

$$O(n|A\rangle + m|B\rangle) = n(O|A\rangle) + m(O|B\rangle)$$

Ex: *Rotation operator* = a map R that assigns to any vector another one that is rotated clockwise by $90°$ about vector $|C\rangle$, is a linear operator.



2. Matrix representation of linear operators

- An operator can be represented by its *components* in a given basis.
- The *components* O_{ij} $(i, j = 1, ..., N)$ of an operator O in the basis $|A_1\rangle, ..., |A_N\rangle$ are defined by:

$$O_{ij} \equiv \langle A_i|O|A_j\rangle$$

Note: These are *numbers*: the result of taking the inner-product of the *vectors* $|A_i\rangle$ and $O|A_j\rangle$.

- The components O_{ij} of a linear operator form the elements of a *matrix*.

ith row — jth column

Ex: 2-dim operator O in $|A_1\rangle, |A_2\rangle$ basis.

$$O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$$

- Column and row vectors are special cases of matrices:
  - A column vector is a matrix with a single column.
  - A row vector is a matrix with a single row.

Matrix multiplication

- To multiply two matrices A and B:

  (1) The number of *columns* of A must be equal to the number of *rows* of B.

  (2) *Row into Column Rule*: If A is an $n \times m$ matrix and B is an $m \times r$ matrix, their product is an $n \times r$ matrix C with entries given by:

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{im}B_{mj}$$

$$\underbrace{\begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix}}_{n \times m} \underbrace{\begin{pmatrix} B_{11} & \cdots & B_{1r} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mr} \end{pmatrix}}_{m \times r} = \underbrace{\begin{pmatrix} C_{11} & \cdots & C_{1r} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nr} \end{pmatrix}}_{n \times r}$$

For instance: $C_{11} = A_{11}B_{11} + A_{12}B_{21} + \cdots + A_{1m}B_{m1}$

Ex: $O = \underbrace{\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}}_{2 \times 2}$ $\quad |B\rangle = \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{2 \times 1}$ $\quad O|B\rangle = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underbrace{\begin{pmatrix} O_{11}b_1 + O_{12}b_2 \\ O_{21}b_1 + O_{22}b_2 \end{pmatrix}}_{2 \times 1}$

3. Eigenvectors and Eigenvalues

- An *eigenvector* of an operator O is a vector $|\lambda\rangle$ that does not change its orientation when O acts on it: $O|\lambda\rangle = \lambda|\lambda\rangle$, for some number λ.
- An *eigenvalue* λ of an operator O is the number that results when O acts on one of its eigenvectors.
- *Convention*: Eigenvectors $|\lambda\rangle$ are labeled by their eigenvalues λ.

- *Note*: λ and $|\lambda\rangle$ are different mathematical objects: λ is a *number* and $|\lambda\rangle$ is a *vector*.
- *Also note*: $|\lambda\rangle$ and $\lambda|\lambda\rangle$ are two *different* vectors. They point in the same direction but have different lengths.

Example:

- Let O be a 4-dim operator with matrix representation in a particular basis given by:

$$O = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

- Then it has 4 eigenvectors given by:

$$|A\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |B\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |C\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |D\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- *Check*:

$$O|A\rangle = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 5|A\rangle$$

- *Similarly*: $O|B\rangle = 3/2|B\rangle$, $O|C\rangle = 2|C\rangle$, $O|D\rangle = -7|D\rangle$

- We say: "$|A\rangle$ is an eigenvector of O with eigenvalue 5, $|B\rangle$ is another eigenvector of O with eigenvalue 3/2, *etc*..."