Bell Experiment

Here's a slightly more formal derivation of the literal *QM* prediction discussed in class.

The initial state is an *EPR* state $|\Psi\rangle$ that anti-correlates the spin-orientations of the particles along the same spin axis. There are 3 ways to represent it in terms of axes V, R, L:

$$\sqrt{½}|\uparrow\rangle_1|\downarrow\rangle_2 - \sqrt{½}|\downarrow\rangle_1|\uparrow\rangle_2 \quad or \quad \sqrt{½}|\nwarrow\rangle_1|\searrow\rangle_2 - \sqrt{½}|\searrow\rangle_1|\nwarrow\rangle_2 \quad or \quad \sqrt{½}|\nearrow\rangle_1|\swarrow\rangle_2 - \sqrt{½}|\swarrow\rangle_1|\nearrow\rangle_2$$

Spin states along different axes are related *via* the technical result (see next page for proof):

$$|\uparrow\rangle = \cos(\theta/2)|\uparrow'\rangle + \sin(\theta/2)|\downarrow'\rangle$$
$$|\downarrow\rangle = \sin(\theta/2)|\uparrow'\rangle - \cos(\theta/2)|\downarrow'\rangle$$

$\theta = angle\ between \uparrow and \uparrow'$

Examples:

$$|\uparrow\rangle = \cos(120°/2)|\nearrow\rangle + \sin(120°/2)|\swarrow\rangle$$
$$= \tfrac{1}{2}|\nearrow\rangle + \tfrac{\sqrt{3}}{2}|\swarrow\rangle$$

$$|\uparrow\rangle = \cos(90°/2)|\rightarrow\rangle + \sin(90°/2)|\leftarrow\rangle$$
$$= \sqrt{½}\,|\rightarrow\rangle + \sqrt{½}\,|\leftarrow\rangle$$

Recall: One of the relations for the eigenvectors of Albert's *Color* and *Hardness*

NOW: Suppose the initial state is $|\Psi\rangle = \sqrt{½}|\uparrow\rangle_1|\downarrow\rangle_2 - \sqrt{½}|\downarrow\rangle_1|\uparrow\rangle_2$ and we want to measure particle 1's spin along V and particle 2's spin along L. To apply the *Born Rule*, we must first expand $|\Psi\rangle$ in eigenvector bases of V and L. It's already in an eigenvector basis of V; but we need to expand it's particle 2-part in terms of L states:

$$|\downarrow\rangle_2 = \tfrac{\sqrt{3}}{2}|\nearrow\rangle_2 - \tfrac{1}{2}|\swarrow\rangle_2$$

$$|\uparrow\rangle_2 = \tfrac{1}{2}|\nearrow\rangle_2 + \tfrac{\sqrt{3}}{2}|\swarrow\rangle_2$$

SO:

$$|\Psi\rangle = \tfrac{1}{\sqrt{2}}|\uparrow\rangle_1\left\{\tfrac{\sqrt{3}}{2}|\nearrow\rangle_2 - \tfrac{1}{2}|\swarrow\rangle_2\right\} - \tfrac{1}{\sqrt{2}}|\downarrow\rangle_1\left\{\tfrac{1}{2}|\nearrow\rangle_2 - \tfrac{\sqrt{3}}{2}|\swarrow\rangle_2\right\}$$

$$= \tfrac{\sqrt{3}}{2\sqrt{2}}|\uparrow\rangle_1|\nearrow\rangle_2 - \tfrac{1}{2\sqrt{2}}|\uparrow\rangle_1|\swarrow\rangle_2 - \tfrac{1}{2\sqrt{2}}|\downarrow\rangle_1|\nearrow\rangle_2 - \tfrac{\sqrt{3}}{2\sqrt{2}}|\downarrow\rangle_1|\swarrow\rangle_2$$

Now we can apply the *Born Rule*:

$$\Pr(\textit{spin-orientations of \#1 and \#2 differ along } V, L \textit{ in state } |\Psi\rangle) = \left|-\tfrac{1}{2\sqrt{2}}\right|^2 + \left|-\tfrac{1}{2\sqrt{2}}\right|^2 = 1/4$$

By rotational symmetry, this holds for spin measurements along all other pairs of axes as well.

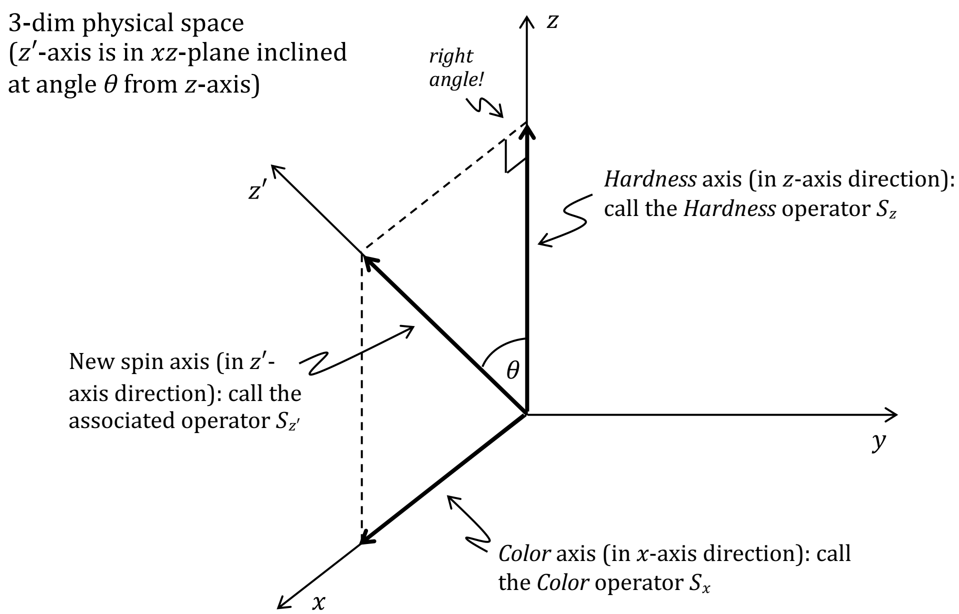
Now let's derive the technical result from the previous page:

<u>How to relate spin axes (technical result): In general,</u>

$$\Big[\text{spin-up on } z\Big] = \underset{\text{Prob} = \cos^2(\theta/2)}{\Big[\text{spin-up on } z'\Big]} + \underset{\text{Prob} = \sin^2(\theta/2)}{\Big[\text{spin-down on } z'\Big]}$$

$\theta =$ angle between axes z and z'

<u>First recall</u>: We're talking about a particular property: *Spin-along-an-axis*. For electrons (and other spin-½ particles), this property has only two values: *spin-up-along-the-axis*, *spin-down-along-the-axis*. Since the axis can point in any direction in 3-dim physical space, there can be an infinite number of such properties, one for each axis/direction in 3-space. Albert calls two of them *Hardness* and *Color*. It turns out that these are spin properties along axes that are at 90° to each other. Here's one way to represent their relation to each other, and to another spin axis.



What is the relation between $S_{z'}$, S_z, and S_x? We have:

$$S_z = S_{z'}\cos\theta \tag{1}$$

$$S_x = S_{z'}\sin\theta \tag{2}$$

Now multiply (1) on both sides by $\cos\theta$, and (2) on both sides by $\sin\theta$, and then add them together:

$$S_{z'} = S_x\sin\theta + S_z\cos\theta$$

In terms of matrices, recall:

$$S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Color operator   *Hardness* operator

Eigenvectors of S_z (*Hardness*):

$$|hard\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |soft\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

call it *"up-along-z-axis"*: $|\uparrow\rangle$ call it *"down-along-z-axis"*: $|\downarrow\rangle$

So:

$$S_{z'} = S_x \sin\theta + S_z \cos\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

The general form of the operator for spin along an axis z' inclined at θ from the z-axis, in the S_z eigenvector basis. (*Note*: when $\theta = 90°$, we get the *Color* operator!)

The eigenvectors of $S_{z'}$ then turn out to be:

$$|\uparrow'\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \qquad |\downarrow'\rangle = \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix}$$

Explicitly, $S_{z'}|\uparrow'\rangle = |\uparrow'\rangle$, and $S_{z'}|\downarrow'\rangle = -|\downarrow'\rangle$. To check, recall the trigonometric half-angle formulas:

$\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$

$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$

(*Note*: when $\theta = 90°$, we get the *Color* eigenvectors!)

SO: The relations between the eigenvectors of S_z (*Hardness*) and those of $S_{z'}$ are:

$$\boxed{\begin{array}{ll} |\uparrow\rangle = \cos(\theta/2)|\uparrow'\rangle + \sin(\theta/2)|\downarrow'\rangle & |\uparrow'\rangle = \cos(\theta/2)|\uparrow\rangle + \sin(\theta/2)|\downarrow\rangle \\ |\downarrow\rangle = \sin(\theta/2)|\uparrow'\rangle - \cos(\theta/2)|\downarrow'\rangle & |\downarrow'\rangle = \sin(\theta/2)|\uparrow\rangle - \cos(\theta/2)|\downarrow\rangle \end{array}}$$

And this produces the result we wanted (in particular, the result we used is given by the first equality). For instance, if you're a *hard* electron (an electron with *spin-up-along-z-axis*) in state $|\uparrow\rangle$, then the probability that a measurement of spin along the z' axis yields the value *spin-up* is:

$$\Pr(\textit{value of Spin-along-}z'\textit{-axis is up in state } |\uparrow\rangle) = |\cos(\theta/2)|^2 = \cos^2(\theta/2)$$