Moore Chap 11: The Löwenheim-Skolem Theorem

Recall: Set Theory -- attempt to mathematically codify the concept of infinity

I. Zermelo-Fraenkel (*ZF*) Formal Set Theory

Primitives of *ZF*: Individuals: Sets ("pure" iterative sets)
Property: set-membership

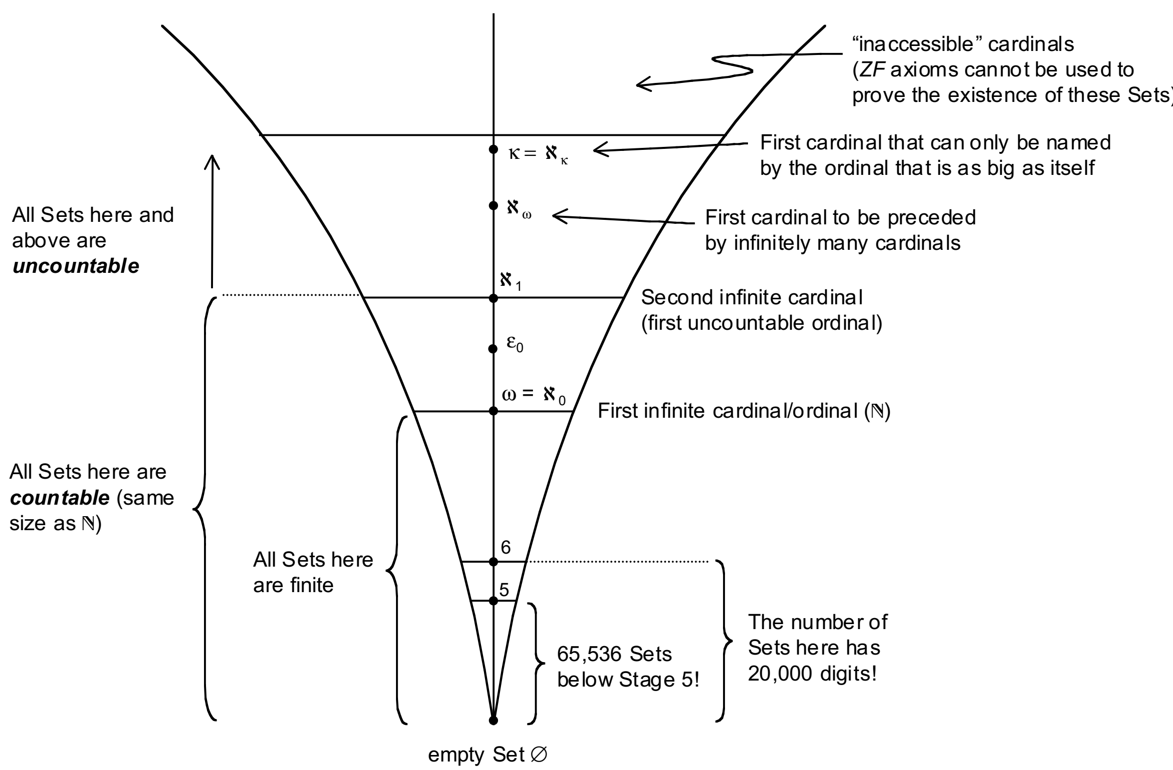
Formal Rules of *ZF*: (I) First-Order Logic
(II) *ZF* axioms

ZF axioms:

(ZF1) Two sets are equal if and only if they have the same members. (*Axiom of Extensionality*)
(ZF2) The empty set exists. (*Empty Set Axiom*)
(ZF3) Given any sets *x* and *y*, there is a set *z* whose members are *x* and *y*. (*Axiom of Pairing*)
(ZF4) Given any set *x*, there is a set *y* which has as its members all members of members of *x*. (*Axiom of Unions*)
(ZF5) Given any set *x*, there is a set y which has as its members all the subsets of *x*. (*Powerset Axiom*)
(ZF6) Given any set *x* and a function on *x*, there is a set *y* which has as its members all the images of members of *x* under this function. (Any "set-sized" collection of sets is a set.) (*Axiom Scheme of Replacement*)
(ZF7) An infinite set exists. (*Axiom of Infinity*)
(ZF8) Every non-empty set *x* contains a member which is disjoint from *x*. (*Axiom of Foundation*)
(AC) For any non-empty set *x*, there is a set y which has precisely one element in common with each member of *x*. (*Axiom of Choice*)

II. Advantages:

(A) Precise notion of "set": avoids paradoxes of the One and the Many (Russell's paradox, Set of Sets Paradox, etc)
(B) Precise notions of infinity: *ZF* Set theory includes Cantor's ordinals and cardinals:



III. Problems:

(A) (Gödel) *ZF* Formal Set Theory includes formal arithmetic as a part (recall that the natural numbers can be defined in terms of Sets).

<u>*SO*</u>:

[Gödel's 1st Incompleteness Theorem] ⇒ [There are truths about Sets that cannot be proven within *ZF*.]

[Gödel's 2nd Incompleteness Theorem] ⇒ [If *ZF* is consistent, then its consistency cannot be proven within *ZF*.]

(B) Do we know what we're talking about? (Do we really know what *ZF* is *about*?)

<u>*Problem of how to interpret ZF*</u>.

<u>First note</u>: "Ordinary" languages: Interpretations are (usually) easy to fix. The speaker can always point to the objects being referred to in the language (ostensive definitions).

ZF Set Theory language: We can't point to Sets. Is there a way to fix the subject matter of the language of *ZF* to *unambiguously* be about *Sets*? The Löwenheim-Skolem Theorem says "No":

<u>Löwenheim-Skolem Theorem:</u>

If a *first order* formal system *S* is consistent, then *S* has a <u>model</u> whose <u>domain</u> is a ***countable*** set.

- <u>model</u>: an interpretation that makes all theorems of *S* true
- <u>domain</u>: the set of Things that the interpretation is talking about

Applies to ZF Set Theory: ZF is a first-order formal system.

<u>*SO*</u>: If *S* is consistent, then we can always interpret it as describing only ***countably*** many Things.

<u>Consequences for *ZF*</u>:

(1) No matter how many true statements from the language of *ZF* we are given, we could never tell if the speaker was talking about Sets or natural numbers (or *any* ***countable*** collection of Things).

(2) <u>*BUT*</u>: This seems to be a Big Problem: <u>*What about uncountable Sets?*</u>

[Under the "intended" interpretation of *ZF*, there are such Things as uncountable Sets (the Vast majority of Sets).]

[*L-S* Theorem says *ZF* can be interpreted as being about only countable Things.]

[***How can an <u>uncountable</u> Thing be interpreted as a <u>countable</u> Thing?***]

IV. The Skolem Paradox

Under its intended interpretation, *ZF* refers to uncountable Sets.
BUT: The *L-S* Theorem says we can always interpret *ZF* as only referring to countable sets.
SO: How can we interpret an uncountable Set in terms of a countable set?

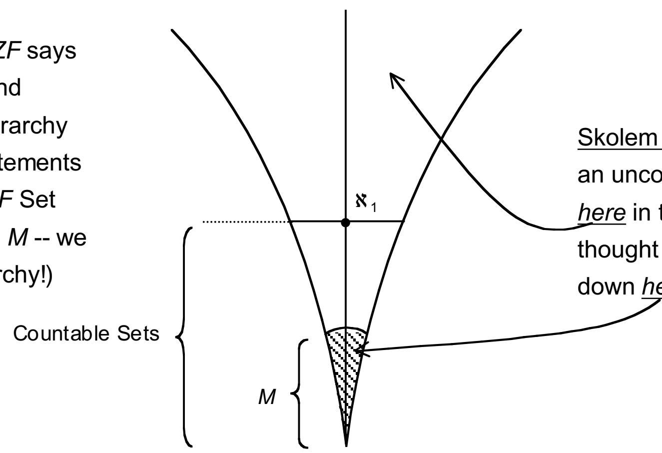
> Example: How can we interpret the sentence "The powerset $\wp(\mathbb{N})$ of $\mathbb{N}$ is uncountable" *only* in terms of countable sets?

General idea: The *L-S* Theorem allows us to do the following:

Take a small slice *M* off the bottom of the Set Hierarchy such that:

(1) *M* is a countably infinite Set
(2) *M* serves as an interpretation of *ZF*: The members of *M* can be interpreted as the subject matter of *ZF*. Under this interpretation, an "*M*-set" corresponds to a "*ZF* Set".

L-S Theorem: *Everything ZF* says about all Sets (countable and uncountable) in the Set Hierarchy can be reinterpreted as statements about *M*-sets. (*i.e.*, to do *ZF* Set theory, all we really need is *M* -- we don't need the entire Hierarchy!)



Skolem paradox: How can an uncountable Set way up *here* in the Hierarchy be thought of as an *M*-set way down *here*?

Formal Resolution of Skolem Paradox:

Recall: To say "Set *A* is uncountable" means "There is another set *B* such that the members of *A* cannot be *paired in 1-1 fashion* with the members of *B*".

under the "intended" interpretation, *B* is $\mathbb{N}$

... and this just means "Another set *C* exists whose members are the pairs of *A* and *B* members"

SO: Statements about uncountable sets are interpreted in *M* as statements about whether or not certain *M*-sets exist.

> Example: The statement "The powerset $\wp(\omega)$ of ω is uncountable" is interpreted in *M* as a statement about certain *M*-sets: "There is an *M*-set M_1 (corresponding to $\wp(\mathbb{N})$) and there is an *M*-set M_2 (corresponding to $\mathbb{N}$) and there is not an *M*-set corresponding to the set of pairs of members of M_1 and M_2"
>
> *i.e.*, "*Within M*, there is a set M_1 that looks like $\wp(\mathbb{N})$ and another M_2 that looks like $\mathbb{N}$, and these can't be paired."

Outside of *M*, we can see that all *M*-sets are really only countable. The *M*-set M_1 that *M* says is $\wp(\mathbb{N})$ really isn't: outside *M*, M_1 and $\mathbb{N}$ can be paired, but this requires the existence of a "pairing" Set that isn't in *M*.

Lingering Conceptual Problems:

The *L*-*S* Theorem says there is nothing intrinsic to *ZF* that can determine what its intended interpretation is.

In particular: Anything you can do in *ZF*, you can do in *M*. But *we* know that *ZF* extends to Things outside *M* (*i.e.*, it extends to Sets in the full Set Hierarchy).

BUT: How do we know that what we take to be the full Hierarchy *really* is the intended interpretation of *ZF*? What if what we think is the full hierarchy is *really* a small slice, call it *M'*, near the bottom of an even larger hierarchy?

In particular: What we think are ***uncountable*** sets in our hierarchy may *really* be ***countable*** *M'*-sets in the larger hierarchy.

Suggests a relativism of the following sort (Skolem):

> A set can only be said to be countable or uncountable *relative* to an interpretation of *ZF*.

But recall: The distinction between countable and uncountable Sets is the basic distinction between types of infinity:

Countably infinite Sets: $\mathbb{N}$, ω, $\aleph_0$ -- "first level" of infinity
Uncountable Sets: $\aleph_1$, $\aleph_2$, $\aleph_3$, ... -- each labels the next higher level of infinity

Are we thus left with a relative concept of infinity?

Optional: Zermelo-Fraenkel (*ZF*) Formal Set Theory

Primitives of *ZF* (intended interpretation):	Individuals (infinite):	Sets ("pure" iterative sets)
	One Property:	set-membership (denoted by "$\in$")

Formal Rules of *ZF*: (I) First-Order Logic
(II) *ZF* axioms

***ZF* axioms (8 + Axiom of Choice):**

(ZF1) Two sets x, y are the same if and only if they have the same members. (*Axiom of Extensionality*)

$$x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$$

(ZF2) A set x exists that has no members (*i.e.*, the empty set). (*Empty Set Axiom*)

$$(\exists x)(\forall y){\sim}(y \in x)$$

Notation: (ZF1) and (ZF2) entail there is a unique empty set: Call it $\varnothing$.

(ZF3) Given any sets x and y, there is a "pair" set z whose members are x and y. (*Axiom of Pairing*)

$$(\forall x)(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow (w = x \vee w = y))$$

Notation: (ZF1) and (ZF3) entail there is a unique pair set for any given x, y: Call it $\{x, y\}$.

(ZF4) Given any set x, there is a "union" set y which has as its members all members of members of x. (*Axiom of Unions*)

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\exists w)(w \in x \bullet z \in w))$$

Notation: (ZF1) and (ZF4) entail there is a unique union for any set x: Call it $\cup x$.
Let $x \cup y$ represent the union set $\cup\{x, y\}$ of the pair set of x and y. (This is used in ZF6.)

(ZF5) Given any set x, there is a set y which has as its members all sets whose members are also members of x (*i.e.*, y contains all the "subsets" of x). (*Powerset Axiom*)

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\forall w)(w \in z \rightarrow w \in x))$$

Notation: (ZF1) and (ZF4) entail there is a unique powerset for any set x: Call it $\wp(x)$.
Define $x \subseteq y$ ("x is a subset of y") as $\forall z(z \in x \rightarrow z \in y)$. Then (ZF5) can be written as:

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \subseteq x)$$

(ZF6) An infinite set exists. (*Axiom of Infinity*)

$$(\exists x)(\varnothing \in x \bullet (\forall y)(y \in x \rightarrow y \cup \{y\} \in x))$$

<u>*Comment*</u>: This axiom guarantees the existence of a set x such that $\varnothing$ is a member of x, and for any set y, if y is a member of x, then so is $y \cup \{y\}$. The set x thus takes the following form:

$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots\}$

$\varnothing$ is in x. This implies $\varnothing \cup \{\varnothing\}$, or $\{\varnothing\}$, is in x. This implies $\{\varnothing\} \cup \{\{\varnothing\}\}$ is in x. *Etc*...

(ZF7) Every non-empty set x contains a member which is disjoint from x. (*Axiom of Foundation*)

$$(\forall x)(\sim x = \varnothing \rightarrow (\exists y)(y \in x \bullet \sim(\exists z)(z \in y \bullet z \in x)))$$

<u>*Comment*</u>: This axiom says that for any set x other than the empty set, there is a "minimal" member y of x that has no members in common with members of x. This rules out circular chains of sets (*e.g.*, $x \in y$ and $y \in z$ and $z \in x$) and infinitely descending chains of sets. In particular, it rules out the possibility of a set being a member of itself (if there was such a set x, then the infinitely descending chain ... $x \in x \in x \in x$ would be possible). So Russell-type paradoxes are avoided.

(ZF8) Given a function $\mathcal{A}(x, y)$ (*i.e.*, a map that relates every set x to a unique set y), then for any set z, we can form a new set w which has as its members all the images of members of z under this function. (*Axiom Scheme of Replacement*)

$$(\forall x)(\exists! y)\mathcal{A}(x, y) \rightarrow (\forall z)(\exists w)(\forall v)(v \in w \leftrightarrow (\exists u)(u \in z \bullet \mathcal{A}(u, v)))$$

<u>*Comment*</u>: The members of w are formed by collecting together all the sets to which the members of z are mapped by the function $\mathcal{A}$. You start with the set z and get the set w by replacing all the members of z with their counterparts under the function $\mathcal{A}$. This is called an "Axiom Scheme" since it holds for all possible functions $\mathcal{A}$ (so there's really one axiom per function $\mathcal{A}$: you can build a new set from an original by using any appropriate available function).

(AC) For any non-empty set x, there is a set y which has precisely one element in common with each member of x. (*Axiom of Choice*)

<u>*Comment*</u>: *AC* doesn't tell you *how* to construct y; *i.e.*, it doesn't say what the "choice" function is that you use to pick out the members of y from members of x. All the other axioms *do* give you recipes for the construction of new sets. For this reason, the status of *AC* as an axiom is sometimes debated. It is needed in order to prove that all sets can be well-ordered, so it's important for Cantor's theory of ordinals and cardinals.