Moore Chap 10: Transfinite Mathematics

Topics
- I. Iterative Conception of a Set
- II. Ordinals as Sets
- III. Cardinal Numbers
- IV. Transfinite Arithmetic and the Continuum Hypothesis

Set Theory -- allows for a mathematical theory of infinity.
Fundamental Question: What *exactly* is a set?

Motivation: Members of a set are more fundamental than the set itself. Members "exist first".

(Allows resolution of Russell's Paradox) : (No set can be a member of itself. So Russell's R cannot be a set. Cantor: It is an "inconsistent totality".) → Note: A set *can* be a member of *other* sets.

Zermelo-Frankel (*ZF*) Set Theory

Formal system based on 9 axioms and two primitive notions: "set" and "is a member of".

(ZF1) Two sets are equal if and only if they have the same members. (Axiom of Extensionality)

(ZF2) The empty set exists. (Empty Set Axiom)

(ZF3) Given any sets x and y, there is a set z whose members are x and y. (Axiom of Pairing)

(ZF4) Given any set x, there is a set y which has as its members all members of members of x. (Axiom of Unions)

(ZF5) Given any set x, there is a set y which has as its members all the subsets of x. (Powerset Axiom)

(ZF6) Given any set x and a function on x, there is a set y which has as its members all the images of members of x under this function. (Any "set-sized" collection of sets is a set.) (Axiom Scheme of Replacement)

(ZF7) An infinite set exists. (Axiom of Infinity)

(ZF8) Every non-empty set x contains a member which is disjoint from x. (Axiom of Foundation)

(AC) For any non-empty set x, there is a set y which has precisely one element in common with each member of x. (Axiom of Choice)

ZF reproduces all results in set theory. But still doesn't answer question: What is a "set"?

I. The Iterative Conception of a Set

First, restrict attention to "pure" sets, call them "Sets". Sets are sets that would exist if nothing existed except sets.

Can iteratively construct Sets in stages:

> Stage 1: Construct the empty Set $\varnothing$.
>
> At any stage n, construct all possible Sets (that haven't yet been constructed) whose members are taken only from prior stages.

Stage 1: First Set is the empty Set: $\varnothing$

Stage 2: Only one Set can be constructed: $\{\varnothing\}$

Stage 3: Two Sets can be constructed: $\{\{\varnothing\}\}$ and $\{\varnothing, \{\varnothing\}\}$

Stage 4: Twelve Sets can be constructed:

2 new single-member Sets: $\{\{\{\varnothing\}\}\}$
$\{\{\varnothing, \{\varnothing\}\}\}$

5 new two-member Sets: $\{\varnothing, \{\{\varnothing\}\}\}$
$\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$
$\{\{\varnothing\}, \{\{\varnothing\}\}\}$
$\{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$
$\{\{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$

4 new three-member Sets: $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$
$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$
$\{\varnothing, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$
$\{\{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$

1 new four-member Set: $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$

In general, if number of Sets constructed by stage n is k, then number of Sets constructed by stage $n + 1$ is $2k$.

<u>SO</u>: Stage 4 has $2^4 = 16$ Sets total.
Stage 5 has $2^{16} = 65, 636$ Sets
Stage 6 has $2^{65,636}$ Sets!
etc...

For each ordinal α, there is a Stage α!

Key features of "construction" metaphor:

(1) No end to Set construction: Hierarchy of Sets has no top!
(2) Members of a Set "exist" before the Set itself.
(3) *Potential* infinity of Sets:
Hierarchy is infinite in the sense that, for any Stage α, we can always progress to the next Stage $\alpha + 1$.

Can the Set hierarchy be thought of as actually infinite?
Is the "construction" metaphor just a manner of speaking about Sets?

<u>Note</u>: If the hierarchy is actually infinite, then there might seem to be a problem with the Set of all Sets.
If the hierarchy exists as a complete whole, then we can consider the Set that contains all Sets in it.
But this Set of all Sets both is and is not a member of itself!

<u>Cantor-Way-Out</u>: The Set of all Sets is an "inconsistent totality"; a "misbehaving Set.

II. Ordinals as Sets

Recall: Ordinals are measures of the "length" of Sets:
ω is the length of $\{ 0, 1, 2, 3, ... \}$
$\omega + 1$ is the length of $\{ 0, 1, 2, 3, ..., \omega \}$
etc.

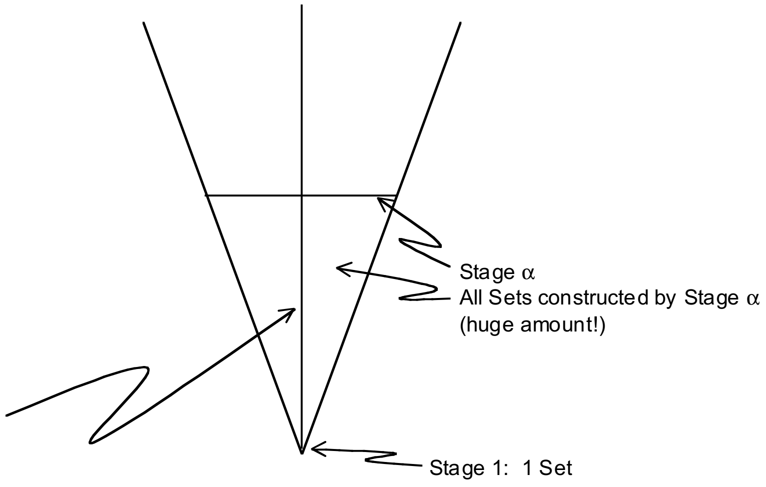
Note: Natural numbers are ordinals.

<u>SO</u>: 5 is the length of $\{ 0, 1, 2, 3, 4 \}$
1 is the length of $\{ 0 \}$
0 is the length of $\{ \}$ or $\varnothing$

<u>Motivates following identification</u>:

$0 = \varnothing$
$1 = \{\varnothing\}$
$2 = \{\varnothing, \{\varnothing\}\}$
$3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$
⋮
$\omega = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, ... \}$

<u>SO</u>: Ordinals can be considered as Sets -- part of the Set Hierarchy



Recall: Ordinals label the stages of Set construction. Represent them as the "back-bone" of the *V*-shaped Set Heirarchy

III. Cardinal Numbers

Motivation: No limit to how big an infinite set can be.

BUT: There *is* a limit to how *small* it can be.

example: All infinite sets are at least as big as $\mathbb{N}$ (think of $\mathbb{N}$ as the first infinite size).

Terminology: A ***countable*** set is any set that is either finite or the same size as $\mathbb{N}$.
An ***uncountable*** set is any set bigger than $\mathbb{N}$.

Cardinal numbers measure the "size" of sets. They are the smallest ordinals of a given set size.

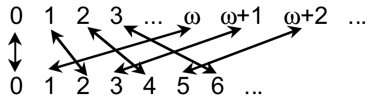
- The *finite cardinals* are the natural numbers -- measure the size of finite sets.

example: The size of the set of planets,
{Mercury, Venus, Earth, Mars, Saturn, Jupiter, Uranus, Neptune, Pluto}
is the Cardinal 9.

- The first *infinite cardinal* is called $\aleph_0$ and measures the size of $\mathbb{N}$.

All ordinals between ω and ε_0 are countable: they *all* have the same size as $\mathbb{N}$ (but different "lengths").
ω is the smallest ordinal of this size.

example: *Claim*: ω has the same size as $\omega + \omega$
Proof: Pair first ω ordinals in $\omega + \omega$ onto even members of ω.
Pair second ω ordinals in $\omega + \omega$ onto odd members of ω.



SO: $\mathbb{N}$ and ω and $\aleph_0$ all name the *same* set; namely, the set of natural numbers.
ω measures its "length".
$\aleph_0$ measures its "size" -- tells us how many members it has.

- The next infinite cardinal is called $\aleph_1$ and measures the size of the ordinal that succeeds all countable ordinals (must exist!).

Between $\aleph_0$ and $\aleph_1$ there are many ordinals.
$\omega+1, \omega+2, \ldots$
$\omega\times 2, \ldots$
$\omega^\omega \ldots$
All have the same size $\aleph_0$, but all have different lengths.

- Cardinals are labeled by ordinals:

$$\underbrace{\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots, \aleph_{\varepsilon_0}, \ldots, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}}, \ldots$$

All of these are cardinals that come much after their labels -- *eg.*, $\aleph_0$ comes much after 0

first cardinal to succeed all of these is labeled by the ordinal that it is:
$\kappa = \aleph_\kappa$

Incredibly big! So big that it needs itself to say how big it is!

III. Transfinite Arithmetic and the Continuum Hypothesis

Some results:

(1) If κ and λ are cardinals, at least one of which is infinite, and $\kappa \geq \lambda$, then

$$\kappa + \lambda = \lambda + \kappa = \kappa \text{ x } \lambda = \lambda \text{ x } \kappa = \kappa$$

example: $\aleph_7 + \aleph_5 = \aleph_7 \times \aleph_5 = \aleph_7$

(2) If κ is a cardinal, finite or infinite, then $\kappa < 2^\kappa$

Motivation: If set *A* has n members, then $\wp(A)$ has 2^n members.

example: $A = \{a, b, c\}$
$\wp(A) = \{ \{ \}, \{a\}, \{b\}, \{c\}, \{a, b\}. \{a, c\}, \{b, c\}, \{a, b, c\} \}$

Result (2) then follows from the claim that $\wp(A)$ is always bigger than *A*, for any *A*.

Particular Consequence of Result (2): $\underset{\substack{\nearrow \\ \text{size of } \mathbb{N}}}{\aleph_0} < \underset{\substack{\nwarrow \\ \text{size of } \wp(\mathbb{N})}}{2^{\aleph_0}}$

Further Claim: $\mathbb{R}$ and $\wp(\mathbb{N})$ have the same size.

SO: $\aleph_0 < 2^{\aleph_0}$ says that $\mathbb{R}$ is bigger than $\mathbb{N}$.

Recall *Cantor's Unanwered Question*: How much bigger than $\mathbb{N}$ is $\mathbb{R}$?

The "Continuum Hypothesis" (*CH*) states that $\mathbb{R}$ is the "next infinite size up" from $\mathbb{N}$. More precisely:

$$\boxed{CH\text{: } 2^{\aleph_0} = \aleph_1}$$

Why it's called the "Continuum" Hypothesis:

The points on a line form a "***continuum***", which has two basic properties:

(1) Between any two points is another (*denseness* property).

(2) There are no "gaps" between points.

The rationals are *dense*: between any two rationals is another.

BUT: There are gaps in the rationals (where the irrationals would be).

The reals include both rationals and irrationals: they "fill in the gaps" between the rationals.

So $\mathbb{R}$ measures the "size" of a continuum: measures how "close" together the points on a line are.

And *CH* claims that this "closeness" is given by $\aleph_1$.

Is *CH* true?

Technical results: Godel (1938): Can't prove that *CH* is false within *ZF* set theory.
Cohen (1963): Can't prove that *CH* is true within *ZF* set theory.

Apparent Paradox

Claim: The infinity of the Infinitely Big is *smaller* than the infinity of the Infinitely Small!

Proof: $\aleph_0$ is the size of $\mathbb{N}$. And $\mathbb{N}$ is the infinity of the Infinitely Big; *i.e.*, the infinity of the "infinite by addition".
$2^{\aleph_0}$ is the size of $\mathbb{R}$. And $\mathbb{R}$ is the infinity of the Infinitely Small; *i.e.*, the infinity of the "infinite by division".
SO: $\aleph_0 < 2^{\aleph_0}$ says that the infinity of the Infinitely Big is smaller than the infinity of the Infinitely Small!

Should we be convinced?

Probably not:

(A) Can't really identify $\mathbb{R}$ with the infinity of the "infinite by division". The "infinite by division" only requires that between any two points there is another. And this is the property of denseness, for which we only need the rational numbers, not the reals. And the size of the rationals is $\aleph_0$, not $2^{\aleph_0}$ (see the Paradox of the Pairs for proof).

(B) Can't really identify $\mathbb{N}$ with the infinity of the "infinite by addition". We can extend addition to ordinals beyond ω.