Moore Chap 8: Cantor and the Mathematics of the Infinite

Topics
I. Sets and Paradoxes of the Infinitely Big
II. Cantor and Diagonal Arguments
III. Cantor's Theory of Ordinal Numbers

I. Sets and Paradoxes of the Infinitely Big

Recall: Paradox of the Even Numbers

Claim: There are just as many even natural numbers as natural numbers

natural numbers = non-negative whole numbers (0, 1, 2, ..)

Proof:

$$\begin{array}{cccccccc} \{\,0, & 1, & 2, & 3, & 4, & \ldots\ldots, & n, & \ldots\ldots\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ \{\,0, & 2, & 4, & 6, & 8, & \ldots\ldots, & 2n, & \ldots\ldots\} \end{array}$$

In general: 2 criteria for comparing sizes of sets:

(1) *Correlation criterion*: Can members of one set be paired with members of the other?

(2) *Subset criterion*: Do members of one set belong to the other?

Can now say:

(a) There are as many even naturals as naturals in the *correlation* sense.

(b) There are less even naturals than naturals in the *subset* sense.

Moral: (To talk about the infinitely big, just need to be clear about what's meant by size)

notion of sets makes this clear

Bolzano (1781-1848)

Promoted idea that notion of infinity was fundamentally *set-theoretic:*

(To say something is infinite is just to say there is some set with infinite members)

"God is infinite in knowledge"
means
"The set of truths known by God has infinitely many members"

SO: Are there *infinite* sets?

"a many thought of as a one" -Cantor

Bolzano: *Claim*: The set of truths is infinite.

Proof: Let p_1 be a truth (ex: "Plato was Greek")

Let p_2 be the truth "p_1 is a truth".

Let p_3 be the truth "p_3 is a truth".

In general, let p_n be the truth "p_{n-1} is a truth", for any natural number n.

Dedekind: (1831-1916) *Claim*: The set of thoughts is infinite.

Proof: Let s_1 be a thought.

Let s_2 be the thought that s_1 is a thought.

Let s_3 be the thought that s_3 is a thought.

In general, let s_n be the thought that s_{n-1} is a thought, for any natural number n.

BUT: Can these sets *really* be treated as complete wholes?

Problem #1 with sets:
Russell's Paradox

Let *R* be the set of all sets that do not belong to themselves.

There are sets that *do* belong to themselves:
- The set of sets
- The set of things written on this lecture note

But most sets do not belong to themselves.
So *R* is very likely an infinite set.

Claim: *R* belongs to itself if and only if *R* does not belong to itself.

Proof: (1) Suppose *R* belongs to *R*.
Then *R* is a set that does not belong to itself.
So *R* does not belong to *R*.
(2) Suppose *R* does not belong to *R*.
Then *R* is a set that belongs to itself.
So *R* does belong to *R*.

Russell's Paradox is a paradox of the One and the Many: It looks like *R* can't be thought of as a "one".
SO: Sets were introduced initially (in part) to address paradoxes of the Infinitely Big. But now it seems we've just replaced them with paradoxes of the One and the Many!

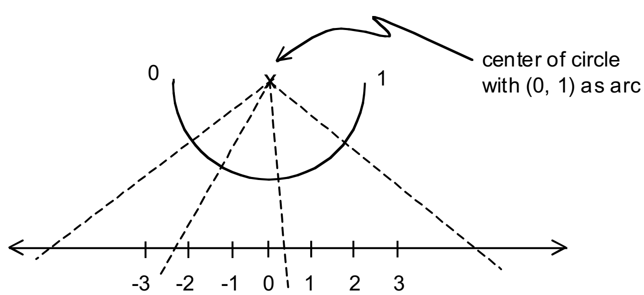
II. Cantor and Diagonal Arguments

- adopted correlation criterion for set-size:
  (1) Set *A* has the same size as set *B* just when members of *A* can be paired with members of *B*.
  (2) Set *A* is bigger than set *B* just when all members of *B* can be paired with some members of *A*, but not with all of them.

Some results:

1. There are as many even natural numbers as natural numbers.
2. There are as many real numbers between 0 and 1 as there are real numbers.

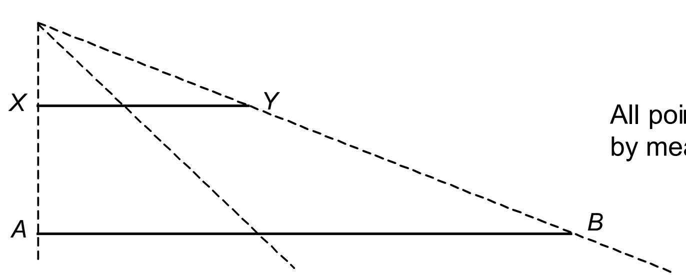
Proof:



Every point on the arc (i.e., real number between 0 and 1) is paired with a real number on the real number line by means of the dashed projection lines that originate at the circle's center.

3. All line segments have the same number of points.

Proof:



All points in $\overline{XY}$ are paired with all points in $\overline{AB}$ by means of dashed projection lines.

4. There are more real numbers between 0 and 1 than there are natural numbers.

Proof: (**"Diagonal" Argument**)

(i) Pair natural numbers with decimal expansions of reals between 0 and 1. There are many ways to do this. One particular way is the following:

0 $\leftrightarrow$	0 . 3 3 3 3 ...	= 1/3	all reals between 0 and 1 can be given an infinite decimal expansion
1 $\leftrightarrow$	0 . 1 4 1 5 ...	= π - 3	
2 $\leftrightarrow$	0 . 4 1 4 2 ...	= $\sqrt{2}$ - 1	
3 $\leftrightarrow$	0 . 5 0 0 0 ...	= 1/2	
$\vdots$	$\vdots$		

(ii) Construct a real between 0 and 1 that is not listed in the table:

(a) Go down the "diagonal" of the table starting at the first digit in the decimal expansion of the first real.

(b) Write "3" if the digit in the diagonal is a 4; write "4" if the digit in the diagonal is anything else.

our example: 0.4334...

(iii) This real is not listed in the table!
By construction, it differs from the first real in its first decimal place; it differs from the second real in its second decimal place, etc. In general, it differs from all listed reals (no matter how they are listed).
BUT: The table contains all the natural numbers (in its first column).
SO: There are more real numbers between 0 and 1 than there are natural numbers.

Recall: There are just as many reals between 0 and 1 as there are reals.
SO: There are more real numbers than there are natural numbers (even though both are infinite).

$\mathbb{R}$ = set of real numbers

$\mathbb{N}$ = set of natural numbers

5. There are more sets of natural numbers than there are natural numbers.

Proof: (**"Diagonal" Argument**)

(i) Pair natural numbers with sets of natural numbers.

Represent a set of natural numbers as a sequence of "yeses" and "noes" depending on whether successive naturals belong to it or not.

Examples:

Represent	{0, 2, 4, 6, 8, ... }	as	<yes, no, yes, no, yes, ... >
Represent	{1, 2}	as	<no, yes, yes, no, no, no, no, ... >

SO: One way to do Step (i) is the following:

	0	1	2	3 ...
0 $\leftrightarrow$	yes	no	yes	...
1 $\leftrightarrow$	no	no	yes	...
2 $\leftrightarrow$	no	no	no	...
3 $\leftrightarrow$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

(ii) Construct a set of natural numbers that is not listed in the table in the following way:
Go down the diagonal. Write “no” for each “yes”, and “yes” for each “no”.

our example: <no, yes, yes, ... >

(iii) By construction, this set of naturals is not listed in the table: It differs from the first listed set in its first member; it differs from the second listed set in its second member, etc. It differs from all listed sets of naturals, no matter how they are listed.
BUT: The table lists all natural numbers (in its first column).
SO: There are more sets of natural numbers than there are naturals.

Terminology: Let A be a set.
The powerset $\wp(A)$ of A is the set of all subsets of A.

ex: $A = \{a, b\}$
$\wp(A) = \{ \{ \}, \{a\}, \{b\}, \{a, b\} \}$

SO: Result #5 can be stated as: $\wp(\mathbb{N})$ is larger than $\mathbb{N}$.

Further Claim: For any set A, $\wp(A)$ is larger than A.

Consequence: No limit to how large an infinite set can be!
- $\mathbb{N}$ is infinite.
- $\wp(\mathbb{N})$ is larger.
- $\wp(\wp(\mathbb{N}))$ is larger still, *etc*...

One Big Question (“Cantor’s Unanswered Question”):

How much larger than $\mathbb{N}$ is $\mathbb{R}$?
(a) Is it the “next infinite size up” from $\mathbb{N}$?
(b) Are there intermediate sizes between $\mathbb{N}$ and $\mathbb{R}$?

The “***Continuum Hypothesis***” is the claim that the answer is (a).

III. Cantor’s Theory of Ordinal Numbers

Motivation: The size of a set doesn’t depend on what its members are or how they are ordered.

Important definition:
A ***well-ordering*** of a set X (finite or infinite) is an imposition of order on the members of X that
(1) singles out one member as the first (unless X is the empty set)
(2) for each member or set of members already specified, singles out its successor (unless no members are left).

Examples:

1. < ..., -2, -1, 0, 1, 2, ... >   whole numbers
   - *Not a well-ordering*: Doesn't specify a first.

2. < 0, ..., 1/4, ..., 1/2, ..., 1, ..., 1 1/2, ..., 2, ... >   non-negative rational numbers
   - *Not a well-ordering*: Doesn't specify a successor of 0.

3. $\langle \underbrace{0, 1, 2, 3,}_{\text{naturals}} \underbrace{\ldots, -3, -2, -1}_{\text{negative wholes}} \rangle$ whole numbers

   - *Not a well-ordering*: Doesn't specify a successor to $\mathbb{N}$.

4. < 0, 1, 2, 3, ... >   natural numbers $\mathbb{N}$
   - *Well-ordering*.

   no successor specified at the end; but this is okay, since no members are left.

5. < 1, 2, 3, ..., 0 >   naturals $\mathbb{N}$
   - *Well-ordering*. (0 is specified as the successor to all the non-negative naturals.)

6. $\langle \overbrace{0, 2, 4, 6, \ldots,}^{\text{even naturals}} \overbrace{1, 3, 5, \ldots}^{\text{odd naturals}} \rangle$ naturals $\mathbb{N}$

   some still left (i.e., odds)

   successor to all evens

   none left -- no need for successor

   - *Well-ordering*.

Note: Well-orderings of different sets may have the same "shape" ("length").

examples:			
	< 1, 2, 3, 4, ..., 0 >	well-ordering of $\mathbb{N}$	same "shape"/"length"
	< 3, 5, 7, ..., 1 >	well-ordering of odd naturals	
	<1, 2, 4, 5, ..., 0, 3>	well ordering of $\mathbb{N}$ with "one more member"	slightly "longer"!

Ordinal numbers: measure "length" of well-ordered sets in correlation sense:

By definition, ordinals are well-ordered:

(i) One ordinal is first.

(ii) For each ordinal, there is another which is its successor.

(iii) For each set of ordinals (finite or infinite), there is an ordinal that succeeds them all.

- First ordinals are the natural numbers { 0, 1, 2, 3, ... }.
- The ordinal that succeeds $\mathbb{N}$ is called "ω".

the "length" of $\mathbb{N}$ *not* a natural number

- Next ordinal is called "$\omega + 1$". just a name at this point; *not* a sum

  Next is called "$\omega + 2$". length of < 1, 2, 3, 4, ..., 0 >

  etc... length of < 1, 2, 4, 5, ..., 0, 3 >

- The ordinal that succeeds all "ω + _" ordinals is called "$\omega + \omega$" or "$\omega \times 2$". length of < 0, 2, 4, ..., 1, 3, 5, ... >

- Next is "$(\omega \times 2) + 1$". length of < 2, 4, 6, ..., 1, 3, 5, ..., 0 >

  etc...

<u>A small part of the ordinals</u>:

$0, 1, 2, 3, \ldots$

$\omega, \omega + 1, \omega + 2, \ldots$

$\omega \times 2, (\omega \times 2) + 1, (\omega \times 2) + 2, \ldots$

$\omega \times 3$

$\vdots$

$\omega^2, \omega^2 + 1, \omega^2 + 2, \ldots$

ω^3

$\vdots$

$\omega^\omega, \omega^\omega + 1, \omega^\omega + 2, \ldots$

$\vdots$

$\omega^{\omega^\omega}, \ldots, \omega^{\omega^{\omega^\omega}}, \ldots$

First ordinal to succeed all of these is called "ε_0".

<u>Claim</u>: There are as many ordinals preceding ω as there are preceding ε_0.

<u>Burali-Forti Paradox</u>

Consider the set of all ordinals -- call it Ω. As a set, Ω must have a "length".

<u>SO</u>: There must be an ordinal that "measures" this "length".

<u>BUT</u>: It can't be in Ω (no members of Ω are big enough).

<u>SO</u>: Ω is *not* the set of all ordinals! (No set can be "big enough" to contain all the ordinals.)

<u>OR</u>: Since Ω is a set of ordinals, there must be an ordinal that is the successor to Ω.
So Ω cannot be the set of all ordinals.

Possible Responses:

Burali-Forti: Ordinals cannot themselves be well-ordered.

Cantor: No such set as Ω.
Some totalities are too big to be considered as sets! ← Also Cantor's response to Russell's *R*

These totalities are "absolutely infinite".

"inconsistent totalities"
"truly infinite"

maybe "well-behaved" sets like $\mathbb{N}$ aren't really infinite after all

Problem: Are we back to the basic distinction between

(a) finite sets (now including "well-behaved" infinities labeled by ordinals)

(b) infinite sets (misbehaving "true" infinities)

Cantor's ordinals are used to resolve paradoxes of the Infinitely Big.

BUT: Are these paradoxes just replaced by paradoxes of the One and the Many?