Moore Chap 4: The Calculus

Topics
I. The Calculus and Infinitesimals
II. The Status of Infinitesimals
III. The Concept of a Limit
IV. Limits and Infinite Sums

I. The Calculus and Infinitesimals

2 Sets of Problems

Properties of Curves
- area under a curve
- slope of tangent to curve
- *etc.*

(*geometrical*)

Continuously-varying Quantities
- *example*: How do distance and speed vary under constant acceleration?

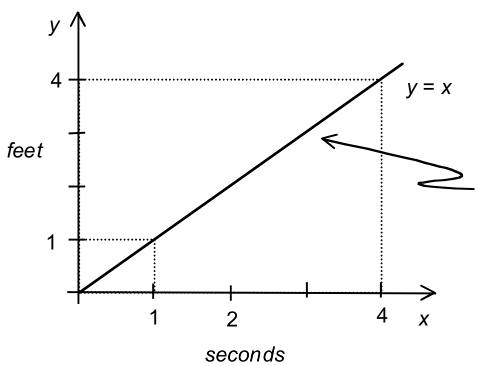
(*motion*)

Example 1:

First: constant speed motion:

Constant speed motion = motion for which the rate at which distance changes per time is constant.

Can represent geometrically by a curve on a graph with vertical axis labeled in units of distance and horizontal axis labeled in units of time. One example:



Curve $y = x$ (*i.e.*, straight line) represents relation between time and distance for object moving at constant speed of x feet in x seconds.

slope = rise/run = distance/time = speed

For such a graph:

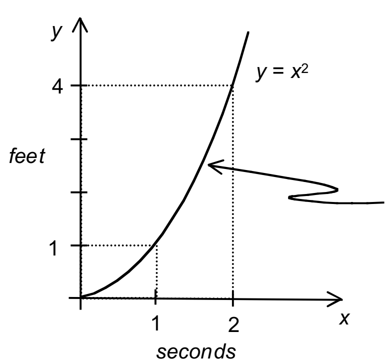
constant slope = constant speed

Now: Accelerated (non-constant speed) motion:

Accelerated motion = motion for which the rate at which distance changes per time (speed) is itself changing; *i.e.*, non-constant speed.

From above: **non-constant speed = non-constant slope**

One example:



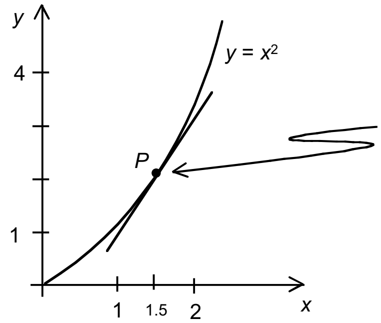
Curve $y = x^2$ represents relation between time and distance for object moving at constant acceleration of x^2 feet in x seconds.

slope is non-constant: rise/run is not constant

i.e., object's speed is constantly changing between 0 seconds and 2 seconds

Question: For accelerated motion, what is the object's speed at a given *instant*? (Say, at $x = 1.5$ seconds?)

Geometrically:

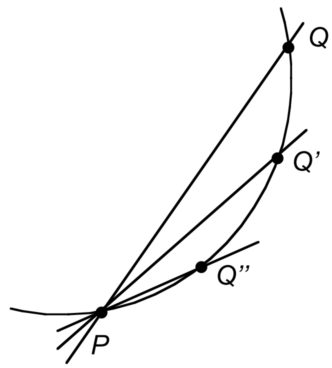


What is the slope (speed) of the *tangent line* to the curve $y = x^2$ at $x = 1.5$ sec?

In general, how do you determine the slope of the tangent line to a curve at a point *P*?

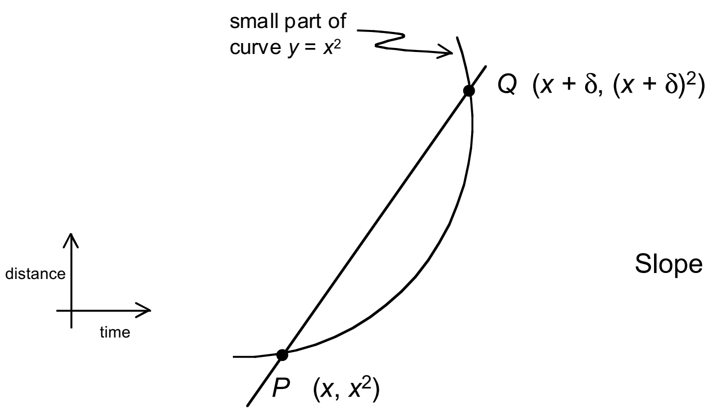
Claim: Tangent at *P* is the line joining *P* to a point *Q* that is *infinitely close* to *P*.

Idea: As *Q* approaches *P*, the line joining *Q* to *P* approaches the tangent line at *P*



Can now calculate the slope of the tangent to P in the following manner:

Assume *Q* is infinitely close to *P* and both are on curve $y = x^2$:



Let δ be an *infinitely small* ("***infinitesimal***") quantity

$$\text{Slope of } \overline{PQ} = \frac{\text{rise}}{\text{run}} = \frac{(x+\delta)^2 - x^2}{(x+\delta) - x}$$

$$= \frac{(x^2 + 2x\delta + \delta^2 - x^2)}{\delta}$$

$$= 2x + \delta$$

infinitely small quantity: ***Ignore it***!

<u>SO</u>: Slope of $\overline{PQ} = 2x$

<u>SO</u>: "instantaneous" speed of object at 1.5 sec is $2 \times 1.5 = 3$ ft/sec

Coherent Reasoning?

What exactly is δ?

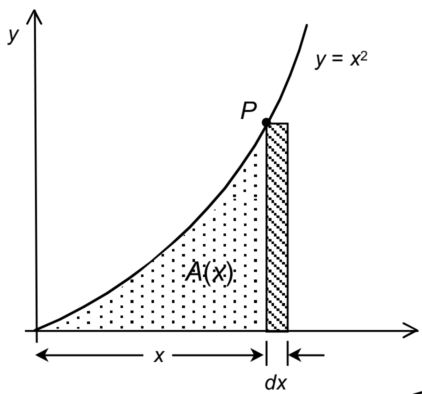
(a) Enough like 0 to be discarded?

(b) But can't be exactly 0!

Can't divide by 0:
If we could, then since $n \times 0 = m \times 0$ for any numbers n, m, we would have $n = m$ for any n, m.

Example 2:

Find area $A(x)$ under curve $y = x^2$.



Reasoning:

[An infinitesimal increase in x (call it dx)] *leads to* [An infinitesimal increase in area $A(x)$ (call it dA) by an infinitely thin strip of height x^2 and width dx]

OR: $dA = x^2 dx$ (area of infinitely thin strip)

SO: $\dfrac{dA}{dx} = x^2$ Divide increase in $A(x)$ by increase in x: get x^2.

The function $A(x)$ is the *anti-derivative* or the *indefinite integral* of x^2.

As the *anti-derivative* of x^2, $A(x)$ is found by using the method of example 1 "backwards; *i.e.*, *there* we knew the ratio on the left and wanted the value on the right; *here*, we have the value on the right and want (part of) the ratio on the left.

As the indefinite integral of x^2, view $A(x)$ as the infinite sum of infinitely small rectangles, each with infinitely small area $dA = x^2 dx$:

$$A(x) = \int dA = \int x^2 dx$$

II. The Status of Infinitesimals

Leibniz (1646-1716): "useful fictions"

Newton (1642-1727): "These ultimate ratios with which the quantities vanish are indeed not ratios of ultimate [*sc*. infinitesimal] quantities, but limits to which the ratios of quantities vanishing without limit always approach, to which they may come up more closely than by any given difference but beyond which they can never go." - Moore pg. 65

i.e., infinitesimals are not *actual*, but *potential*.

Berkeley (1685-1753):

The Calculus is incoherent:

The Analyst; or, A Discourse addressed to an Infidel mathematician. Wherein it is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

L' Hôpital (1661-1704:

Calculus textbook:

"A quantity which is increased or decreased by a quantity which is infinitely smaller than itself may be considered to have remained the same."

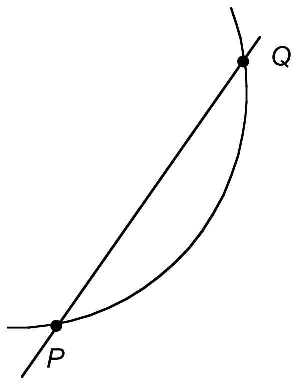
"A curve may be regarded as the totality of an infinity of straight segments, each infinitely small: or... as a polygon with an infinite number of sides. - Moore pg. 65

Cauchy (1789-1857)
Weierstrauss (1815-1897)
} provide rigorous foundations for the Calculus based on the concept of a limit

think of as representing potential infinity, not actual; a (rigorous) way of talking that does not explicitly refer to infinitesimals.

II. The Concept of a Limit

Example: Slope of tangent to curve $y = x^2$



$$\text{slope of } \overline{PQ} = \frac{2x\delta + \delta^2}{\delta}$$

Claim: Tangent to curve at P is the ***limit*** of all lines $\overline{PQ}$ as Q approaches P

This means:

(1) The smaller δ is, the closer $\frac{2x\delta + \delta^2}{\delta}$ is to the slope of the tangent; *and*,

(2) You can get as *arbitrarily close* to the slope of the tangent as you care to specify:

> For any number ε no matter how small, you can always find a (finite!) number δ such that $\frac{2x\delta + \delta^2}{\delta}$ lies within ε of the slope of the tangent.

Terminology: "$\lim\limits_{\delta \to 0} \frac{2x\delta + \delta^2}{\delta} = 2x$" means "The limit as δ goes to zero of $\frac{2x\delta + \delta^2}{\delta}$ is $2x$".

Two points:

(1) δ is *never* zero -- it's always a finite number.

(2) δ is *never* "fully" infinitesimally small -- it's always a finite number.

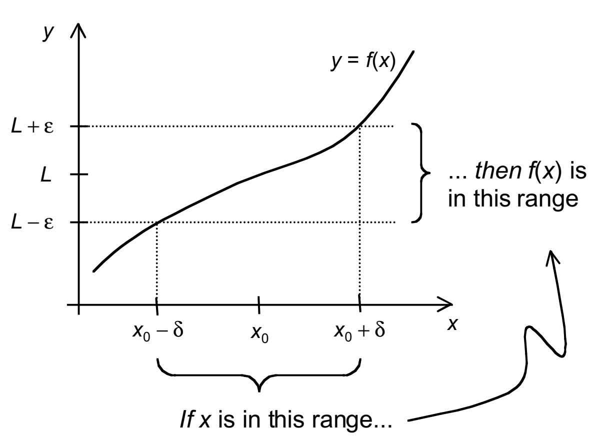
"ε-δ" Definition of a Limit (Cauchy)

Let $f(x)$ be defined in a neighborhood of x_0. Then,

$$\lim_{x \to x_0} f(x) = L$$

if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that,

if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.



> No matter how small ε is chose, δ can be chosen small enough so that $f(x)$ is within ε of L.

III. Limits and Infinite Sums

The concept of a limit allows a precise definition of an infinite sum.

Note first: Addition is only defined for "finite" input.

Recall: In the formal system of arithmetic, addition is simply a 2-place function; it takes two pieces of input and outputs a sum.

SO: The following "infinite" sum makes no sense without further ado:

$$1/2 + 1/4 + 1/8 + \ldots \equiv \sum_{k=1}^{\infty} 1/2^k$$

BUT: We *can* calculate "***partial sums***": $S_n \equiv \sum_{k=1}^{n} 1/2^k$

$S_1 = 1/2$

$S_2 = 1/2 + 1/4 = 3/4$

$S_3 = (1/2 + 1/4) + 1/8 = 7/8$

$S_4 = ((1/2 + 1/4) + 1/8) + 1/16 = 15/16$

all of these partial sums are *finite* sums: only involve 2 pieces of input

Now form a *sequence* of all these partial sums:

$$\{\ 1/2, 3/4, 7/8, 15/16, \ldots., S_n, \ldots\} \equiv \{S_n\} \equiv f(n)$$

where each member is given by the *function* $f(n) = \dfrac{2^n - 1}{2^n}$

Define the ***infinite sum*** S_∞ as the limit of the sequence $\{S_n\}$ of its partial sums.

Two steps:

(1) (Definition of the limit of a sequence) The sequence $\{S_n\}$ generated by the function $f(n)$ has a limit L if, for every $\varepsilon > 0$, there is an $N > 0$ such that

$$\text{if } n \geq N, \quad \text{then } |f(n) - L| < \varepsilon.$$

Write: $\lim_{n\to\infty} \{S_n\} = L$.

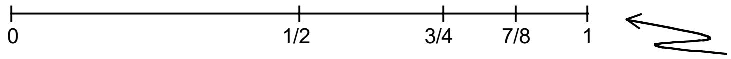
$\{S_n\}$ has a limit L if there's a point N in the sequence afterwhich $f(n)$ stays within ε of L, for any ε.

(2) (Definition of an infinite sum) The infinite sum $\sum_{k=1}^{\infty} g(k) \equiv S_\infty$ is the limit of the sequence of its partial sums, if such a limit exists:

$$S_\infty = \lim_{n\to\infty} \{S_n\}.$$

Can now address Zeno's Paradoxes:

Runner Paradox:



Racetrack: Starting line at 0 and finish-line at 1.

Claim: Achilles will never cross the finish-line.

Assumptions: (a) The track is infinitely divisible.

SO: It's length = an infinite sum of finite pieces

(b) An infinite sum of finite pieces is infinite.

Aristotle: Rejected (b). As an infinite sum of finite pieces, the racetrack is *potentially* infinite, and not *actually* infinite.

Calculus: Rejects (b). Some infinite sums are finite (depends on whether or not the sequence of their partial sums has a finite limit).

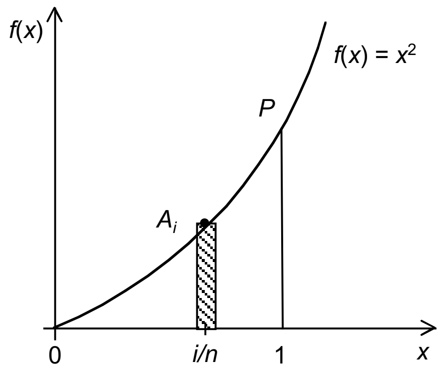
Runner case: $S_\infty = 1/2 + 1/4 + 1/8 + \ldots \equiv \sum_{k=1}^{\infty} 1/2^k$

The sequence of partial sums $\{S_n\}$ is generated by the function $f(n) = \frac{2^n - 1}{2^n} = 1 - 1/2^n$

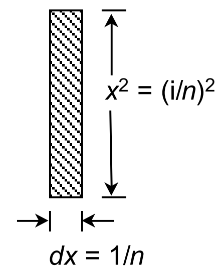
So $S_\infty = \lim_{n \to \infty} (1 - 1/2^n) = 1$. (*i.e.*, as we expect, the racetrack's length is 1)

Optional

How to calculate the *definite* integral $\int_0^1 dA = \int_0^1 x^2 dx$ (using the notion of a limit)



*i*th rectangle:



Divide interval [0, 1] into n segments of equal length $dx = \frac{1-0}{n} = 1/n$

Construct n rectangles with these segments as bases.

Identify the location of the ith rectangle as at x = i/n (the endpoint of the ith segment).

SO: The height of the ith rectangle is $f(i/n) = (i/n)^2$.

AND: The area A_i of the ith rectangle is $A_i = (i/n)^2(1/n)$

Total area A = sum of all A_i as n gets very large:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} A_i = \lim_{n\to\infty} \sum_{i=1}^{n} (i/n)^2 1/n$$

$$= \lim_{n\to\infty} 1/n^3 \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} 1/n^3 \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{2 + (3/n) + (1/n^2)}{6} = 1/3$$

Riemann (1826-1866) Integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n}$$

where x_i^* is the location of the ith rectangle

for continuous functions $f(x)$