Algebraic Substantivalism and the Hole Argument

JONATHAN BAIN

Department of Humanities and Social Sciences
Polytechnic University
Brooklyn, NY 11201
jbain@duke.poly.edu

word count: 14362

ABSTRACT. Algebraic substantivalism, as an interpretation of general relativity formulated in the Einstein algebra formalism, avoids the hole argument against manifold substantivalism. In this essay, I argue that this claim is well-founded. I first identify the hole argument as an argument against a specific form of semantic realism with respect to spacetime. I then consider algebraic substantivalism as an alternative form of semantic realism. In between, I justify this alternative form by reviewing the Einstein algebra formalism and indicating the extent to which it is expressively equivalent to the standard formalism of tensor analysis on differential manifolds.

Draft

1 Introduction

In this essay, I look at algebraic substantivalism as a way to avoid the hole argument. Algebraic substantivalism is an interpretation of the Einstein algebra formalism invented by Geroch ([1972]) and most recently developed by Heller ([1992]), and Heller and Sasin ([1995a], [1995b], [1994]). General relativity (GR) is usually formulated in tensor analysis on differential manifolds (TADM), but Geroch ([1972]) and Heller ([1992]) have demonstrated that it can be rendered in the algebraic formalism as well. In Section 2, I set the stage by reviewing the hole argument and identifying the particular interpretation of GR formulated in TADM that falls victim to it, viz. manifold substantivalism. In Section 3, I review the notion of an abstract Einstein algebra as given in Heller ([1992]) and indicate how Geroch's ([1972]) original definition fits into the more abstract scheme of the former. It will be seen that any TADM model of GR is fully equivalent to an Einstein algebra; however the converse is not the case - there are Einstein algebras that cannot be represented by TADM models of GR. In Section 4, I further substantiate this point by reviewing how singular spacetimes may be described within the framework of Einstein algebras and consider the advantages of doing so. Finally, in Section 5, I consider Earman's ([1989], [1986], [1977]) version of algebraic substantivalism and review Rynasiewicz's ([1992]) critique of its application to the hole argument. I then suggest an alternative version of algebraic substantivalism that avoids this critique. This version may be seen as taking its cue from recent discussions on the viability of a structural realist position in the debate over scientific realism. My conclusion is that algebraic substantivalism, adequately construed, legitimately avoids the hole argument.

2 Formalism, Interpretation and the Hole Argument

The Hole Argument is, in general, an argument against a particular form of scientific realism with respect to spacetime. I take scientific realism to be composed of two components: a semantic component and an epistemic component. The semantic component characterizes the realist's desire to read the theoretical claims of certain theories literally, while the epistemic component characterizes the realist's contention that there can be good reasons to believe the theoretical claims of certain theories.[1] If we take the semantic component seriously, then the manner in which a theory is formulated is important. In particular, to be a semantic realist with respect to spacetime entails advocating some sort of literal construal of a theory of spacetime presented in a particular formalism. When the formalism is that of tensor analysis on differential manifolds (TADM), one such literal construal is provided by the position known as manifold substantivalism.

2.1 Semantic Realism and Manifold Substantivalism

In the following, I will implicitly distinguish between: (a) a theory; (b) the formalism in which a theory is presented; and, (c) an interpretation of a theory. For simplicity's sake, I will identify a theory with a set of dynamical equations. Hence I will identify GR with the Einstein equations. A model of a theory with respect to a formalism is then a solution to the set of dynamical equations rendered in that formalism. In particular, a TADM model of GR is a pair (M, g), where M is a smooth (i.e., C^{∞}) differential manifold (with all the attendant properties: 4-dim, connected, Hausdorff, paracompact and without boundary), and g is a metric field defined on M and satisfying the Einstein equations. The question of what a formalism amounts to is a bit more tricky. Again, for simplicity's sake, I will take a mathematical formalism to be a category - very generally, a collection of objects and maps between the objects.[2] The TADM formalism can then be identified with the category of smooth differential manifolds. A TADM model of GR is a specific element of this category; namely, a smooth differential manifold that satisfies the Einstein equations (i.e., a manifold on which is defined a metric that satisfies the Einstein equations). Finally, I understand an interpretation of a theory to involve two things: (1) A description of what the world would be like if the theory were true; i.e., an answer to the question, What does the theory say about the world?; and, collaterally, (2) A way of (informally) declaring what parts of the formalism the theory is presented in represent real things in the world, according to the theory.

In this context, manifold substantivalism is a particular interpretation of models of field theories presented in a particular formalism (TADM). Manifold substantivalism holds that the differential manifold that appears in models of spacetime theories in the TADM formalism should be interpreted realistically (i.e., should be literally construed). In particular, it holds that manifold points have real correlates in the world; namely, spacetime

[1] This characterization follows those of Earman ([1993]) and Horwich ([1982]). The distinction between the components, as I see it, roughly follows the distinction between the two separate enterprises of interpretation and confirmation. For any theory T, the former enterprise endeavors to give an account of what the world would be like if T were true. The latter enterprise endeavors to give an account of the conditions under which we are justified in believing T's claims. The scientific realist should be able to give accounts of both endeavors.

[2] In barest outline, a category **C** consists of a collection of objects and a collection of morphisms between them such that composition is defined on the latter. This amounts to a rule that assigns to any two morphisms $f: C_1 \to C_2$ and $g: C_2 \to C_3$, the morphism $g \circ f: C_1 \to C_3$, where C_1, C_2, C_3 are objects in **C**.

Draft

points. Semantic realist sympathies have been influential in informing the manifold substantivalist position. One way to make such sympathies explicit is as follows:

A central motivation for a manifold substantivalist interpretation of GR comes from the way physical fields are represented in TADM (see, e.g., Earman [1989], p. 158; Field [1989], p. 181). A physical field is represented by a C^∞-tensor field. This latter is a mathematical object that quantifies over the points of M.[3] This property allows such an object to be acted on by differential operators defined on M, and it is the action of differential operators on such tensor fields that give us the local differential equations of physics that model natural phenomena. By definition, mathematical tensor fields can only exist if there is an underlying manifold M to support them.

To move from this observation to the manifold substantivalist claim that the points of M represent real, substantival spacetime points, it suffices to adopt two semantic realist assumptions. The first claims that realism with respect to a given physical object entails a literal construal of the representation of that object provided by the formalism in which the theory governing the behavior of the object is presented. The second is the following principle of ontological commitment: Ineliminable quantification over a type of object requires ontological commitment to it (see, e.g., Horwich [1978], p. 410; Butterfield [1989], p. 2). The upshot then is, if you are a semantic realist with respect to fields, you have to be a realist with respect to spacetime points. This argument for manifold substantivalism may be reconstructed as follows:

(1) Realism with respect to fields entails a literal construal of the representations of fields provided by the formalism of the theory under consideration.
(2) Ineliminable quantification over a sort of object entails ontological commitment to it.
(3) Representations of fields require ineliminable quantification over manifold points.

∴ Realism with respect to fields entails an ontological commitment to manifold points.

Think of (1) as a statement of semantic realism. Think of (2) as an interpretive principle that informs the semantic component of realism. Think of (3) as intended as a purely mathematical observation.[4]

2.2 The Hole Argument

Manifold substantivalism, then, minimally involves an ontological commitment to manifold points. If it is further characterized by the claim that diffeomorphically related TADM models represent distinct possible worlds (what is known in the literature as the denial of "Leibniz Equivalence"), then the hole argument (Earman and Norton [1987]) indicates that, in the context of general relativity, it is at odds with determinism. One first notes that solutions to the Einstein equations are unique up to diffeomorphism. A "hole" diffeomorphism is then

---

[3]Under the standard definition, a vector field (i.e., a (1, 0)-tensor field) on M is a map $X : M \rightarrow T(M)$ such that $X(p) = X_p$ is a tangent vector in the tangent space $T_p(M)$, $\forall p \in M$. In words, a vector field on M assigns a tangent vector to every point p in M. Here $T(M) = \bigcup T_p(M)$ is the tangent bundle over M. In general, an (r, s)-tensor field on M is then a map that associates to every point $p \in M$ an (r, s)-tensor in the tensor product space $T_p(M)^r_s$ (composed of r copies of $T_p(M)$ and s copies of its dual $T_p{}^*(M)$) above p. Literally, then, tensor fields quantify over the points of M.

[4]I do not deny that there are other motivations for manifold substantivalism, or for realism with respect to spacetime points in general. Such motivations may be linked to realist positions other than the semantic realism described above (Friedman [1983], for instance, advocates spacetime realism on the basis of an epistemic realist's appeal to unifying power). The purpose of this essay is simply to indicate that the semantic realist who *is* motivated by the realist convictions above (i.e., premises 1 and 2), need not despair over the Hole Argument.

constructed that is the identity outside a region of the manifold (the hole) and smoothly differs from the identity inside the region. Two TADM models of GR related by a hole diffeomorphism agree everywhere outside the hole and differ inside. In the context of GR, this constitutes a violation of determinism: specification of the conditions outside the hole via the Einstein equations fails to uniquely specify the conditions inside the hole.

To make this a bit more explicit requires fleshing out the notion of determinism at stake. I will take determinism to be a property of possible worlds in the following sense: Let **W** be the collection of all physically possible worlds and let w_i stand for an initial segment of the possible world w. Then $w \in \mathbf{W}$ is deterministic just when, for all $w' \in \mathbf{W}$,

$$(w_i =_A w_i') \Rightarrow (w =_A w') \qquad \text{(Det)}$$

In words, if w and w' agree on initial segments, then they agree everywhere. Particular versions of (Det) will depend on how the relation $=_A$ of agreement between worlds is cashed out. To translate this definition in terms of possible worlds into a definition that applies to models of a theory T in a given formalism F, we need a way of stating how such models represent possible worlds. This may be given by a map f, from the space $\mathbf{M_F}$ of F-models of T to the space **W** of possible worlds (i.e., f is what might be considered a first-run attempt at formalizing the notion of an interpretation). The preimage under f of $=_A$ is simply the Leibniz Equivalence relation $=_L$. Let $\mathfrak{m}, \mathfrak{m}' \in \mathbf{M_F}$. Then to say that $\mathfrak{m}$ and $\mathfrak{m}'$ represent the same possible world is to say that they are Leibniz Equivalent: $\mathfrak{m} =_L \mathfrak{m}'$. In general, then, an intepretation f of a theory T in a formalism F is deterministic just when,

$$(\mathfrak{m}_i =_L \mathfrak{m}_i') \Rightarrow (\mathfrak{m} =_L \mathfrak{m}') \qquad \text{(Det)}_F$$

where $\mathfrak{m}_i$ and $\mathfrak{m}_i'$ are the pull-backs under f of w_i and w_i'.

This can now be specialized to models of GR in the TADM formalism. Initial segments of such models can be given by appropriate restrictions to Cauchy surfaces defined on the manifolds that appear in them. Let m and m' be TADM models of GR that share a common manifold M, and let S be a Cauchy surface on M. Then determinism requires,

$$(m|_S =_L m'|_{S'}) \Rightarrow (m =_L m') \qquad \text{(Det)}_{TADM}$$

In words: If the restrictions $m|_S$ and $m'|_{S'}$ represent the same physically possible world, then m and m' represent the same physically possible world.

Insofar as the manifold substantivalist denies Leibniz Equivalence for diffeomorphic TADM models (i.e., she claims that $m \neq_L m'$ for any m, m' related by a diffeomorphism), she violates (Det)_{TADM} when $m' = \phi^*m$, where ϕ is a hole diffeomorphism. Informally, then, the Hole Argument can be given by the following:

$$\text{(Manifold Substantivalism)} \Rightarrow \sim\text{(LE)}_{TADM} \Rightarrow \sim\text{(Det)}_{TADM}$$

where $\sim\text{(LE)}_{TADM}$ denotes the claim that diffeomorphic TADM models of GR represent distinct physically possible worlds (i.e., are not Leibniz Equivalent). This suggests three distinct options for avoiding the hole argument:

(A) Reject Manifold Substantivalism.

(B) Reject the implication from Manifold Substantivalism to Denial of Leibniz Equivalence.

(C) Reject the implication from Denial of Leibniz Equivalence to Indeterminism.

Draft

This essay is concerned with a particular variant of Option (A), a variant that attempts to uphold the realist intuitions underlying manifold substantivalism, and hence, arguably, can be considered as an alternative version of substantivalism.[5] This alternative version claims that it is possible to represent physical fields in other formalisms in which differential manifolds do not appear. Hence, we are not forced to adopt the manifold substantivalist position on the basis of the semantic realist's position described above. In particular, we can reject Premise (3) in the above reconstructed argument for manifold substantivalism, and yet still retain the semantic realist intuitions underlying Premises (1) and (2).

One particular version of the resulting semantic realism with respect to spacetime is what I will call algebraic substantivalism. As with manifold substantivalism, it is a particular interpretation of models of GR in a particular formalism. In this case, the latter is what I will call the Einstein algebra (EA) formalism. In Sections 3 and 4 below, I provide the details of EA, indicating the extent to which it is expressively equivalent with the TADM formalism. In Section 5, I will attempt to identify the interpretive principles that inform an algebraic substantivalism that is robust enough to avoid the Hole Argument.

3 Formalism: Einstein Algebras

To begin, note that the following chain of successive structures appears in the TADM formalism in the construction of a model of a field theory:

set of points $\rightarrow$ topology $\rightarrow$ maximal atlas $\rightarrow$ differential manifold

Simply put, the Einstein algebra formalism is based on the alternative chain,

commutative ring $\rightarrow$ differential structure $\rightarrow$ differential manifold

Specifically, the Einstein algebra formalism takes advantage of an alternative to the standard definition of a differential manifold as a set of points embued locally with topological and differential properties. The manifold substantivalist's gloss of this definition awards ontological status to the sub-sets of points comprising local regions on the manifold. It is on these regions that the differential equations of field-theoretic physics are defined.

The alternate definition emphasizes the differential structure that supports field-theoretic equations, as opposed to the regions of M on which such structure is predicated. It is motivated by the following considerations: The set of all C^∞-real-valued functions on a differential manifold M forms a commutative ring $C^\infty(M)$ under pointwise addition and multiplica-

[5]On the other hand, one can take Option (A) by rejecting spacetime realism *in toto* and subscribing to a relationalist interpretation of GR (see, e.g., Belot [1999], and references therein). Other responses to the Hole Argument appearing in the literature take the forms of Options (B) and (C). In particular, Option (B) observes that the Hole Argument requires that the manifold substantivalist must hold that TADM models m and m' related by a hole diffeomorphism are *ontologically distinct*, hence they represent *different possible worlds*. Option (B)'ers attempt to sever the link between ontological distinctness and modal distinctness by engaging in metaphysical excursions into the notions of identity and possibility. Ontologically intrepid B'ers claim (Bi) Realism with respect to spacetime points need not entail that m and m' are ontologically distinct (e.g., Hoefer [1996]). Modally intrepid B'ers claim (Bii) If m and m' are ontologically distinct, it need not follow that they represent different possible worlds (e.g., Maudlin [1990]; Butterfield [1989], [1984]; Brighouse [1994]).

Option (C) has been advocated by those who subscribe to a "syntactic" solution to the hole argument. Such folk claim that determinism is a formal property of a theory; in particular, it does not depend on how the theory is interpreted. Hence the type of indeterminism that is entailed by manifold substantivalism must be a phony type that we shouldn't be worried about (Leeds [1995]; Mundy [1992]). Another version of Option (C) claims that denial of Leibniz Equivalence merely entails a purely semantic underdetermination of co-intended interpretations of a formal language, and has nothing to do with physical indeterminism (Liu [1996]; Rynasiewicz [1994], [1992]). This version views the hole argument as a variant of Quine- and Putnam-style inscrutability of reference arguments.

tion.[6] Let $C^c(M) \supset C^\infty(M)$ be the subring of constant functions on M. A derivation on the pair $(C^\infty(M), C^c(M))$ is a map $X : C^\infty(M) \to C^\infty(M)$ such that $X(af + bg) = aXf + bXg$ and $X(fg) = fX(g) + X(f)g$, and $X(a) = 0$, for any $f, g \in C^\infty(M)$, $a, b \in C^c(M)$. The set $\mathcal{T}(M)$ of all such derivations on $(C^\infty(M), C^c(M))$ forms a module over $C^\infty(M)$ and can be identified with the set of smooth contravariant vector fields on M. A metric g can now be defined as an isomorphism between the module $\mathcal{T}(M)$ and its dual $\mathcal{T}^*(M)$. Tensor fields may be defined as multi-linear maps on copies of $\mathcal{T}(M)$ and $\mathcal{T}^*(M)$, and a covariant derivative can be defined with its associated Riemann tensor. Thus all the essential objects of the TADM formalism necessary to build a model of GR may be constructed from a series of purely algebraic definitions based ultimately on the ring $C^\infty(M)$. Geroch's ([1972]) observation at this point is that the manifold only appears initially in the definition of $C^\infty(M)$. This suggests viewing C^∞ and C^c as algebraic structures in their own right, with M as simply one representation of them. To see how this may be worked out explicitly, I now follow Heller ([1992], pp. 278-283) and construct an abstract algebraic model of GR from the ground up with no reference at all to a manifold M.

Let $\mathcal{C}$ denote the abstract linear algebra $(\mathcal{C}+\cdot, \mathbb{K}+\cdot, *)$, where $(\mathcal{C}+\cdot)$ is a commutative ring and $(\mathbb{K}+\cdot)$ is a field. A $\mathcal{C}$-module is a module $\mathbb{W}$ over the ring $(\mathcal{C}+\cdot)$. Let $\mathfrak{X}(\mathcal{C})$ be the $\mathcal{C}$-module of $\mathcal{C}$-derivations. A scalar product on $\mathbb{W}$ is a non-degenerate, symmetric $\mathcal{C}$-linear map $g : \mathbb{W} \times \mathbb{W} \to \mathcal{C}$. A covariant derivative ∇ in $\mathbb{W}$ is a map $\nabla : \mathfrak{X}(\mathcal{C}) \to L_{\mathbb{R}}(\mathbb{W}; \mathbb{W})$, $X \mapsto \nabla_X$, defined by $\nabla_X(\alpha W) = X(\alpha)W + \alpha\nabla_X W$, where $X \in \mathfrak{X}(\mathcal{C})$, $W \in \mathbb{W}$, $\alpha \in \mathcal{C}$ and $\nabla_X : \mathbb{W} \to \mathbb{W}$ is a map assigning the directional derivative $\nabla_X W$ to every $W \in \mathbb{W}$ (here $L_{\mathbb{R}}(\mathbb{W}; \mathbb{W})$ is the set of $\mathbb{R}$-linear maps from $\mathbb{W}$ to itself). The Riemann tensor R of ∇ is the map $R : \mathfrak{X}(\mathcal{C}) \times \mathfrak{X}(\mathcal{C}) \to L_{\mathcal{C}}(\mathbb{W}; \mathbb{W})$, defined by $R(X, Y) \equiv R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$, where $R_{XY} \in L_{\mathcal{C}}(\mathbb{W}; \mathbb{W})$. The trace tr L of a $\mathcal{C}$-linear map $L : \mathbb{W} \to \mathbb{W}$ is the map $\mathrm{tr} : L_{\mathcal{C}}(\mathbb{W}; \mathbb{W}) \to \mathcal{C}$, defined by $\mathrm{tr}\, L = W^i(LW_i) \in \mathcal{C}$, where W^i, W_i, $i = 1, ..., n$, are bases for $\mathbb{W}$ and its dual $\mathbb{W}^*$. The Ricci tensor Ric is a $\mathcal{C}$-linear map $\mathrm{Ric} : \mathfrak{X}(\mathcal{C}) \times \mathfrak{X}(\mathcal{C}) \to \mathcal{C}$, defined by $\mathrm{Ric}(Y, Z) = \mathrm{tr}_X(R_{XY}Z)$, for any $X \in \mathfrak{X}(\mathcal{C})$. ($R_{XY}Z$ is a $\mathcal{C}$-linear map $\mathbb{W} \to \mathbb{W}$ for Y, Z fixed and X varying.)

An abstract Einstein algebra $\mathcal{A}$ is defined by Heller ([1992], pp. 281-282) as an abstract linear algebra $\mathcal{C}$ such that,

(i) The $\mathcal{C}$-module $\mathbb{W} = \mathfrak{X}(\mathcal{C})$ of all $\mathcal{C}$-derivations admits a scalar product g with Lorentz signature;

(ii) A covariant derivative ∇ exists in $\mathbb{W}$ such that $\nabla g = 0$;

(iii) $\mathrm{Ric} = 0$.[7]

Every TADM model (M, g) of a spacetime satisfying the Einstein equations is a realization of an abstract Einstein algebra $\mathcal{A}$ but not vice versa: Einstein algebras are more general. To see this explicitly, I shall now consider three concrete representations of $\mathcal{A}$, what Heller ([1992], pp. 282-283) refers to as the Gelfand, Geroch, and Sikorski representations.

3.1 Representations of Einstein Algebras

[6]See appendix for relevant definitions.

[7]In Geroch's ([1972], p. 274) original definition, an Einstein algebra (I, R, g) consists of (a) a commutative ring I, (b) a subring R of I isomorphic with the real numbers, and (c) an isomorphism g from the space D of derivations on (I, R) to its dual D*, such that the contraction property is satisfied and the Ricci tensor vanishes. To include solutions with sources, Heller modifies his condition (iii) to (iii') $\mathrm{Ein} + \Lambda g = T$, where Ein is the Einstein tensor, Λ is the cosmological constant, and T is a suitable energy-momentum tensor.

Draft

Heller first notes that every abstract linear algebra $\mathcal{A}$ admits a *Gelfand representation* defined by $\rho : \mathcal{A} \to \mathbb{K}^{\mathcal{A}^*}$, $\rho(x)(\phi) = \phi(x)$, where $x \in \mathcal{A}$, $\phi \in \mathcal{A}^*$, and $\mathcal{A}^*$ is the algebraic dual of $\mathcal{A}$ (i.e., the set of homomorphisms $\phi : \mathcal{A} \to \mathbb{K}$) and $\mathbb{K}^{\mathcal{A}^*}$ is the algebra of $\mathbb{K}$-valued functions on $\mathcal{A}^*$. Intuitively, the Gelfand representation turns the abstract object $\mathcal{A}$ into a "concrete" algebra of functions; moreover, it is universal: every representation of $\mathcal{A}$ is a subrepresentation of the Gelfand representation (Heller [1992], p. 282; Heller and Sasin [1995b], p. 3644).

Now suppose C is a subalgebra of the algebra $\mathbb{K}^M$ of $\mathbb{K}$-valued functions on an arbitrary nonempty set M.[8] If M is a completely regular space with respect to C, then the pair (M, C) is referred to as a *ringed space* over $\mathbb{K}$.[9] M can be said to be structured by C. The pair ($\mathcal{A}^*$, $\rho(\mathcal{A})$) is a ringed space over $\mathbb{K}$, which Heller refers to as an Einstein ringed space.[10] A C^∞-differential manifold (M, $\mathfrak{A}$), where M is a topological space and $\mathfrak{A}$ is a maximal atlas on M, defines a ringed space (M, C^∞(M)), where C^∞(M) is the algebra of smooth $\mathbb{R}$-valued functions on M. This motivates what Heller calls the *Geroch representation* of an Einstein algebra, defined by the map $\kappa : \mathcal{A} \to C^\infty(M)$, where $M \subset \mathcal{A}^*$, and $C^\infty(M) \subset \mathbb{R}^M$ is the set of smooth $\mathbb{R}$-valued functions on M. A space structured by the Geroch representation of an Einstein algebra is referred to by Heller as a *Geroch ringed space* (M, C^∞(M)) and was seen to be isomorphic to a TADM model of GR by explicit construction above.[11] While all TADM models of GR are isomorphic to Geroch ringed spaces, not all spaces structured by representations of Einstein algebras are isomorphic to TADM models of GR, as can be seen by the following considerations.

Any differential manifold defines a ringed space (M, C^∞(M)). Conversely, to turn an arbitrary ringed space (M, C) into a differential manifold, it suffices that the following conditions are satisfied:

(i) C is closed under localization. (Let f be a function defined on a region $A \subset M$, where M is endowed with the topology τ_C induced by C. f is a *local C-function* if for every point $p \in A$ there is a neighborhood B of p in the topological space (A, $\tau_C|_A$) and a function $g \in C$ such that $f|_B = g|_B$ (where $\tau_C|_A$ is the topology induced by the restriction of τ_C to A and $f|_B$, $g|_B$ are the restrictions of f and g to B). C is closed under localization just when $C = C_M$, where C_M is the set of all C-local functions defined on M.)[12]

Draft

---

[8]In the following, M will, in general, denote an arbitrary nonempty space, or a topological space, and the pair (M, $\mathfrak{A}$), where $\mathfrak{A}$ is a maximal atlas, will denote a differential manifold. It should be clear by context what the letter M denotes.

[9]M is a completely regular space with respect to a family of functions C if M is Hausdorff and for every open set $A \subseteq M$ and for every $p \in A$, there is a "bump" function $f \in C$ such that f(p) = 1 and $f|_{(M - A)} = 0$. C is then said to separate points in M (see, e.g., Gillman and Jerison [1960], p. 36).

[10]The Gelfand representation can always be made to separate points in $\mathcal{A}^*$. Note also that it is an established result that the maximal ideals of $\mathcal{A}$ are in 1-1 correspondence with the elements of $\mathcal{A}^*$ (i.e., the "characters" of $\mathcal{A}$). This anticipates Rynasiewicz's observations below in Section 5.2.

[11]A Geroch ringed space defines a smooth differential manifold that satisfies the Einstein equations. Hence perhaps it is more accurate to denote it by the triple (M, C^∞(M), g), where g is a scalar product defined on derivations on C^∞(M) and satisfying the Einstein equations. In the following, however, I shall stick with the notation introduced by Heller and denote a Geroch ringed space by (M, C^∞(M)), with the understanding that it is not just a differential manifold (i.e., a space structured by C^∞(M)), but a space structured by a Geroch representation of an Einstein algebra (i.e., a space structured by C^∞(M) constrained by the Einstein algebra axioms).

[12]The topology τ_C induced by C is the weakest topology in which the functions of C are continuous. (Recall that $f : X \to \mathbb{R}$ is continuous *iff* the preimage under f of each open set in $\mathbb{R}$ is open. τ_C is then the smallest topology containing the family $\mathcal{S}$ of open sets of X that are preimages of all elements of C; i.e., a subset U of X belongs to $\mathcal{S}$ *iff* there is an $f \in C$ and an open set V in $\mathbb{R}$ such that U = f[V]. See, e.g., Gillman and Jerison [1960], p. 38.)

(ii) C is closed with respect to superposition with smooth Euclidean functions; i.e., for any integer n and any Euclidean function $\omega \in C^{\infty}(\mathbb{R}^n)$, $f_1, ..., f_n \in C$ entails $\omega \circ (f_1, ..., f_n) \in C$.

(iii) M is locally diffeomorphic to $\mathbb{R}^n$; i.e., for every $p \in M$ there is a neighborhood $V \in \tau_C$ of p and a neighborhood $U \in \tau_{C^{\infty}(\mathbb{R}^n)}$ in $\mathbb{R}^n$ such that $(U, C^{\infty}(\mathbb{R}^n)_U)$ is diffeomorphic to (V, C_V).[13]

If C only satisfies (i) and (ii), then it is referred to as a differential structure and the ringed space (M, C) is called a *differential space*. It turns out that an Einstein algebra can be represented by a differential space, and this defines the *Sikorski representation*, $\sigma : \mathcal{A} \to C$, with C a differential structure. Gruszczak, et. al. ([1988]) indicate how a Sikorski representation can be constructed. Again, this involves explicitly constructing the algebraic equivalents of the relevant tensor fields on a differential space (M, C).[14] Thus, insofar as a differential space is more general than a differential manifold, an Einstein algebra is more general than a TADM model of GR.

The above constructions may be summarized by the following sequence:

$((M, \mathfrak{A}), g)$	$\leftrightarrow$	$(M, C^{\infty}(M))$	$\to$	(M, C)
TADM model of GR		Geroch ringed space		Einstein differential space

Here a TADM model of GR is the pair $((M, \mathfrak{A}), g)$, where $(M, \mathfrak{A})$ is a smooth differential manifold ($\mathfrak{A}$ being a maximal atlas) and g a metric satisfying the Einstein equations. A Geroch ringed space is a space structured by the Geroch representation of an Einstein algebra; and an Einstein differential space is a space structured by the Sikorski representation of an Einstein algebra. The map "$\to$" may be thought of as a forgetful functor (or a partial isomorphism) that "forgets" the structure represented by condition (iii) above; while the map "$\leftrightarrow$" indicates total isomorphism.

At this point, a note on the terminology introduced in the above is in order. Heller implicitly distinguishes between a representation of an Einstein algebra $\mathcal{A}$ as a map from $\mathcal{A}$ to, in general, a space of functions (hence his Gelfand, Geroch, and Sikorski representations above); and a space structured by such a representation (respectively, an Einstein ringed space, a Geroch ringed space, or an Einstein differential space). In more common terminology, the map and the space are together referred to as a representation. Briefly, in standard parlance, a representation of an abstract object A consists of a map from that object to a collection of morphisms Mor(P', P') defined on a second object P'. The elements of A are represented by the morphisms (maps between elements of P'), the actions of which, in a sense, are being used to

---

[13]The meaning of "diffeomorphic" is the following: Let (M, C) and (N, D) be two ringed spaces satisfying (i) and (ii). A 1-1 map $f : M \to N$ is a diffeomorphism iff $h \circ f \in C$ and $g \circ f^{-1} \in D$, for each $h \in D$, $g \in C$. (M, C) and (N, D) are then said to be diffeomorphic.

The conditions (i), (ii), (iii) are those used by Penrose and Rindler ([1984], pp. 180-182) after Nomizu ([1956], p. 1). They are equivalent to the standard definition of a differential manifold in terms of the pair $(M, \mathfrak{A})$ with $\mathfrak{A}$ a maximal atlas on the topological space M. For such a manifold, one can construct C as the set of all functions $f : M \to \mathbb{R}$ such that the composition $f \circ g^{-1}$ is a smooth (C^{∞}) function for every $g \in \mathfrak{A}$. Conversely, if (M, C) satisfies (i)-(iii), then an atlas for M can be constructed as consisting of all maps $g : U \to \mathbb{R}^n$, $U \subset M$, such that $h \circ g^{-1} \in C^{\infty}(\mathbb{R}^n)$.

[14]In general, a differential space (M, C) admits a Lorentz metric and curvature tensor (and hence is potentially an "Einstein differential space"; i.e., a space structured by a Sikorski representation of an Einstein algebra) provided it is of constant differential dimension (i.e., there is an open covering $\mathcal{B}$ of M such that on every open set $B \in \mathcal{B}$ there exist n smooth tangent vector fields forming a vector basis). See Gruszczak, et. al. ([1988], p. 2579), or Heller ([1992], p. 283) and references therein.

represent the internal structure of the elements of A. More precisely, after Geroch ([1985], p. 120), let **C** and **C**' be categories (see footnote 2). A *representation* of object A in **C** consists of the pair (P', π), where P' is an object in **C**' and π : A $\rightarrow$ Mor(P', P') is a morphism in **C** that associates to every element of A an element of Mor(P', P'). Intuitively, the elements of A are represented by maps, which might be operators or functionals, on P'.

Hence, in standard terminology, the Gelfand representation of an abstract Einstein algebra $\mathcal{A}$ can be taken to consist of the morphisms defined on its algebraic dual $\mathcal{A}$*, such morphisms being elements of $\mathbb{K}^{\mathcal{A}^*}$ (i.e., functionals that take $\mathbb{K}$-valued functions on $\mathcal{A}$ to $\mathbb{K}$). Here **C** is the category of abstract Einstein algebras, **C**' is the category of Einstein ringed spaces, and the Gelfand representation map ρ associates each element of $\mathcal{A}$ with an element of $\mathbb{K}^{\mathcal{A}^*} \subset$ Mor(P', P') (where P' is an Einstein ringed space). We can then speak of the following "concrete" ringed space representations of an abstract Einstein algebra:

(1) Einstein ringed space representation: ($\mathcal{A}$*, ρ);

(2) Geroch ringed space representation: (M, κ);

(3) Einstein differential space representation: (M, σ);

where M $\subset \mathcal{A}$*, and ρ, κ and σ are Heller's Gelfand, Geroch and Sikorski representation maps.

Given that an Einstein differential space is nontrivially distinct from a TADM model of GR, it does not necessarily follow that we should henceforth become devout algebraicists. This might be argued for if the generalization to Einstein differential spaces is significant. In the next section I will indicate how this argument might proceed; in particular, one can claim that the move to differential spaces, and their generalizations, is conceptually significant when it comes to singularities. In brief, there are models of GR that can be represented by generalizations of Einstein differential spaces (referred to as Einstein structured spaces), and that cannot, strictly speaking, be formulated in TADM (i.e., equivalently - cannot be represented by Geroch ringed spaces). Such models are given by spacetimes containing nontrivial singularities. These GR models indicate that the algebraic formalism is not completely expressively equivalent to the TADM formalism in the context of GR; in fact, in one sense, the algebraic formalism is "more expressive".

Draft

4 Expressive Equivalence: Singularities

One initial worry for the algebraic substantivalist comes in the form of spacetime singularities. Under one construal, a singularity is associated with a missing point. If manifold points have been done away with in the algebraic formalism, how can such singularities be represented? It might appear that expressive equivalence fails for models of GR that admit singularities. However, Heller ([1992]) and Heller and Sasin ([1995a], [1995b], [1994]) claim that not only can singularities be represented in the algebraic formalism, but moreover, the nature of singularities becomes more easily understood if one switches to the latter. In particular, pathological features of b-incomplete singularities in some GR solutions are better understood in the algebraic paradigm. In this section, I review their programme and indicate what it entails about the expressive equivalence between TADM and the algebraic formalism. I shall argue that, strictly speaking, GR spacetimes that admit singularities can be represented in the algebraic formalism, and cannot be represented in the TADM formalism.

4.1 The b-boundary Construction in TADM

Earman ([1995], p. 28) identifies two distinct ways of conceiving the notion of a singularity. Taken as a noun, a singularity is, in some sense, a missing point; viz. a localized entity. Taken as an adjective, a singular spacetime is one which exhibits certain global features; under the most popular construal, the latter allow for the existence of incomplete curves (under a suitable notion of incompleteness). Intuitively, insofar as a differential space is a global alternative to a differential manifold, the algebraic substantivalist should fair better on the second global, adjectival construal of a singularity. This is in fact the conception Heller and Sasin work with. I now indicate first how this conception is cashed out in the TADM formalism.

The standard approach to treating singularities as incomplete curves is to collection them into a set $\partial M'$ considered as a boundary to the differential manifold M. Given a suitable notion of incompleteness, an equivalence relation ~ is defined on $\partial M'$ to divide it into classes of incomplete curves such that each class terminates at the same point. These points then constitute the singular boundary $\partial M = \partial M'/\sim$ of spacetime. Finally, a topology on $\overline{M} \equiv M \cup \partial M$ is needed to allow for "contact" with the interior M. The main task for such a project is in defining such a topology. Minimally, one requires that M be open and dense in $M \cup \partial M$.

Different notions of incompleteness have been proposed: geodesic incompleteness, bounded acceleration incompleteness, causal incompleteness, b-incompleteness (see, e.g., Earman [1995], Clarke [1993]). Schmidt's ([1971]) notion of b-incompleteness identifies a curve γ as incomplete just when γ has finite *generalized affine parameter* (g.a.p.) length.[15] The singular b-boundary $\partial_b M$, consisting of equivalence classes of b-incomplete curves is constructed in the following manner (see, e.g., Clarke [1993], pp. 34-36). One considers the bundle of orthonormal frames O(M) and forms its Cauchy completion $\overline{O}(\overline{M})$.[16] The projection map π on O(M) is then extended to $\overline{O}(\overline{M})$ and $\overline{M} \equiv M \cup \partial_b M$ is defined by $\overline{M} = \overline{O}(\overline{M})/\pi$. It can now be shown that the "bundle" boundary $\partial_b M$ consists of the endpoints of every b-incomplete curve in M.

While boundary constructions in general are motivated by a global adjectival construal of the notion of a singularity, they can also be construed as localizing singularities, not as entities in spacetime, but rather as ideal boundary points. However, this desire to retain a sense of localization in the notion of a singularity produces rather unpleasant conceptual difficulties in some b-boundary constructions. Bosshard ([1976]) and Johnson ([1977]) demonstrated that in the closed Friedman universe, the b-boundary $\partial_b M$ consists of a single point corresponding to both the initial and final singularities, and that, in the closed Friedman and Schwarzschild solutions, the b-boundary is not Hausdorff-separated from M (for details, see Clarke [1993], pp. 40-45). These results are hard to reconcile with any notion of localization. Intuitively, some degree of separation (temporally and spatially, if not causally) between the initial and final singularities in the Friedman solution should obtain. Moreover, that the points of the b-boundary are not Hausdorff-separated from the points of the interior implies counter intuitively that every event in spacetime is in the neighborhood of a singularity.

It is these conceptual pathologies that Heller and Sasin claim are explained in the algebraic formalism. To construct the b-completion of a spacetime in the algebraic formalism requires a generalization of an Einstein algebra and a corresponding generalization of a

[15] The g.a.p. λ of a curve $\gamma(t)$ is defined by $\lambda = \int(\sum_i X^i(t)^2)^{1/2}dt$, where $X^i(t)$ are components of the tangent vector $(\partial/\partial t)$ to $\gamma(t)$ with respect to an orthonormal frame field $e_i^a(t)$ that is parallelly propagated along $\gamma(t)$. The significance of g.a.p. length resides in the facts that a g.a.p. can be defined on any (C^1) curve, and, while a g.a.p. depends on choice of frame e_i^a, it can be shown that for any other frame e'^a_i and g.a.p. λ', the λ-length of a curve γ is finite if and only if the λ'-length of γ is finite. Thus the notion of b-incompleteness is well-defined.

[16] A space Ω is Cauchy complete if every Cauchy sequence in Ω converges to a point in Ω. This requires the existence of a distance function on Ω. It turns out that such a function can be defined in the frame bundle O(M) by d(x, y) = [greatest lower bound of G(x, y)], where G(x, y) is a positive definite Euclidean metric on O(M). Such a metric can be defined by constructing a basis for the tangent space to O(M). (Note that such a metric cannot be defined on the base space M insofar as it admits only an indefinite Lorentzian spacetime metric.)

differential space. These generalizations yield respectively a sheaf of Einstein algebras and a structured space. In brief, the conceptual advantages Heller and Sasin claim accrue to the algebraic formalism are the following. In moving to the category of structured spaces, one gains a conceptual unification of the notions of a singular and a nonsingular spacetime, in contrast to the differential manifold category. Second, the difficulties associated with the b-boundary construction for the closed Friedman and Schwarzschild solutions are pathologies only when viewed from within the differential manifold category and its emphasis on local properties. Within the structured space category, in contrast, the emphasis throughout is on sheaf-theoretic global features. These features allow a natural distinction between the decidedly non-local behavior of fields on the b-boundary and the local behavior of fields on the interior M.

In sum, according to Heller and Sasin, the move to structured spaces provides both conceptual unity and conceptual clarity when it comes to singularities.

4.2 Sheaves of Einstein algebras and structured spaces

Let $(\overline{M}, \tau)$ be a topological space where $\overline{M} \equiv M \cup \partial M$ with differential manifold M open and dense in $\overline{M}$. A *sheaf of Einstein algebras* is a sheaf[17] C of linear function algebras over $(\overline{M}, \tau)$ such that, for any $p \in U \in \tau$, $C(U)$ is an Einstein algebra. The pair (M, C) is called an *Einstein structured space*. It is a space structured by a sheaf of Einstein algebras.

In general, for a topological space (M, τ), a *structured space* is a pair (M, C) where C is a sheaf of real τ-continuous functions on (M, τ) such that C is closed under composition with Euclidean functions (i.e., for any open set $U \in \tau$ and any functions $f_1, \ldots, f_n \in C(U)$, and $\omega \in C^\infty(\mathbb{R}^n)$, the superposition $\omega \circ (f_1, \ldots, f_n)$ is in $C(U)$). Note that a sheaf of functions automatically satisfies closure under localization, hence a sheaf closed under Euclidean composition is automatically a differential structure (see above Section 2).[18] A structured space, then, is very much like a differential space, differing only in generality: for a structured space, there are no restrictions on the topology τ (recall that a differential space (M, C) requires M to have the topology τ_C (see footnote 12)). It can be shown that when $\tau = \tau_C$, the structured space (M, C) reduces to the differential space (M, C), where $C = C(M)$ (Heller and Sasin [1995a], p. 390). Equivalently, since $\tau = \tau_C$ if and only if C separates points in M, a structured space becomes a differential space just when C separates points in M (Heller and Sasin [1995b], p. 3647; [1994], p. 801; see also Gillman and Jerison [1960], p. 40). The upshot of moving from a differential space to a structured space is that the latter now encompasses the boundary ∂M.[19]

Finally, for any two structured spaces (M, C_1), (N, C_2), one can define a *structured map* $f : (M, C_1) \rightarrow (N, C_2)$ as a continuous map $f : M \rightarrow N$ such that, for any $g \in C_2(U)$, one has $g \circ f|_{f^{-1}(U)} \in C_1(f^{-1}(U))$. Structured spaces and structured maps together form a category, viz. the category of structured spaces.[20]

Draft

[17]For a brief outline of the notion of a sheaf, see Ward and Wells ([1990], p. 166). Intuitively, whereas a fiber bundle over a topological space X assigns a type of algebraic object to every point of X, a sheaf over X assigns a type of algebraic object to every open set U of X.

[18]A differential structure can thus be defined as a sheaf of continuous functions closed under composition with Euclidean functions.

[19]In Schmidt's b-boundary construction, singularities can be neatly ordered according to a number of types (see Earman [1995], pp. 37-40, for a bestiary). Heller and co-workers have demonstrated that spacetimes that contain regular singularities can be represented by Einstein differential spaces in the algebraic formalism. Quasi-regular singularities, and the more frightening curvature singularities, on the other hand, which motivate the move to the b-boundary construction in TADM, require a corresponding move to structured spaces in the algebraic formalism. (See, e.g., Heller [1992], p. 283, and references therein.)

[20]Heller and Sasin ([1995b]) develop differential geometry on structured spaces in a manner analogous to the development on abstract Einstein algebras and Geroch ringed spaces given in Section 2. The key ingredient in the structured space approach is the identification of a vector field as an $\mathbb{R}$-linear sheaf morphism that obeys the Leibniz rule.

4.3 The b-boundary construction in the category of structured spaces

I now review two theorems within the category of structured spaces and indicate how these are used by Heller and Sasin in their resolutions of the conceptual pathologies afflicting the b-boundary conception of singularities in the category of differential manifolds. The first theorem demonstrates how the b-boundary construction is performed in terms of structured spaces.

Theorem 1. Let (M, g) be a spacetime. On the b-completion $\overline{M}$ of M there exists the differential structure $\overline{C}$ such that $\overline{C}(M) = C^{\infty}(M)$ and if (M, g) is a solution to the Einstein equations, then $(\overline{M}, \overline{C})$ is an Einstein structured space.

Sketch of proof (Heller and Sasin [1994], [1995a], [1995b]): Given the orthonormal frame bundle O(M) equipped with a positive definite metric G, for large integers n, there exists an embedding $\iota : O(M) \hookrightarrow \mathbb{R}^n$ (G is the pull-back of the Euclidean metric on $\mathbb{R}^n$). A differential structure C(O(M)) on O(M) can be defined by the set of functions $\{\pi_1|_{O(M)}, \ldots, \pi_n|_{O(M)}\}$, where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the projection of the ith coordinate on $\mathbb{R}^n$. In the standard way, the Cauchy completion $\overline{O}(\overline{M})$ is constructed. One then defines a differential structure $(C^{\infty}(\mathbb{R}^n))_{\overline{O}(\overline{M})}$ on $\overline{O}(\overline{M})$ induced from the structured space $(\mathbb{R}^n, C^{\infty}(\mathbb{R}^n))$. One now has the structured space $(\overline{O}(\overline{M}), (C^{\infty}(\mathbb{R}^n))_{\overline{O}(\overline{M})})$. From this one can construct the b-completion of M as $\overline{M} = \overline{O}(\overline{M}) / O(3, 1)$. To get the b-boundary $\partial_b M$, define the projection $\pi : \overline{O}(\overline{M}) \to \overline{M}$. One then has $M = \pi(O(M))$. Since O(M) is open and dense in $\overline{O}(\overline{M})$, M is open and dense in $\overline{M}$. Hence $\partial_b M$ is given by $\overline{M} - M = \pi(\overline{O}(\overline{M})) - \pi(O(M))$.

Theorem 1 indicates that, in the structured space category, singular spacetimes are the same type of object as nonsingular spacetimes. Both are represented by structured spaces. In the manifold category, a singular spacetime is not the same type of object as a nonsingular spacetime. The former is represented by a manifold with boundary, which is not the same type of object (viz., does not belong to the same category) as a manifold.[21]

The next theorem, when applied to the Schwarzschild and closed Friedman solutions, is taken by Heller and Sasin to explain the results of Bosshard and Johnson.

Theorem 2. Let (M, C) be a structured space with topology τ_C (the weakest topology in which the functions in C are continuous). Let $(\overline{M}, \tau)$ be a topological space such that $\overline{M} = M \cup \{x_0\}$, $x_0 \notin M$. If the following conditions are satisfied,

(i) $\tau|_M = \tau_{C(M)}$

(ii) $(x_0 \in U \in \tau) \Rightarrow (U = \overline{M})$

then,

(1) there is no differential structure $\overline{C}$ on $(\overline{M}, \tau)$ such that $\tau = \tau_{\overline{C}(\overline{M})}$ and $\overline{C}(M) = C(M)$; and

(2) there is exactly one differential structure $\overline{C}$ on $(\overline{M}, \tau)$ such that $\overline{C}(M) = C(M)$; and one has $\overline{C}(\overline{M}) \cong \mathbb{R}$.

[21]See, e.g., Bishop and Goldberg ([1980], p. 22). Unlike a manifold with boundary, a C^{∞} differential manifold is differentiable at all points; intuitively, it has no "edge points" at which differentiation may break down.

Draft

For proofs, see Heller and Sasin ([1994], pp. 807-808; [1995b], pp. 3659-3660). The theorem remains valid if the point x_0 is replaced by a set of points X_0 such that $\overline{M} = M \cup X_0$ and $M \cap X_0 = \emptyset$. Conclusion (1) of Theorem 2 says that the differential structure on $\overline{M}$ cannot be defined in such a way that its topology is compatible with the topology on the interior M (in the sense $\tau = \tau_{\overline{C}}$). Conclusion (2) says, if this compatibility condition $\tau = \tau_{\overline{C}}$ is dropped, then $\overline{C}(\overline{M}) \cong \mathbb{R}$ is the only prolongation of the differential structure C from M to $\overline{M}$; hence only constant functions can be prolonged from M to $\overline{M}$.[22]

As indicated above, Bosshard and Johnson demonstrated that, for the closed Friedman solution, conditions (i) and (ii) are satisfied, hence conclusions (1) and (2) apply. The latter indicate that the points in X_0 are both topologically and differentiably indistinguishable from each other (i.e., they cannot be distinguished by either continuous or smooth functions). Specifically, since only constant functions can be prolonged to $\overline{M}$, the only global vector fields defined on $(\overline{M}, \overline{C})$ are zero vector fields. Thus any curve joining points in the b-boundary $\partial_b M$ with other points in $\partial_b M$ or with points in the interior M will have zero g.a.p. length. However, the restriction of the prolongation $\overline{C}(U)$, for U any open subset of M, will in general admit non-trivial local cross-sections and thus admit non-zero local vector fields. Thus $\overline{C}(M)$ will likewise be non-trivial since M is open in $\overline{M}$. Hence, from the global sheaf-theoretic perspective, the topological and differentiable "pathologies" of the b-boundary are consistent with non-pathological behavior of fields in the interior M. As Heller and Sasin state,

> The situation seems to be pathological if we look at it from within the manifold category, but if we use the category of structured spaces, we gain the global insight into what is going on... (Heller and Sasin [1994], p. 810).

Draft

The constructions described above may be summarized by the following extension of the sequence from Section 3.1:

$((M, \mathfrak{A}), g)$	$\leftrightarrow$	$(M, C^\infty(M))$	$\rightarrow$	(M, C)	$\rightarrow$	$(M, \mathcal{C})$
TADM model of GR		Geroch ringed space		Einstein differential space		Einstein structured space

Again, the map "$\rightarrow$" may be taken to be a forgetful functor that removes structure, leading us from the category of smooth differential manifolds to that of differential spaces to that of structured spaces. The significance of moving to the right in the sequence is that in doing so, one is able to include more GR models under a single category. To describe nonsingular GR models, one may remain within the category of smooth differential manifolds. To describe GR models containing regular singularities, *and remain within a single category*, one must move to the category of differential spaces.[23] To describe GR models containing regular, quasi-regular, and/or curvature singularities, *and remain within a single category*, one must move to the category of structured spaces. This indicates that, in a nontrivial sense, the TADM formalism is

[22] A prolongation of the differential structure C on M to $\overline{M}$ is a sheaf $\overline{C}$ on $\overline{M}$ such that $\overline{C}(M) = C(M)$. Such a prolongation induces prolongations of the algebraic objects derived from C (see Heller and Sasin [1994], p. 805).

[23] In the classificatory scheme associated with the b-boundary construction, regular points of the boundary $\partial_b M$ (i.e., "regular singularities") are those that can be made to vanish by a suitable global extension of the manifold M. Hence, arguably, one can represent "regular" singular spacetimes within the category of smooth manifolds, provided one stipulates that only maximally extended M's count as capable of representing spacetime. Certainly, however, for quasi-regular (which are removable only via local extensions) and curvature singularities (which cannot be "removed"), one must introduce a boundary construction, and hence move out of the smooth manifold category.

not expressively equivalent to the algebraic formalism. The latter is, in fact, "more expressive".

5 Interpretation: Algebraic Substantivalism

In this section, I will consider some of the options when it comes to providing an interpretation of a model of GR in the Einstein algebra (EA) formalism. I will be specifically concerned with substantivalist interpretations. In particular, I will be concerned with the following questions:

(1) Can the semantic realist intuitions that underlie manifold substantivalism be upheld by a suitably defined notion of algebraic substantivalism?

(2) Can such a suitably defined algebraic substantivalist interpretation of EA models of GR legitimately avoid the hole argument?

I first consider the interpretation used by Earman ([1989], [1986], [1977]) to avoid the hole argument. I then review Rynasiewicz's ([1992]) critique of Earman. Finally, I will attempt to make explicit two distinct ways to interpret models in the EA formalism.

5.1 Earman's Algebraic Substantivalism

Earman ([1989], [1986], [1977]) describes a way to avoid the hole argument that makes use of the Einstein algebra formalism. In short, Earman views the hole argument as presenting the manifold substantivalist with a proliferation of diffeomorphically related TADM models all of which she must interpret literally. Earman's algebraicist handles this TADM proliferation by declaring such diffeomorphic TADM models to be equivalent in virtue of representing a single Einstein (or, in Earman's terminology, "Leibniz") algebra. This allows her to adopt Leibniz Equivalence for TADM models and hence avoid violating $(\mathrm{Det})_{\mathrm{TADM}}$.

Earman describes this tactic as a substantivalist one:

> While the Leibniz algebras provide a solution to the problem of characterizing the structure common to a Leibniz-equivalence class of [TADM] models, and the solution eschews substantivalism in the form of space-time points, the solution is nevertheless substantival, only at a deeper level (Earman [1989], pp. 192-193).

An initial question Earman leaves unanswered is, how can a substantivalist position be defined with respect to "structure". As will be seen below, omitting the gory details leaves Earman's algebraic substantivalist open to attack.

On the surface, then, Earman's algebraic substantivalist resolution of the hole argument can be represented schematically by,

$$(\text{Algebraic Substantivalism}) \not\Rightarrow \sim(\mathrm{LE})_{\mathrm{TADM}}$$

In brief, suppose ϕ is a hole diffeomorphism and m is a TADM model of GR. Earman's algebraic substantivalist wants to claim that ϕ^*m and m are equivalent by virtue of the fact that they represent the same Leibniz algebra $\mathcal{L}$, which itself represents a single physically possible world. In short, Earman's algebraic substantivalist simply transfers her semantic realism from TADM models to EA models: Only the latter, and not the former, are to be literally interpreted (viz., taken at face value).

However, given the isomorphism between Geroch ringed spaces and TADM models of GR, the proliferation of diffeomorphically related TADM models is matched by an equinumerous proliferation of Leibniz algebras. Earman claims this latter proliferation is harmless. Whereas the TADM proliferation threatens determinism (under a manifold substantivalist interpretation), the Leibniz algebra proliferation does not. It does not, according to Earman, given that we restrict the meaning of determinism to TADM models. In particular, for a given Leibniz algebra $\mathcal{L}$, let $R(\mathcal{L})$ be the collection of all TADM representations m of $\mathcal{L}$. Determinism is then defined in the following manner (Earman [1989], p. 218, footnote 17): Let S(m) be a Cauchy surface in the TADM model $m \in R(\mathcal{L})$. Let $R(\mathcal{L})|_S$ be the restriction of the members of $R(\mathcal{L})$ to those times up to and including the Cauchy surfaces S(m) for each $m \in R(\mathcal{L})$. Then the field theory in question is deemed deterministic just when,

$$(R(\mathcal{L})|_S = R(\mathcal{L}')|_{S'}) \Rightarrow (R(\mathcal{L}) = R(\mathcal{L}'))$$

Note that this is simply a specialized version of Section 2.2's $(\mathrm{Det})_{\mathrm{TADM}}$; specialized to a particular version of Leibniz Equivalence. Hence it is safe to say that the meaning of determinism for Earman's algebraic substantivalist is the same as it is for the manifold substantivalist. Again, the only difference is that Earman's algebraic substantivalist claims Leibniz Equivalence for TADM models, whereas the manifold substantivalist does not. This is not a change in the meaning of determinism; rather, it is a change in the interpretation of the TADM formalism.

5.2 Rynasiewicz on Earman

Rynasiewicz ([1992]) mounts a two-stage attack against Earman's algebraic substantivalist. He first argues that the latter runs into problems when it comes to making the representation map between Leibniz algebras and TADM models do what it is intended to do. He then claims that, regardless of how Earman's algebraic substantivalist interprets the representation map, the hole argument can simply be translated into the EA formalism, hence nothing is gained by adopting it. I shall claim that Rynasiewicz's stage 1 attack can be partially deflected by Earman's algebraic substantivalist. However, his stage 2 attack presents the latter with fundamental difficulties. These difficulties arguably can be overcome, but doing so requires fleshing out the semantic realist committments of Earman's algebraic substantivalist in a bit more detail.

Draft

Stage 1. Rynasiewicz first claims that there is no principled way of defining a representation map that does the work required of it by Earman's algebraic substantivalist; i.e., that allows a diffeomorphism class of TADM models to indirectly represent a single physically possible world by means of representing a *single* distinct Leibniz algebra. His argument runs as follows: Let $\mathcal{L}$ be a Leibniz algebra and m a TADM model of GR that represents $\mathcal{L}$ by means of the map ψ. Let ϕ be a Leibniz Shift defined on TADM models. Define a map θ on Leibniz algebras via $\theta \equiv \psi^{-1} \circ \phi \circ \psi$. This definition entails,

(1) $(\theta^*\mathcal{L} = \mathcal{L}) \Rightarrow (\phi^*m = m)$

Now if we assume,

(2) $\phi^*m \neq m$;

then, according to Rynasiewicz,

> ... since ϕ^*m and m are distinct but equivalent substantival models, they represent the same physical situation, and since each Leibniz algebra directly characterizes a single physical reality, $\mathcal{L}$ and $\theta^*\mathcal{L}$ represent distinct physical situations (Rynasiewicz [1992], pp. 581-82).

This implies that Earman's algebraic substantivalist, like the manifold substantivalist, is also saddled with the denial of Leibniz Equivalence. I will denote this by,

(3) $\theta^*\mathcal{L} \neq_L \mathcal{L}$

where, again, "$=_L$" is the relation of Leibniz Equivalence. If this is the case, then, according to Rynasiewicz, there is a problem with identifying what $\mathcal{L}$ is common to a collection of Leibniz Equivalent TADM models. In Rynasiewicz's words:

> The algebraicist owes us some principle telling us what reality a spacetime model represents in terms of what algebras it realizes (Rynasiewicz [1992], p. 582).

There seem to be two charges being leveled here:

(1) Earman's algebraic substantivalist faces a problem of identifying the real Leibniz algebra from a plethora of isomorphic impostors.

(2) Earman's algebraic substantivalist fails to explain in virtue of what the members of a homomorphic equivalence class of Leibniz algebras are Leibniz Equivalent.

The first claim is readily defused. As indicated above, Earman acknowledges the existence of a plethora of isomorphic Leibniz algebras. Again, he claims that this does not, in and of itself, impugn a literal interpretation of Leibniz algebras. The difference between Earman's algebraic substantivalist's literal interpretation of Leibniz algebras and the manifold substantivalist's literal interpretation of TADM models is that the latter violates $(\mathrm{Det})_{\mathrm{TADM}}$ whereas the former does not.[24]

Moreover, it is not all that clear that Earman's algebraic substantivalist is forced into denying Leibniz Equivalence for Leibniz algebras in the manner suggested by Rynasiewicz's argument above. What follows from (1) and (2) directly is $\theta^*\mathcal{L} \neq \mathcal{L}$. This is not a statement of the denial of Leibniz Equivalence. Ontological distinctness need not entail modal distinctness, as Option (B)ers to the hole argument claim (see Section 2 above). Hence it is entirely within the rights of Earman's algebraic substantivalist to adopt a version of Option (B) to the hole argument (suitably adapted to the EA formalism), and thereby provide an answer to the question, "In virtue of what do homomorphic Leibniz algebras represent a single physically possible world?" (see footnote 5 for two ways this could be done). However, and here I agree with the sentiments of Rynasiewicz's second charge, the question now arises as to why not simply adopt Option (B) in the TADM formalism to begin with - why introduce an alternative formalism?[25] In fact, Rynasiewicz claims that such an introduction is completely redundant insofar as the hole argument can be completely translated into the EA formalism.

Hence, the second charge is a bit more problematic than the first, insofar as Earman does not provide explicit details concerning how a Leibniz algebra is to be literally interpreted. Failing to provide such details places Earman's algebraic substantivalist in the same situation as a manifold substantivalist who simply states by fiat that hole diffeomorphic TADM models are equivalent (i.e., satisfy Leibniz Equivalence) without further qualification.

---

[24] Rynasiewicz ([1992], p. 580) argues that this resolution of the Problem of the Plethora of Leibniz algebras based on restricting the notion of determinism to the TADM level "...changes the meaning of determinism". That this is not the case was argued above: The meaning of determinism for both Earman's ASer and the manifold substantivalist is cashed out in terms of Section 2.2's $(\mathrm{Det})_{\mathrm{TADM}}$.

[25] A committed algebraicist might claim that the EA formalism is to be preferred, given that it out-performs TADM in representing singular spacetimes. However, such a devotee still needs to articulate the version of Option B that allows her to avoid the hole argument.

Stage 2. Rynasiewicz claims that, regardless of how Earman's algebraic substantivalist relates TADM models to Leibniz algebras, she still faces the problem of dealing with an EA version of the hole argument. In particular, Rynasiewicz maintains that Earman's algebraic substantivalism is manifold substantivalism cast in a different language, hence it fails to avoid the hole argument, which can simply be translated into the EA formalism:

> ... [T]he hole argument can be reformulated directly in terms of Leibniz algebras. As it turns out, a Leibniz algebra is just a substantival model in disguise (Rynasiewicz [1992], pp. 582-583).

The basis of this claim is the existence of a bijection between the points of a completely regular realcompact topological space X and the set of real maximal ideals of the algebra C(X) of continuous, real-valued functions on X.[26] Hence, for any Leibniz algebra[27] $\mathcal{L} = (\mathcal{R}^\infty, \mathcal{R}, A_i)$, the set of real maximal ideals of the ring $\mathcal{R}$ defines a realcompact topological space X, and the ring $\mathcal{R}^\infty$ defines a maximal C^∞-atlas on X. Thus to any $\mathcal{L}$, there corresponds a unique smooth differential manifold, call it $M(\mathcal{L})$. Hence any TADM model (M, O_i) corresponds to a unique algebraic model $(C^\infty(X), C(X), A_i)$, and each Leibniz algebra $(\mathcal{R}^\infty, \mathcal{R}, A_i)$ corresponds to a unique TADM model $(M(\mathcal{L}), O_i)$. The hole argument now applies to the Leibniz algebras $(\mathcal{R}^\infty, \mathcal{R}, A_i)$ insofar as each is now uniquely associated with a TADM model $(M(\mathcal{L}), O_i)$; since determinism is defined in terms of structures on the latter, it has now effectively been brought down to the level of the Leibniz algebras.[28]

I think this is a bit too fast of an indictment of algebraic substantivalism in general. At most, it is an indictment of a particular version of algebraic substantivalism. Before attempting to substantiate this claim, I'd like to make a few remarks concerning the expressive equivalence between EA and TADM.

From Section 3, we already knew that a Geroch ringed space is equivalent to a TADM model of GR insofar as differential structure is concerned. Rynasiewicz observes that their equivalence extends to the topological level.[29] Note, too, that the existence of a bijection between Geroch ringed spaces and TADM models indicates that, in moving to the former from the latter, manifold points are only done away with in name, not in structure. In moving to a Geroch ringed space, we replace the label "point" with the label "maximal ideal of $\mathcal{R}$". Otherwise, nothing really has changed. One might say that the point structure is preserved between TADM models and Geroch ringed spaces. However, for singular general relativistic

Draft

---

[26]For details, see Rynasiewicz ([1992], pp. 583-584) or Gillman and Jerison ([1960], p. 115). The realcompact qualification amounts to the following: For a compact space X, it can be shown that the maximal ideals I of C(X) are all of the form $I_x = \{f \in C(X) : f(x) = 0, x \in X\}$, and are said to be fixed. This establishes a bijection ¶ : $X \to \mathfrak{M}$, $¶(x) = I_x$, between a compact X and the space $\mathfrak{M}$ of maximal ideals in C(X). For realcompact X's, there is a further distinction between real maximal ideals (all I such that C(X)/I is isomorphic to the real field $\mathbb{R}$), and hyperreal ideals (all I such that C(X)/I contains a copy of $\mathbb{R}$). In the context of realcompact X's, it can be shown that only the real ideals are fixed, and hence can be put into 1-1 correspondence with the points of X (Gillman and Jerison [1960], p. 56). The significance of realcompact spaces is just that the spaces of interest in spacetime theories are normally connected and paracompact, and these properties entail realcompactness.

[27]After Geroch (see footnote 7 above), Rynasiewicz takes a Leibniz algebra $\mathcal{L}$ to be a tuple $\mathcal{L} = (\mathcal{R}^\infty, \mathcal{R}, A_i)$, where $\mathcal{R}^\infty$ is a commutative ring, $\mathcal{R}$ is a subring isomorphic with the real numbers, and the A_i are algebraic objects defined as in Section 2 (the correlates of geometrical objects defined on the corresponding manifold).

[28]More precisely, for any homomorphism $\theta : \mathcal{R} \to \mathcal{R}$, there corresponds a diffeomorphism $\phi : M(\mathcal{L}) \to M(\mathcal{L})$, and vice-versa, such that $M(\theta^*\mathcal{L}) = M(\mathcal{L})$ (the existence of such a ϕ is guaranteed by the fact that maximal ideals are preserved under homomorphism). Now let ϕ be a hole diffeomorphism. Then $(M(\mathcal{L}), O_i)$ and $(M(\mathcal{L}), \phi^*O_i)$ are hole diffeomorphs with corresponding *distinct* Leibniz algebras $(\mathcal{R}^\infty, \mathcal{R}, A_i)$ and $(\mathcal{R}^\infty, \mathcal{R}, \theta^*A_i)$. If ontological commitment is placed at the level of Leibniz algebras, then determinism fails.

[29]Note that there are two ways to view the reconstruction of points of a differential manifold. One can reconstruct the points of a topological space X from the maximal ideals of C(X), and then impose a differential structure on X to obtain a differential manifold. Alternatively, one can directly reconstruct the points of M from the maximal ideals of $C^\infty(M)$. See, e.g., Demaret, et al ([1997], p. 29).

spacetimes, this is not as evident. In moving from a TADM model to an Einstein structured space, the concept of spacetime point arguably changes radically. There is a "globalization" of the concept that makes it hard to definitely say that the point structure of a TADM model is preserved in an Einstein structured space. Recall from Section 4.2 that there are no restrictions on the topology of a structured space; in particular, the collection of structuring functions need not separate the points of M (i.e., M need not be completely regular). Hence it is not immediately clear how a reconstruction of points can be accomplished at the topological level.

However, the point is taken that, at least for nonsingular GR solutions, EA and TADM are completely expressively equivalent.[30] Hence, for such solutions, the hole argument can be re-expressed in terms of EA. However, it does not necessarily follow that all substantivalist interpretations of the EA formalism are thereby put in jeopardy. Recall that the hole argument, as originally formulated, is an argument against manifold substantivalism; i.e., an argument against a specific interpretation of models of GR in the TADM formalism. To say that it translates into EA is not, inandof itself, informative. What remains to be identified is the specific version of algebraic substantivalism that falls victim to the EA-reformulated hole argument; i.e., the correlate in EA of manifold substantivalism. On first blush, this correlate simply denies Leibniz Equivalence for homomorphic Leibniz algebras. In the next section, I will attempt to put a more concrete face on this version of algebraic substantivalism, and will offer an alternative version that, arguably, does avoid the EA-translated hole argument, and that, arguably, is a nontrivial interpretation insofar as it has no correlate in the TADM formalism.

5.3 Two Faces of Algebraic Substantivalism

Recall from Section 2.2 the schematized hole argument as originally formulated in TADM:

$$\text{(Manifold Substantivalism)} \Rightarrow \sim\text{(LE)}_{\text{TADM}} \Rightarrow \sim\text{(Det)}_{\text{TADM}} \tag{1}$$

Earman's algebraic substantivalist replaces manifold substantivalism with algebraic substantivalism in (1) and claims:

$$\text{(Algebraic Substantivalism)} \not\Rightarrow \sim\text{(LE)}_{\text{TADM}} \tag{2}$$

thus saving determinism. Rynasiewicz's second-stage attack re-conceives the hole argument as:

$$\text{(Algebraic Substantivalism)} \Rightarrow \sim\text{(LE)}_{\text{EA}} \Rightarrow \sim\text{(Det)}_{\text{EA}} \tag{3}$$

where, (Det)_{EA} is the (Det)_{F} of Section 2.2 specialized to the EA formalism. In brief, Earman's algebraic substantivalist does not provide us with an explanation of why the adoption of Leibniz Equivalence for EA models is appropriate.[31]

In terms of semantic realism, the question asked of Earman's algebraic substantivalist is, What semantic realist convictions can justify the adoption of Leibniz Equivalence for EA models of GR? In particular, can we define the algebraic substantivalist position in such a way that it does not imply the denial of Leibniz Equivalence (for either TADM or EA models), and

[30] This is reflected in Rynasiewicz's ([1992], p. 572) remark: "In fact, for the category of topological spaces of interest in spacetime physics, the program [of Leibniz algebras] is equivalent to the original spacetime approach." My one qualification here is that it is not all that evident that the category of topological spaces of general interest in spacetime physics should be restricted to nonsingular spaces.

[31] In fact, the denial of Leibniz Equivalence (of some shape or form) is taken by many to be the characteristic that distinguishes substantivalism in general from relationalism (see, e.g., Belot [1999], p. 39; Earman and Norton [1987], p. 521; Maudlin [1990], p. 84). Hence, if Earman desires to describe the algebraic position as a substantivalist one, then the above authors would have it be committed to the denial of Leibniz Equivalence, in some shape or form. The alternative is to reject the criterion of Denial of Leibniz Equivalence as essential to the substantivalist position. Option (B)'ers (footnote 5) take this alternative. So does the structuralist interpretation advocated below.

yet maintains the semantic realist intuitions that motivate manifold substantivalism? From Section 2.1, these intuitions were the following:

(a) Ineliminable quantification over a sort of object entails ontological commitment to it.

(b) Realism with respect to fields entails a literal construal of the representations of fields provided by the formalism of the theory under consideration.

I will now consider two interpretations of EA models of GR, both of which, arguably, satisfy (a) and (b), and only the second of which does not imply the denial of Leibniz Equivalence.

(1) Function Literalism. A Geroch representation turns an abstract Einstein algebra $\mathcal{A}$ into a Geroch ringed space (M, C^{∞}(M)). In the Geroch representation, the correlates of tensor fields are derivations on C^{∞}(M), the collection of smooth functions on M. Insofar as a derivation on C^{∞}(M) is a map from C^{∞}(M) to C^{∞}(M), such a derivation may be said to quantify over elements of C^{∞}(M). Hence, the correlates of tensor fields in the Geroch representation quantify over C^{∞}-functions. Taking a cue from Penrose and Rindler ([1984], p. 180), a function literalist might interpret these functions as a system of scalar fields and include it in her ontology.

Evidently, such function literalism is the position that is vulnerable to Rynasiewicz's reformulated hole argument (3) insofar as there is a 1-1 correspondence between certain sets of such functions (i.e.., those that form maximal ideals) defined on a space X and the points of X. At the least, the function literalist must provide an explanation of what it is in virtue of which homomorphic Geroch ringed spaces are equivalent.

Note that function literalism is a semantic realist position with respect to the Geroch representation of an abstract Einstein algebra $\mathcal{A}$ (or, at the least, the Gelfand representation). This suggests applying the semantic realist stance not to a "concrete" representation of $\mathcal{A}$, but rather directly to $\mathcal{A}$ itself.

Draft

(2) Algebraic Structuralism. Function literalism results from a literal construal of a particular representation of an Einstein algebra. Similarly, manifold substantivalism results from a literal construal of a TADM model of GR. What underlies both a Geroch representation and a TADM model is the structure defined by the abstract Einstein algebra that might be said to generate them both. An Einstein algebra is an abstract algebra. Read literally, it refers to abstract structural properties; namely, the algebraic operations that define it, constrained by the Einstein algebra axioms. The algebraic structuralist will claim that it is this structure that should minimally be thought of as constituting spacetime.

Such a position arguably satisfies the semantic realist intuitions (a) and (b) above in the following sense. What is at the base of principles (a) and (b) is the desire to distinguish between essential structure and surplus structure in the representations of physical objects given in a particular formalism. The semantic realist desires to read literally that aspect of the formalism that a given theory is presented in that is essential to the representations, within that formalism, of physical objects the realist is already ontologically committed to. In this context, the algebraic structuralist's position claims that the structure that defines an abstract Einstein algebra (specifically, the formal operations that define it as an abstract algebra, and the constraints imposed by the Einstein algebra axioms) is what is essential to support the representations of fields. It is this structure that underlies homomorphism classes of Geroch ringed spaces; i.e., function spaces that are isomorphic with TADM models of GR. Hence it is also what underlies diffeomorphism classes of TADM models of GR. The algebraic structuralist can thus legitimately say that TADM models related by a hole diffeomorphism represent the same physically possible world by virtue of the fact that they share the structure of a given abstract Einstein algebra. The unique physically possible world they represent possesses the spacetime structure exhibited by this particular abstract Einstein algebra.

Such algebraic structuralism rests primarily on the distinction between an abstract object and a representation of it as given in Section 3.1 above.[32] This distinction may perhaps be made a bit clearer in the context of the following immediate objection: Why can't the isomorphism between Geroch ringed spaces and TADM models of GR be pushed back to the underlying abstract Einstein algebra itself? Instead of asking, In virtue of what do homomorphic Geroch ringed spaces represent a single physically possible world, now ask, In virtue of what do homomorphic abstract Einstein algebras represent a single physically possible world? Indeed, this is the basis of Rynasiewicz's critique of Earman's algebraic substantivalist outlined above (Section 5.2). My response is the following:

The question, "In virtue of what does a class of homomorphic abstract Einstein algebras represent a single possible world?" implies such algebras can potentially be made distinct (otherwise, why ask the question in the first place?). But they can only be made distinct by imparting some characterization on them beyond the structure given by their internal formal operations. Certainly the concept of an abstract object requires further elaboration. Minimally, however, such an object should be unique up to transformations that preserve its internal structure. Arguably this property is what distinguishes an abstract object from a concrete representation of it. Hence, I want to say that, by definition, an abstract Einstein algebra $\mathscr{A}$ is unique up to homomorphism, and its internal structure (which is all there is to it) is what underlies homomorphism classes of concrete representations of it. Such a concrete representation is the Gelfand representation, which turns $\mathscr{A}$ into a "concrete" algebra of functions. This additional characterization of $\mathscr{A}$ (as an algebra of functions) now introduces potential questions of identity, where before there were none. In particular, for Gelfand representations (and the Geroch and Sikorski subrepresentations), it is now a non-trivial question to ask, Do homomorphic Einstein ringed spaces (resp., Geroch ringed spaces and Einstein differential spaces) potentially represent the same possible world? Sentiments of this sort are in fact expressed by Heller:

> The main advantages of the purely algebraic treatment is that no spacetime events appear in it from the very beginning. There are elements of abstract algebras that should be considered as the primary "objects" of the theory. Only after changing to the Gelfand representation of the given Einstein algebra $\mathscr{A}$ (or to some of its subrepresentations) do these elements become real-valued functions on the set ... of strictly maximal ideals of $\mathscr{A}$ (or on some of its subsets) which assume the role of spacetime "events" (Heller [1992], pp. 286-287).

Note, too, that, if acceptable, such algebraic structuralism, motivated by a literal construal of abstract objects, arguably, has no correlate in the TADM formalism. In the TADM formalism, the differential manifold M has a well-defined intended interpretation as a point set embued with topological and differentiable properties. In the EA formalism, an abstract Einstein algebra does not come ready-made with an intended interpretation (i.e., a "concrete" representation). This makes a difference to the semantic realist intent on literal construals of the essential aspects of formalisms. Hence, to the degree that the EA formalism is more abstract than the TADM formalism (in the sense of dealing directly with abstract objects), the structural realist approach more readily finds a home in the former.

6 Conclusion

Algebraic substantivalism, as a realist interpretation of models of GR formulated in the Einstein algebra formalism, avoids the hole argument against the manifold substantivalist

[32]Ladyman ([1998], p. 421) suggests a metaphysical version of "structural realism" that is ontologically committed to the "invariant state" that underlies various equivalent representational schemes. Algebraic structuralism may, in general, be seen as a structural realist interpretation of GR which identifies the relevant invariant state as an abstract Einstein algebra.

interpretation of TADM models of GR. In this essay, I have attempted to demonstrate that this claim is well-founded by first identifying manifold substantivalism as a species of semantic realism about spacetime, and then indicating how the motivations underlying such semantic realism can be upheld by adopting the Einstein algebra (EA) formalism. In Section 2, these motivations were cashed out in terms of a general desire to read literally that aspect of the formalism that a theory is presented in that is essential in the representations of physical objects. Thus, in the context of prior ontological commitment to fields, a semantic realist requires that we reify those mathematical objects necessary to support the representations of fields.

In Section 4, it was seen that the move to the EA formalism allows certain conceptual problems associated with GR models in TADM incorporating singularities to be resolved. In particular, the non-local behavior of the b-boundary for closed Friedman and Schwarzschild models seems more readily assimilated in the global sheaf-theoretic algebraic formalism than in the local TADM formalism. Furthermore, singular and non-singular solutions in the algebraic formalism belong to the same mathematical category (structured spaces). In the TADM formalism, singular and non-singular solutions are not of the same mathematical type. Hence the EA formalism provides both conceptual clarity and conceptual unity in the context of singular spacetimes.

In Section 5, a form of semantic realism with respect to EA models of GR was suggested under the heading algebraic structuralism. An algebraic structuralist desires to read literally the structure defined by a particular abstract Einstein algebra (as given by its formal operations constrained by the Einstein algebra axioms). Such a literal construal does not extend to "concrete" representations of abstract Einstein algebras; in particular, it does not extend to Geroch representations, the function space equivalents of TADM models of GR. Insofar as an abstract Einstein algebra, as an abstract object, is unique up to homomorphism, it is legitimate for the algebraic structuralist to adopt Leibniz Equivalence for Einstein algebras; in particular, she will claim that it is the structure of a given abstract Einstein algebra that uniquely corresponds to the spacetime structure of a given physically possible world. Hence it is legitimate for her to claim that homomorphic Geroch ringed spaces represent the same physically possible world. Hence it is legitimate for her to claim that diffeomorphic TADM models of GR represent the same physically possible world. The hole argument, in either of its EA or TADM formulations, is thus avoided.

Draft

References

Belot, G. [1999]: 'Rehabilitating Relationalism', *International Studies in the Philosophy of Science* **13**, pp. 35-52.

Bishop, R. and S. Goldberg [1980]: *Tensor Analysis on Manifolds*, New York: Dover.

Bosshard, B. [1976]: 'On the b-boundary of the Closed Friedmann Model', *Communications in Mathematical Physics* **46**, pp. 263-268.

Brighouse, C. [1994]: 'Spacetime and Holes', in: *PSA 1994*, East Lansing: PSA, pp. 117-125.

Butterfield, J. [1989]: 'The Hole Truth', *British Journal for the Philosophy of Science* **40**, pp. 1-28.

Clarke, C. J. S. [1993]: *The Analysis of Space-Time Singularities*, Cambridge: Cambridge University Press.

Demaret, J., M. Heller, and D. Lambert [1997]: 'Local and Global Properties of the World', preprint: gr-qc/9702047.

Earman, J. [1995]: *Bangs, Crunches, Whimpers and Shrieks*, New York: Oxford University Press.

Earman, J. [1993]: 'Underdeterminism, Realism and Reason', in P. French, T. Uehling, Jr., and H. Wettstein (eds.), *Midwest Studies in Philosophy, XVIII*, University of Notre Dame Press: Notre Dame, pp. 19-38.

Earman, J. [1989]: *World Enough and Spacetime*, Cambridge: MIT Press.

Earman, J. [1986]: 'Why Space is not a Substance (At Least Not to First Degree)', *Pacific Philosophical Quarterly* **67**, pp. 225-244.

Earman, J. [1977]: 'Leibnizian Spacetimes and Leibnizian Algebras', in *Historical and Philosophical Dimensions of Logic, Methodology and Philosophy of Science*, R. Butts and J. Hintikka (eds.), Dordrecht: D. Reidel, pp. 93-112.

Earman, J. and J. D. Norton [1987]: 'What Price Spacetime Substantivalism? The Hole Story', *British Journal for Philosophy of Science* **38**, pp. 515-525.

Field, H. [1989]: *Realism, Mathematics and Modality*, Oxford: Blackwell.

Friedman, M. [1983]: *Foundations of Spacetime Theories*, Princeton: Princeton University Press.

Geroch, R. [1985]: *Mathematical Physics*, Chicago: University of Chicago Press.

Geroch, R. [1972]: 'Einstein Algebras', *Communications of Mathematical Physics* **26**, 271-275.

Gillman, L. and M. Jerison [1960]: *Rings of Continuous Functions*, New York: Springer-Verlag.

Gruszczak, J., M. Heller, and P. Multarzynski [1988]: 'A Generalization of Manifolds as Spacetime Models', *Journal of Mathematical Physics* **29**, pp. 2576-2580.

Draft

Heller, M. [1992]: 'Einstein Algebras and General Relativity', *International Journal of Theoretical Physics* **31**, pp. 227-288.

Heller, M. and W. Sasin [1995a]: 'Sheaves of Einstein Algebras', *International Journal of Theoretical Physics* **34**, pp. 387-398.

Heller, M., and W. Sasin [1995b]: 'Structured Spaces and their Application to Relativistic Physics', *Journal of Mathematical Physics* **36**, pp. 3644-3662.

Heller, M. and W. Sasin [1994]: 'The Structure of the b-Completion of Space-Time', *General Relativity and Gravitation* **26**, pp. 797-811.

Hoefer, C. [1996]: 'The Metaphysics of Space-Time Substantivalism', *Journal of Philosophy* **43**, pp. 5- 27.

Horwich, P. [1982]: 'Three Forms of Realism', *Synthese* **51**, pp. 181-201.

Horwich, P. [1978]: 'On the Existence of Time, Space, and Spacetime', *Nous* **12**, pp. 397-419.

Johnson, R. [1977]: 'The Bundle Boundary in Some Special Cases', *Journal of Mathematical Physics* **18**, pp. 898-902.

Ladyman, J. [1998]: 'What is Structural Realism?', *Studies in History and Philosophy of Science*, **29**, pp. 409-424.

Leeds, S. [1995]: 'Holes and Determinism: Another Look', *Philosophy of Science* **62**, pp. 425-437.

Liu, C. [1996]: 'Realism and Spacetime: Of Arguments against Metaphysical Realism and Manifold Realism', *Philosophia Naturalis* **33**, pp. 243-263.

Maudlin, T. [1990]: 'Substances and Spacetime: What Aristotle Would Have Said to Einstein', *Studies in History and Philosophy of Science* **21**, pp. 531-561.

Mundy, B. [1992]: 'Space-Time and Isomorphism', in *PSA 1992, Vol. 1*, D. Hull, M. Forbes and K. Okruhlik (eds.), East Lansing: Philosophy of Science Association, pp. 515-527.

Nomizu, K. [1956]: *Lie Groups and Differential Geometry*, Tokyo: Herald Printing.

Penrose, R. and W. Rindler [1984]: *Spinors and Spacetime, Vol. 1*, Cambridge: Cambridge University Press.

Rynasiewicz, R. [1994]: 'The Lessons of the Hole Argument', *British Journal for Philosophy of Science* **45**, pp. 407-436.

Rynasiewicz, R. [1992]: 'Rings, Holes And Substantivalism: On the Program of Leibniz Algebras', *Philosophy of Science* **59**, pp. 572-589.

Schmidt, B. [1971]: 'A New Definition of Singular Points in General Relativity', *General Relativity and Gravitation* **1**, pp. 269-280.

Ward, R. and R. Wells [1990]: *Twistor Geometry and Field Theory*, Cambridge: Cambridge University Press.

Draft

APPENDIX: ALBEGRAIC PRELIMINARIES

Let E be a non-empty set and let + and $\cdot$ be internal binary operations defined on E; i.e., maps $E \times E \to E$. Then for a, c, b $\in$ E, one may consider the following axioms.

A1. $a + b = b + a$

A2. $(a + b) + c = a + (b + c)$

A3. $\exists z \in E$ such that $z + a = a + z = a$

A4. $\forall a \in E, \exists a' \in E$ s.t. $a + a' = a' + a = z$

M1. $a \cdot b = b \cdot a$

M2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M3. $\exists e \in E$ such that $e \cdot a = a \cdot e = a$

M4. $\forall a \in E$ s.t. $a \neq z, \exists a' \in E$ s.t. $a \cdot a' = a' \cdot a = e$

D. $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$

Z. If $a \cdot b = z$ then $a = z$ or $b = z$ (or both)

An *abelian group* is a pair (E+) where + satisfies A1-A4. A *commutative ring* (with unity) is a triple (E+$\cdot$) where +,$\cdot$ satisfy A1-A4, and M1-M3, respectively, and D. A *field* is a triple (E+$\cdot$) where +,$\cdot$ satisfy A1-A4, M1-M4, D, and Z. Now let Ω be a possibly empty set on which +, $\cdot$ may be defined. Let * be an external binary operation on E with respect to Ω; i.e., a map $E \times \Omega \to E$. Then the following axioms may be satisfied for a, b $\in$ E and $\lambda, \mu \in \Omega$.

E1. $\lambda*(\mu*a) = (\lambda\cdot\mu)*a$

E2. $(\lambda + \mu)*a = \lambda*a + \mu*a$, and $\lambda*(a + b) = \lambda*a + \lambda*b$

E3. $\varepsilon*a = a$, where ε is the unit element of Ω.

Draft

A *linear module* over Ω is a sextuple (E+, Ω+$\cdot$, *) where (E+) is an abelian group, (Ω+$\cdot$) is a commutative ring (with unity), and * satisfies E1-E3. A *linear vector space* over Ω is a sextuple (E+, Ω+$\cdot$, *) where (E+) is an abelian group, (Ω+$\cdot$) is a field, and * satisfies E1-E3. A *linear algebra* is a septuple (E+$\cdot$, Ω+$\cdot$, *), where (E+$\cdot$) is a commutative ring, (Ω+$\cdot$) is a field, and * satisfies E1-E3.