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Theories of Newtonian gravity and empirical indistinguishability

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Abstract

In this essay, I examine the curved spacetime formulation of Newtonian gravity known as Newton–Cartan gravity and compare it with flat spacetime formulations. Two versions of Newton–Cartan gravity can be identified in the physics literature—a "weak" version and a "strong" version. The strong version has a constrained Hamiltonian formulation and consequently a well-defined gauge structure, whereas the weak version does not (with some qualifications). Moreover, the strong version is best compared with the structure of what Earman (*World enough and spacetime*. Cambridge: MIT Press) has dubbed Maxwellian spacetime. This suggests that there are also two versions of Newtonian gravity in flat spacetime—a "weak" version in Maxwellian spacetime, and a "strong" version in Neo-Newtonian spacetime. I conclude by indicating how these alternative formulations of Newtonian gravity impact the notion of empirical indistinguishability and the debate over scientific realism.

Keywords: Newtonian gravity; Gauge system; Empirical indistinguishability

1. Introduction

The standard way of formulating Newton's theory of gravity is as the theory of a gravitational field in a background spacetime. The latter is normally taken to be flat Neo-Newtonian spacetime, the spacetime characterized by automorphisms belonging to the Galilei group (the symmetry group for Newtonian dynamics). Newtonian gravity can also be given a curved spacetime formulation by geometricizing the gravitational field and incorporating it into the curvature tensor in a manner similar

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to general relativity. This geometricized version of the theory was first described by Cartan and is usually referred to as Newton–Cartan gravity (NCG, hereafter). While it has been given considerable attention in the philosophical literature (see, e.g., Earman & Friedman, 1973; Friedman, 1983; Earman, 1989; Malament, 1986, 1995; Norton, 1995), I think it deserves a second look, for a number of reasons.

First, the philosophical literature does not reflect the current state of affairs in the physics literature. At least two versions of NCG have appeared in the latter, and these versions affect the status of NCG as an alleged example of empirical indistinguishability. Some authors have claimed that NCG and the standard formulation of Newtonian gravity make identical empirical claims, but subscribe to different ontologies; hence they count as a non-trivial example of empirically indistinguishable theories. This claim is significant in the debate over scientific realism. In particular, an anti-realist may question whether a realist interpretation of Newtonian gravity is possible, given that there are non-trivial empirically indistinguishable versions of it. However, if the standard formulation is only recoverable from NCG under the imposition of certain constraints, and if these constraints effectively reduce the ontology of NCG to the ontology of the standard formulation, then perhaps they are not significantly different after all. In particular, perhaps NCG, so-constrained, is simply the standard formulation in disguise. I will claim that this is not the case—that NCG and the standard formulation, appropriately construed, *are* legitimate non-trivial examples of empirically indistinguishable theories. But this will involve distinguishing between different versions of NCG, as well as different versions of the standard formulation. In particular, it will be seen that "non-geometricized" Newtonian gravity can also be formulated in background spacetimes with less structure than Neo-Newtonian spacetime, and that one version of NCG is the legitimate empirically indistinguishable partner to these theories.

Second, while the symmetries of the standard formulation of Newtonian gravity are relatively straight-forward, those for NCG are, at best, open to debate. Most authors agree that NCG has a gauge structure represented explicitly by a freedom in choosing how to distinguish inertial trajectories from gravitationally accelerated trajectories. But how this gauge structure relates to the standard formulation is a bit cryptic, as is how it relates to other notions of gauge symmetry. In particular, Earman (2002, p. S218) observes that NCG cannot be derived from an action principle; hence, it cannot be formulated as a constrained Hamiltonian system, and to the extent that gauge talk is talk about constrained Hamiltonian systems, gauge talk cannot characterize NCG. On the other hand, a version of NCG as a purported Yang–Mills-type theory has been proposed in the physics literature, primarily by Duval and Künzle (1984). These authors claim that NCG is characterized not by any single symmetry group, but by numerous nested symmetries. Moreover, Christian (1997, 2001) has recently presented a version of NCG formulated explicitly as a constrained Hamiltonian system (which is then transformed into a constraint-free Hamiltonian system by solving for all the constraints). These different versions need to be sorted out. In particular, I will indicate how Christian's version ("strong" NCG) differs from previous versions ("weak" NCG), and how these versions relate to standard formulations of Newtonian gravity in background spacetimes.

Section 2 sets the stage by characterizing three types of classical spacetimes in terms of conditions placed on the curvature tensor. Section 3 looks at theories of Newtonian gravity obtained by placing a Newtonian gravitational field in a background classical spacetime. Section 4 looks at theories of Newtonian gravity obtained by geometricizing the Newtonian gravitational field and making it a part of the background spacetime structure. Finally, Section 5 summarizes the relationships between these theories, identifies among them legitimate instances of empirical indistinguishability, and indicates the impact this discussion has on the debate over scientific realism.

2. Classical spacetimes

The theories of Newtonian gravity considered below will be distinguished in terms of the spacetime structure they posit. Such structure takes the form of privileged global frames of reference, which may be identified intrinsically with congruences of smooth timelike worldlines. In the absence of gravity and other forces, one may identify various classical spacetimes by the frames they minimally admit and the associated group of symmetry transformations between these frames. In this section, I will review three such spacetimes, what Earman (1989, Chapter 2) refers to as Leibnizian spacetime, Maxwellian spacetime, and Neo-Newtonian spacetime. Earman characterizes these spacetimes extrinsically in terms of coordinate transformations between their privileged reference frames. The approach taken below will be to characterize these spacetimes intrinsically by conditions placed on the curvature tensor. This will help to clarify the subsequent discussion of theories of gravity.

To begin, following Malament (1986, p. 183; 1995, p. 493), I will take a classical spacetime to be a structure $(M, h^{ab}, t_a, \nabla_a)$, where M is a smooth differentiable manifold, h^{ab} is a symmetric tensor field on M with signature (0, 1, 1, 1) identified as a degenerate spatial metric; t_a is a covariant vector field on M which induces a degenerate temporal metric $t_{ab} = t_a t_b$ with signature (1, 0, 0, 0); and ∇_a is a smooth derivative operator associated with a connection on M.[1] These objects are required to satisfy the following conditions:

$$h^{ab} t_b = 0 \qquad \text{(orthogonality)}, \tag{1}$$

$$\nabla_c h^{ab} = 0 = \nabla_a t_b \qquad \text{(compatibility)}. \tag{2}$$

As explained in detail by Malament, such a structure serves as the basis for a classical theory of motion in the following manner. The vector field t_a assigns a temporal length to all vectors and thus allows a distinction between timelike and spacelike vectors. The signature of h^{ab} and condition (1) entail that the subspace of

[1] In this essay lower-case Latin letters from the beginning of the alphabet $a, b, c, \ldots$ are abstract indices; lower-case Greek letters $\alpha, \beta, \gamma, \ldots$ are component indices ranging over 0, 1, 2, 3; and lower-case Latin letters from the middle of the alphabet $i, j, k, \ldots$ are component indices ranging over 1, 2, 3.

spacelike vectors is three-dimensional. Condition (2) entails that t_a is closed, so a global time function t exists (given that M is well-behaved topologically). These facts allow M to be decomposed into instantaneous three-dimensional spacelike hypersurfaces Σ_t parametrized by t. Particle trajectories can be associated with timelike curves γ, i.e., curves with everywhere timelike tangent vectors. Such curves can be parametrized by t by requiring their tangent vector fields ξ^a to satisfy $t_a\xi^a = 1$. These tangent fields ξ^a can then be identified as the four-velocity associated with γ. The four-acceleration associated with γ is then given by $\xi^a\nabla_a\xi^b$. The compatibility condition (2) entails that such four-accelerations are spacelike. Finally, the spatial metric h^{ab} assigns a spatial length to spacelike vectors but it does not assign spatial lengths to timelike vectors.[2] This allows acceleration magnitudes to be assigned to particle trajectories but not, in general, velocity magnitudes, and this is minimally what a classical Galilean-invariant theory of motion requires.

At this point, nothing has been assumed about the nature of the connection and, hence, about the curvature of such classical spacetime models. In fact, unlike the Riemannian case in which the compatibility condition $\nabla_a g_{ab} = 0$ on a Lorentzian metric g_{ab} uniquely determines the connection, conditions (1) and (2) fail to uniquely determine a classical connection.[3] Thus, one way to further categorize classical spacetimes is by how they place restrictions on the curvature tensor.

Curvature constraints: For a given connection, there is an associated curvature tensor R^a_{bcd} defined by $R^a_{bcd}x^c y^d z^b = \nabla_c(\nabla_d z^a y^d)x^c - \nabla_c(\nabla_d z^a x^d)y^c$, for arbitrary vector fields x^c, y^d, z^b (here and throughout, the torsion is assumed to vanish identically). Geometrically, $R^a_{bcd}x^c y^d z^b$ measures the difference in z^b upon parallel transport along a (small) closed curve defined by x^c and y^d. Thus, the condition $R^a_{bcd}x^c y^d z^b = 0$ for arbitrary x^c, y^d, z^b represents path independence of parallel transport of an arbitrary z^b along an arbitrary closed curve; in other words: complete path independence of parallel transport. This condition is associated with spacetime flatness and is given by the vanishing of the curvature tensor $R^a_{bcd} = 0$. A slightly less restrictive constraint on the curvature occurs when z^b is required to be spacelike, while x^c and y^d are left arbitrary. In this case, $z^b = h^{be}\omega_e$ for some 1-form ω_e (see footnote 2) and the condition $0 = R^a_{bcd}x^c y^d h^{be}\omega_e = R^{ae}_{cd}x^c y^d\omega_e$ represents path independence of parallel transport for spacelike vectors. Here use has been made of the fact that, while h^{ab} cannot be used to lower indices, it can be used to raise them. So, for instance, $h^{eb}R^a_{bcd}$ can be written as R^{ae}_{cd}. Again, for arbitrary x^c, y^d, and arbitrary spacelike z^b, the preceding condition is equivalent to $R^{ae}_{cd} = 0$. In a similar

[2] It can be shown that a vector ζ^a is spacelike *if and only if* $\zeta^a = h^{ab}\omega_b$ for some 1-form ω_b (Malament, 1986, p. 185). The spatial length of ζ^a is then defined by $(h^{ab}\omega_a\omega_b)^{1/2}$.

[3] As indicated below in Section 4.1, conditions (1) and (2) determine a connection only up to an arbitrary 2-form. In the physics literature, classical spacetimes as defined above are referred to as Galilei manifolds (see, e.g., Künzle, 1972, p. 343). This terminology is motivated by the fact that the homogeneous Galilei group is the most general *linear* group of transformations that preserve the metrics h^{ab} and t_a (see, e.g., Lévy-Leblond, 1971, p. 225). However, if linearity is dropped, the structure-preserving group expands considerably.

vein, the condition $R^{abcd} = 0$ represents path independence of parallel transport of spacelike vectors along closed spacelike curves. To summarize:

$$R^{abcd} = 0 \quad (\textit{spatial flatness}\text{: spacelike vector fields remain unchanged under parallel transport on spacelike hypersurfaces}), \tag{3}$$

$$R^{ab}_{cd} = 0 \quad (\textit{rotation standard}\text{: spacelike vector fields remain unchanged under parallel transport everywhere}), \tag{4}$$

$$R^{a}_{bcd} = 0 \quad (\textit{spacetime flatness}\text{: arbitrary vector fields remain unchanged under parallel transport everywhere}). \tag{5}$$

As restrictions on the curvature tensor, (5) is strongest and (3) weakest, in the sense that (5) ⇒ (4) ⇒ (3). Malament (1986, 1995) refers to condition (3) as spatial flatness, in so far as imposing it on a classical spacetime entails that the three-dimensional spacelike hypersurfaces parametrized by the global time function are flat (i.e., "space" is Euclidean). Condition (4) is equivalent to specifying a standard of rotation (see Section 2.2).[4] Briefly, it requires spacelike vector fields to be covariantly constant throughout spacetime in general (and not just on spacelike hypersurfaces). Hence, it requires spacelike surfaces in M to be "parallel" in the sense that the timelike "rigging" between these surfaces is hypersurface orthogonal. This prohibits "twisting" of the rigging; thus the privileged frames adapted to the rigging are non-rotating with respect to each other (but, as will be seen, can have arbitrary relative acceleration). In this vein, condition (5) not only prohibits relative rotation between adapted frames, but also requires linearity in the time-dependency of translations between such frames; thus it prohibits relative acceleration.

In the remainder of this section, I will distinguish Leibnizian, Maxwellian, and Neo-Newtonian spacetimes in terms of the above three curvature constraints.

2.1. Leibnizian spacetime

Leibnizian spacetime is the classical spacetime with just enough structure to minimally support the existence of rigid Euclidean, arbitrarily rotating, and arbitrarily accelerating reference frames (hereafter referred to as Leibnizian frames). It can be defined as the classical spacetime satisfying

(1) $h^{ab}t_b = 0$ (orthogonality),
(2) $\nabla_c h^{ab} = 0 = \nabla_a t_b$ (compatibility),
(3) $R^{abcd} = 0$ (spatial flatness).

The symmetries of Leibnizian spacetime so defined are generated by vector fields x^a that Lie-annihilate the "absolute objects" h^{ab}, t_a, and $\Gamma^{abc} = h^{ce}h^{bd}\Gamma^{a}_{bc}$, where Γ^{abc}

[4] Condition (4) and its significance to NCG have been given much discussion in the literature. For a sample, see Malament (1986, footnote 11), Ehlers (1997, p. A121), and Christian (1997, pp. 4850–4851; 2001, pp. 329–330, 332–333).

can be viewed as the spatial part of the connection.[5] In particular, the conditions $\mathfrak{L}_x h^{ab} = \mathfrak{L}_x t_a = \mathfrak{L}_x \Gamma^{abc} = 0$ generate an infinite-dimensional Lie group known as the Leibniz group (Leib),[6] which, in terms of coordinates adapted to x^a, consists of transformations of the form:

$$x^i \to x'^i = R^i_j(t)x^j + a^i(t), \quad i,j = 1,2,3,$$

(Leib)

$$t \to t' = t + c,$$

where $R^i_j(t) \in SO(3)$ is an orthogonal rotation matrix for each $t \in \mathbf{R}$, $a^i(t) \in \mathbf{R}^3$ are arbitrary functions of t, and $c \in \mathbf{R}$ is a constant. To see that these are transformations between rigid Euclidean, arbitrarily rotating, and arbitrarily accelerating reference frames, one can first calculate the connection components in "Leibnizian coordinates" to obtain:[7]

$$\Gamma'^i_{00} = R^i_m \ddot{R}^m_j x'^j + R^i_m \ddot{a}^m, \quad \Gamma'^i_{j0} = R^i_m \dot{R}^m_j, \quad \Gamma'^\alpha_{\beta\gamma} = 0 \quad \text{otherwise}, \tag{6}$$

where $a^m = R^m_j a^j$ and the dot denotes differentiation with respect to t. The path of a particle with zero four-acceleration $\xi^a \nabla_a \xi^b = 0$ is then given by

$$\ddot{x}^i + R^i_m \ddot{R}^m_j x^j + R^i_m \ddot{a}^m + R^i_m \dot{R}^m_j \dot{x}^j = 0.$$

This indicates that acceleration and rotation are relative in Leibnizian frames: Any non-zero linear acceleration term on the RHS can be absorbed by an appropriate choice of the functions $\ddot{a}^m(t)$ on the LHS, and any non-zero rotational acceleration term on the RHS can likewise be absorbed by an appropriate choice of the matrices $\dot{R}^i_j(t)$ on the LHS. From a geometric point of view, the degrees of freedom in specifying these functions represent the inability of a Leibnizian connection to distinguish between "straight", "curved", and "twisted" particle trajectories. (In the more familiar context of Neo-Newtonian spacetime (Section 2.3), the second, third and fourth terms on the LHS of the equation of motion above are interpreted as due to centrifugal, linear, and coriolis inertial forces, respectively.)

Note, finally, that conditions (1) and (2) do not guarantee that the constant t spatial slices are flat. This guarantee is secured only with the addition of condition (3). Compatibility of the spatial metric (2), for instance, only guarantees rigidity in

[5] The condition that the fields x^a Lie-annihilate the objects h^{ab}, t_a, and Γ^{abc} entails that the latter remain constant along the integral curves of the former. Hence, the transformations generated by x^a leave the objects h^{ab}, t_a, and Γ^{abc} invariant. In this sense, these objects encode the structure of the frames defined by these transformations and, hence, the structure of the associated spacetime.

[6] This terminology follows Earman (1989, p. 31), who associates these transformations with the spacetime structure proposed in the writings of Leibniz. In the physics literature, this group has been referred to as the Coriolis group (Duval, 1993, p. 2218), or the kinematical group (Künzle, 1972, p. 347).

[7] This follows upon substitution of (Leib) into the general transformation rule for the connection components

$$\Gamma'^\alpha_{\beta\gamma} = \left(\frac{\partial^2 x_\sigma}{\partial x'_\gamma \partial x'_\beta} + \frac{\partial x'_\upsilon}{\partial x'_\gamma} \frac{\partial x'_\mu}{\partial x'_\beta} \Gamma^\sigma_{\mu\upsilon} \right) \frac{\partial x'_\alpha}{\partial x_\sigma}$$

and setting $\Gamma^\sigma_{\mu\upsilon} = 0$. Physically, we pick an arbitrary rigid, non-rotating, geodetic frame in which the connection components vanish, and then perform a Leibniz transformation on it.

the sense that, if two timelike worldlines are at rest relative to each other, then the spatial distance between them remains constant (think of the worldlines as endpoints of a measuring rod).[8] It does not, in particular, guarantee that the state of relative rest of the endpoints can be determined.

2.2. Maxwellian spacetime

Maxwellian spacetime is the classical spacetime with just enough structure to minimally support the existence of rigid Euclidean, non-rotating, and arbitrarily accelerating reference frames (hereafter referred to as Maxwellian frames[9]). It can be defined as the classical spacetime satisfying

(1) $h^{ab}t_b = 0$ (orthogonality),
(2) $\nabla_c h^{ab} = 0 = \nabla_a t_b$ (compatibility),
(4) $R^{ab}_{cd} = 0$ (rotation standard).

The symmetries of Maxwellian spacetime are generated by vector fields x^a satisfying $\mathfrak{L}_x h^{ab} = \mathfrak{L}_x t_a = \mathfrak{L}_x \Gamma^{ab}_c = 0$, where $\Gamma^{ab}_c = h^{bd}\Gamma^a_{bc}$ can be viewed as the rotation part of the connection. One obtains an infinite dimensional Lie group referred to as the Maxwell[10] group (Max) with coordinate representation given by

$$x^i \to x'^i = R^i_j x^j + a^i(t), \quad i,j = 1,2,3,$$

(Max)

$$t \to t' = t + c,$$

where $R^i_j \in \mathrm{SO}(3)$ is a constant orthogonal rotation matrix, $a^i(t) \in \mathbf{R}^3$ are arbitrary functions of $t \in \mathbf{R}$, and $c \in \mathbf{R}$. Maxwellian transformations consist of transformations between rigid Euclidean, non-rotating, and arbitrarily accelerating reference frames, as can be seen by the following. The connection components in Maxwellian coordinates are

$$\Gamma'^i_{00} = R^i_m \ddot{a}^m, \quad \Gamma'^\alpha_{\beta\gamma} = 0 \quad \text{otherwise.} \tag{7}$$

The path of a particle with zero four-acceleration $\xi^a \nabla_a \xi^b = 0$ is thus given in Maxwellian coordinates by

$$\ddot{x}^i + R^i_m \ddot{a}^m = 0.$$

This indicates that acceleration is relative in Maxwellian frames: Any non-zero linear acceleration term on the RHS can be absorbed into the functions $\ddot{a}^m(t)$. Rotation

[8] More precisely, compatibility of the spatial metric entails that "… the h^{ab}-length of all ∇_a-constant spacelike vector fields along arbitrary timelike curves is constant" (Malament, 1986, p. 186). In other words, $\nabla_a(h^{bc}\omega_b\omega_c) = 0$, if both (2) holds *and* $\nabla_a\omega_b = 0$.

[9] In the physics literature, these are sometimes referred to as Galilean frames (e.g., Christian, 1997; Kuchar, 1980).

[10] This terminology follows Earman (1989, p. 31), who associates these transformations with the spacetime structure proposed in the writings of James Clerk Maxwell. In the physics literature, this group is referred to as the Milne group after Milne's work in Newtonian cosmology (Duval, 1993, p. 2218).

terms, however, cannot be so-absorbed, hence rotation is not relative. From a geometric point of view, the degrees of freedom in specifying the $\ddot{a}^m(t)$ represent the inability of a Maxwellian connection to distinguish between "straight", and "curved" particle trajectories. Unlike a Leibnizian connection, however, a Maxwellian connection can distinguish "twisted" from "non-twisted" particle trajectories.

Note that (7) follows immediately from (6) by setting the term $\dot{R}^m_j = 0$ and thus removing the time dependency of the rotation matrices in (Leib). Maxwellian frames may be rotated by a constant amount relative to each other, but they cannot be in rotation (constant or accelerated) with respect to each other over time.

2.3. Neo-Newtonian spacetime

Neo-Newtonian spacetime is the classical spacetime with just enough structure to minimally support the existence of rigid Euclidean, non-rotating and non-accelerating reference frames (hereafter referred to as Neo-Newtonian frames). It can be defined as the classical spacetime satisfying

(1) $h^{ab}t_b = 0$ (orthogonality),
(2) $\nabla_c h^{ab} = 0 = \nabla_a t_b$ (compatibility),
(5) $R^a_{bcd} = 0$ (spacetime flatness).

The symmetries of Neo-Newtonian spacetime are generated by vector fields x^a satisfying $\mathfrak{L}_x h^{ab} = \mathfrak{L}_x t_a = \mathfrak{L}_x \Gamma^a_{bc} = 0$. One obtains a 10-parameter Lie group referred to as the Galilei group (Gal) with coordinate representation given by

$$x^i \to x'^i = R^i_j x^j + v^i t + d^i, \quad i,j = 1,2,3,$$

(Gal)

$$t \to t' = t + c,$$

where $R^i_j \in \mathrm{SO}(3)$ is a constant orthogonal rotation matrix, v^i, $d^i \in \mathbf{R}^3$ and $c \in \mathbf{R}$. Galilei transformations are transformations between rigid Euclidean, non-rotating, and non-accelerating reference frames: In Neo-Newtonian components, the connection takes the familiar form $\Gamma^\alpha_{\beta\gamma} = 0$. Hence, the path of a particle with zero four-acceleration $\xi^a \nabla_a \xi^b = 0$ is given in Neo-Newtonian coordinates by

$$\ddot{x}^i = 0.$$

Any acceleration terms, linear or rotational, that may appear on the RHS cannot be absorbed by appropriate adjustment of parameters on the LHS (there are no degrees of freedom available); hence, acceleration and rotation are absolute in Neo-Newtonian frames. A Neo-Newtonian connection can distinguish between "straight", "curved" and "twisted" particle trajectories.

3. Newtonian gravity in classical spacetimes

Theories of Newtonian gravity in classical spacetimes are obtained by adding a Newtonian gravitational field to a particular classical spacetime. In such theories, one can distinguish between absolute geometrical object fields—those objects that encode the structure of the given classical spacetime; and dynamical geometric object fields—those objects that encode the dynamics of the particular theory. In this context, such objects represent the material contents of the spacetime. Two types of symmetries can thus be identified: spacetime symmetries are the automorphisms of the given spacetime, while dynamical symmetries are the symmetries of the differential equations that hold between the dynamical objects of the theory.

3.1. Newtonian gravity in Neo-Newtonian spacetime

Newtonian gravity in Neo-Newtonian spacetime, Neo-Newt NG for short, is obtained by adding a Newtonian gravitational field to Neo-Newtonian spacetime. Dynamically possible models of Neo-Newt NG are of the form $(M, h^{ab}, t_a, \nabla_a, \phi, \rho)$, where $(M, h^{ab}, t_a, \nabla_a)$ is Neo-Newtonian spacetime, and ϕ and ρ are scalar fields representing the Newtonian gravitational potential and the mass density, respectively. The field equations are

$$\begin{array}{lll} (1) & h^{ab}t_b = 0 & \text{(orthogonality)}, \\ (2) & \nabla_c h^{ab} = 0 = \nabla_a t_b & \text{(compatibility)}, \\ (5) & R^a_{bcd} = 0 & \text{(spacetime flatness)}, \\ & h^{ab}\nabla_a\nabla_b\phi = 4\pi G\rho & \text{(Poisson equation)}, \end{array} \qquad (8)$$

where G is the Newtonian gravitational constant. The equation of motion is

$$\xi^a\nabla_a\xi^b = -h^{ab}\nabla_a\phi \qquad (9)$$

for particle trajectories with four-velocity ξ^a. The spacetime symmetries of Neo-Newt NG are the symmetries of Neo-Newtonian spacetime, namely (Gal). The dynamical symmetries are symmetries of the equation of motion (9), i.e., transformations that send solutions of (9) to other solutions. These are transformations that leave (9) covariant in Neo-Newtonian reference frames, i.e., frames in which (9) takes the form $\ddot{x}^i = -h^{ij}\partial\phi/\partial x^j$. The most general such transformations are elements of (Max) together with the transformation $\phi \mapsto \phi' = \phi - x^i\ddot{a}^i + \varphi(t)$, where φ is an arbitrary function of t.

Neo-Newt NG faces the following conceptual problem (see, e.g., Friedman, 1983, p. 96). The theory states that there are preferred non-accelerating reference frames, i.e., Neo-Newtonian frames. Accordingly, from the point of view of spacetime structure, there is a distinction between these non-accelerated frames and arbitrarily accelerated frames. However, from the point of view of the dynamics, such a distinction cannot be made. To see this, suppose we perform a (Max) transformation on the Neo-Newtonian frame x^a to obtain a frame x'^a that is arbitrarily accelerating with respect to x^a. In this new frame, the equation of motion (9) becomes $\ddot{x}'^i = -h^{ij}\partial\psi/\partial x'^j$, where $\psi = \phi - x'^i\ddot{a}^i$. From the point of view of the dynamics, ϕ and ψ are indistinguishable: if ϕ is a solution to (8) and (9), then so is ψ. Hence, from the point of view of the dynamics, Neo-Newtonian frames cannot be made distinct from Maxwellian frames (in the presence of only gravitational forces)—the dynamics cannot distinguish between ϕ and ψ, and thus cannot distinguish between the gravitational equation of motion in a Maxwellian frame with gravitational potential ψ, from the gravitational equation of motion in a Neo-Newtonian frame with gravitational potential ϕ. In other words, gravitationally accelerated motion cannot be made distinct from non-accelerated motion.

Such a distinction can be made if we impose an additional constraint on ϕ; namely, that it vanish at spatial infinity: $\phi \to 0$ as $x^i \to \infty$. Formally, this entails that $\ddot{a}^i = 0$, and thus reduces the covariance group of (9) to (Gal) plus the transformation $\phi \mapsto \phi' = \phi + \varphi(t)$. Physically, this assumption entails that all the matter in the universe is concentrated in a finite region of space. This may be called the "island universe" assumption, after Misner, Thorne, and Wheeler (1973, p. 295).

Without this additional constraint, Neo-Newt NG suffers from not being well-tuned: its spacetime symmetries (Gal) are smaller than its dynamical symmetries (Max $+\phi$-transformations). Hence, it posits unobservable spacetime fluff; namely, a connection that can distinguish between "inertial" (viz., non-accelerated) motion and gravitationally accelerated motion. To obtain a well-tuned theory, one can fiddle with either the spacetime structure or the dynamics. The dynamics is independently supported by evidence for the equivalence principle (which, in one version, states just that inertial motion is indistinguishable from gravitationally accelerated motion). This indicates that spacetime fiddling is to be preferred. In particular, perhaps moving to Maxwellian spacetime will tune the fiddle.

3.2. Newtonian gravity in Maxwellian spacetime

Newtonian gravity in Maxwellian spacetime, Max NG for short, is obtained by adding a Newtonian gravitational field to Maxwellian spacetime. Dynamically possible models of Max NG are of the form $(M, h^{ab}, t_a, \nabla_a, \phi, \rho)$, where $(M, h^{ab}, t_a, \nabla_a)$ is Maxwellian spacetime, and ϕ and ρ are scalar fields representing the Newtonian gravitational potential and the mass density, respectively. The field equations are

(1) $h^{ab}t_b = 0$ (orthogonality),
(2) $\nabla_c h^{ab} = 0 = \nabla_a t_b$ (compatibility),
(4) $R^{ab}_{cd} = 0$ (rotation standard),
(8) $h^{ab}\nabla_a\nabla_b\phi = 4\pi G\rho$ (Poisson equation),

where G is the Newtonian gravitational constant. The equation of motion is

(9) $\xi^a\nabla_a\xi^b = -h^{ab}\nabla_a\phi$

for particle trajectories with four-velocity ξ^a. The spacetime symmetries of Max NG are (Max). The dynamical symmetries should leave (9) covariant in Maxwellian frames in which it takes the form $\ddot{x}^i + R^i_m\ddot{a}^m = -h^{ij}\partial\phi/\partial x^j$. The most general type of transformation that does this is an element of (Max) with the ϕ-transformation $\phi \mapsto \phi' = \phi - x^i\ddot{a}^i + \varphi(t)$, where φ is an arbitrary function of t.

On first glance, Max NG appears more in tune than Neo-Newt NG in so far as its spacetime symmetries agree with its dynamical symmetries (up to arbitrary φ). It turns out, however, that this alone does not solve the problem afflicting Neo-Newt NG. Max NG still posits an in-principle unobservable distinction between non-accelerated motion and gravitationally accelerated motion. This is due to the fact that, on the one hand, the theory explicitly posits the existence of a gravitational potential field ϕ as the cause of gravitationally accelerated motion; hence, from the point of view of the dynamics, non-accelerated frames are distinct from gravitationally accelerated frames by the absence of ϕ in the former. On the other hand, this distinction is in-principle unobservable since, from the point of view of the spacetime, ϕ can always be transformed away: For any value of ϕ, one can always define a new set of Maxwellian frames by $R^i_m \ddot{a}^m = -h^{ij}\partial\phi/\partial x^j$.

The problem with Max NG is that its spacetime degrees of freedom do not "mesh" with its dynamical degrees of freedom. Formally, the arbitrary functions $a^i(t)$ are not explicitly identified by Max NG with the gravitational potential. Simply put, Max NG does not incorporate a principle of equivalence. Doing so paves the way to the geometricized version of Newtonian gravity known as NCG.

4. Newton–Cartan gravity

NCG identifies the trajectories of objects in free fall (experiencing no other force than the gravitational field) with the geodesics of a non-flat connection. This is done explicitly in two steps: First, one replaces the Poisson equation (8) with a generalized Poisson equation

$$R^c_{abc} = R_{ab} = 4\pi G\rho t_a t_b, \tag{10}$$

which identifies the source ρ of the Newtonian gravitational potential with the curvature tensor R^a_{bcd}. Second, one replaces the equations of motion (9) with the geodesic equation for the connection associated with R^a_{bcd}

$$\xi^a \nabla_a \xi^b = 0 \tag{11}$$

for particle trajectories with four-velocity ξ^a.

At this point, several observations are pertinent. First, the problem afflicting Max NG does not occur in NCG. In effect, the gravitational potential has been absorbed into the curvature of the spacetime; hence, the equation of motion for particles, given by (11), does not posit the existence of a physical gravitational field whose influences are indistinguishable from non-accelerated motion. Rather, non-accelerated motion now includes the special case of "gravitationally accelerated" motion. (As will be seen, the distinction between non-accelerated motion and "gravitationally accelerated" motion is still in-principle unobservable. What (10) and (11) effect is simply the explicit elimination of the gravitational potential term from the equation of motion.)

Second, the generalized Poisson equation (10) indicates that the NCG connection is dynamic in the sense that it is determined in part by the mass density ρ. On the other hand, note that not all of it is dynamic: a large part of it remains absolute (in the sense of being independent of matter terms). Just how much remains absolute is important in so far as this will determine what the spacetime symmetries of NGC are. Unlike general relativity, NGC does have absolute spacetime structure that remains unaffected by matter. In particular, part of the NGC connection contributes to this absolute spacetime structure.

It turns out that there is some lee-way in implementing this geometrization procedure. For a given derivative operator ∇_a and a scalar function ϕ, the compatibility and orthogonality constraints (1), (2), pick out a unique connection given by $\Gamma'^a_{bc} = \Gamma^a_{bc} + h^{ad}\nabla_d\phi t_b t_c$ that satisfies $\xi^a\nabla'_a\xi^b = 0$ *if and only if* $\xi^a\nabla_a\xi^b = -h^{ab}\nabla_b\phi$, for any unit timelike vector field ξ^a (Malament, 1995, p. 498). In general, ∇_a may be an arbitrary classical connection. In the special case in which ∇_a is Neo-Newtonian (i.e., spatiotemporally flat), the new connection associated with ∇'_a satisfies further constraints. In general, however, it need not. These additional constraints become important in considering geometricized theories that are the "Newtonian limit" of general relativity, or that reproduce, for instance, the standard form of the Poisson equation (8). Thus, there arises the possibility of different versions of NCG, depending on what additional constraints one imposes on the curvature. In what follows, I will consider two versions; what I will refer to as weak NCG and strong NCG. I will be primarily concerned with their relationship to each other and to the theories of Newtonian gravity in classical spacetimes described above.

4.1. Weak NCG

Versions of weak NCG that have appeared in the physics literature include Künzle (1972), Duval and Künzle (1984), Künzle and Duval (1994), and De Pietri, Lusanna, and Pauri (1995). In the following, I will first characterize the features common to all these presentations and then look briefly at the version given in Duval and Künzle (1984).

Weak NCG can be characterized by dynamically possible models of the form (M, h^{ab}, t_a, ∇_a, ρ). Here the dynamical objects are a scalar field mass density ρ, and part of a connection associated with the derivative operator ∇_a. The absolute objects include the spatial metric h^{ab}, the temporal metric defined by t_a, and part of the connection associated with ∇_a. These objects are required to satisfy the following field equations:

(1) $h^{ab}t_b = 0$ (orthogonality),
(2) $\nabla_c h^{ab} = 0 = \nabla_a t_b$ (compatibility),
(10) $R^c_{abc} = R_{ab} = 4\pi G\rho t_a t_b$ (generalized Poisson equation),
(11) $\xi^a\nabla_a\xi^b = 0$ (equation of motion),
$R^{[a\ \ c]}_{[b\ \ d]} = 0$ (Curl-freeness).[11] (12)

As indicated above, conditions (10) and (11) implement the equivalence principle. Again, together they imply that particles experiencing forces with mass density

[11] Explicitly, $R^{ac}_{bd} = R^{ca}_{db}$ with indices raised by h^{ab}. The label "curl−freeness" is explained below. Some early presentations of NCG impose a slightly weaker condition $R^{[a\ \ c]}_{(b\ \ d)} = 0$ (Künzle, 1972, p. 350; Misner et al.,1973, p. 301).

sources follow geodesics of the weak NCG connection. This entails that gravitational acceleration terms in the equations of motion can always be absorbed; hence gravitational accelerations are relative.

Condition (12) is an additional constraint on the connection that can be motivated in a number of ways. First, Dixon (1975) has shown that it is the only additional constraint that is consistent from a group theoretic point of view with conditions (1), (2), and (10). Second, it is necessary in demonstrating that weak NCG is the $c \to \infty$ limit of general relativity.[12] Third, it goes *part way* in allowing recovery of the standard formulation of Newtonian gravity in Neo-Newtonian spacetime (Neo-Newt NG). This last motivation will become important in the discussion of Strong NCG in the next section, so it bears fleshing out. The following is adapted in slightly modified form from Künzle (1972, pp. 351–352).

To recover Neo-Newt NG from weak NCG, one can first show that conditions (1) and (2) determine the connection up to an arbitrary 2-form F_{ab}. In particular, given (1) and (2), the connection components can be decomposed according to

$$\Gamma^a_{bc} = {}^u\Gamma^{\ a}_{bc} + t_{(b}F_{c)d}h^{da}, \tag{13}$$

where ${}^u\Gamma^a_{bc}$ is the unique connection for which the arbitrary unit timelike vector field u^a is geodetic, $u^a\,{}^u\Gamma_a u^b = 0$, and curl-free, $h^{a[b}\,{}^u\nabla_a u^{c]} = 0$.[13] Condition (12) requires locally that the 2-form be closed: $\nabla_{[a}F_{bc]} = 0$ (in this sense, it imposes a "curl-free" condition). It follows that, locally, it can be given by $F_{ab} = 2\nabla_{[a}A_{b]}$ for arbitrary 1-form A_b. Hence, a connection satisfying (1), (2) and (12) is determined up to an arbitrary 1-form A_a. Intuitively, such a 1-form does not uniquely determine a scalar function that we could associate with the Newtonian gravitational potential. To see this more concretely, choose a coordinate chart adapted to the temporal and spatial metrics.[14] The connection components are then given by

$$\Gamma^i_{00} = 2h^{ik}F_{0k}, \quad \Gamma^i_{0j} = h^{ik}F_{jk}, \quad \Gamma^\alpha_{\beta\gamma} = 0 \quad \text{otherwise}, \tag{14}$$

and the components of the Ricci tensor are

$$R_{00} = 2\partial_i F^i_0 - F_{ij}F^{ij} = 4\pi G\rho, \quad R_{\alpha\beta} = 0 \quad \text{otherwise}. \tag{15}$$

If we now introduce the field $A^i \equiv 2F^i_0$, condition (12) then entails $\partial_{[i}A_{j]} = 0$; hence, A_i can be given locally by $A_i \equiv \partial_i\phi$, for some scalar function ϕ. The Ricci tensor components then become

$$R_{00} = \partial_i\partial^i\phi - F_{ij}F^{ij} = 4\pi G\rho, \quad R_{\alpha\beta} = 0 \quad \text{otherwise}. \tag{16}$$

Thus, while condition (12) allows us to introduce a scalar function ϕ, we cannot yet identify it as a Newtonian gravitational potential. This is only possible if we can

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[12] Condition (12) imposes a Riemannian symmetry on the classical connection that makes it possible to recover it as a $c \to \infty$ limit of a Riemannian connection. See, e.g., Malament (1986, pp. 194–196) who demonstrates this holds for the weaker $R^{[a\ c]}_{(b\ d)} = 0$ case by way of holding for the stronger case (12).

[13] The "flat-for-u^a" connection ${}^u\Gamma^a_{bc}$ is given explicitly by ${}^u\Gamma^a_{bc} = h^{ad}(\partial_{(b}{}^u h_{c)d} - 1\backslash 2\partial_d{}^u h_{bc}) + u^a\partial_{(b}t_{c)}$, where ${}^u h_{ab}$ is the projection of h^{ab} relative to u^a, defined by the conditions ${}^u h_{ab}h^{bc} = \delta^c_a - t_a u^c$, and ${}^u h_{ab}u^b = 0$ (see, e.g., Künzle, 1972, pp. 348–349; Christian, 1997, p. 4847).

[14] In such a chart $\{t, x^i\}$, $t_a = (dt)_a$, $u^a = (\partial/\partial t)^a$, and $h^{ab} = \delta^{ij}(\partial/\partial x_i)^a(\partial/\partial x_j)^b$; the latter since the generalized Poisson equation (10) entails spatial flatness (see, e.g., Malament, 1986, p. 188).

recover the Poisson equation (8), and this is blocked by the appearance of terms in R_{00} depending on the "spatial part" F_{ij} of the 2-form F_{ab}. To recover the Poisson equation, such terms must be forced to vanish. Two options can be considered:[15]

(a) We can require space to be asymptotically flat.
(b) We can impose condition (4) on the curvature. This entails we can further specialize to Maxwellian coordinates in which $\Gamma^i_{0j} = 0$.

Hence, while weak NCG is the $c \to \infty$ limit of general relativity, it does not constitute the geometricized version of the standard formulation of Newtonian gravity in Neo-Newtonian spacetime, in so far as it cannot recover the Poisson equation (8) without the imposition of additional assumptions. Note, further, that, if option (a) is adopted, what is recovered is not, strictly speaking, Neo Newt NG. Option (a) is equivalent to the "island universe" assumption, which requires the scalar function ϕ to vanish at spatial infinity: $\phi \to 0$ as $x^i \to \infty$. Recall that this assumption reduces the dynamical symmetries of Neo Newt NG from (Max) to (Gal) (with accompanying ϕ-transformations). Thus, strictly speaking, weak NCG plus option (a) recovers a restricted version of Neo Newt NG.

Finally, note that conditions (1) and (2) are sufficient for an "inertial/gravitational split" of the connection, up to an arbitrary timelike vector field u^a. They allow us to identify a "flat-for-u^a" (i.e., "inertial") part of the connection, and a "non-flat-for-u^a" (i.e., "gravitational") part (although, strictly speaking, the "gravitational" part should not be associated with Newtonian gravity, given that the Newtonian potential cannot be recovered from it). This indicates that such a split is *not* sufficient to recover the Poisson equation (8). Again, what is explicitly required for such a recovery is an additional assumption of the form of (a) or (b) above.

4.1.1. Duval and Künzle's (1984) "Gauge" theory of weak NCG

As noted above, conditions (1), (2) and (12) only determine a weak NCG connection up to an arbitrary 1-form A_a, or, equivalently, up to a unit timelike vector field u^a.[16] This motivates Duval and Künzle's (1984) version of weak NCG which identifies the degrees of freedom of the connection as a gauge given (redundantly) by the pair (u^a, A_a). Any other pair (u'^a, A'_a) reproduces the same weak NCG connection, so long as u'^a is a unit timelike vector field. Duval and Künzle demonstrate that this condition holds, and thus weak NCG connections are invariant, under transformations of the following form:

$$u^a \mapsto u'^a = u^a + h^{ab} w_b,$$
$$A_a \mapsto A'_a = A_a + \partial_a f + w_a - (u^b w_b + 1/2 h^{cd} w_c w_d) t_a, \tag{17a}$$

[15] See Künzle (1972, p. 352). Künzle's second option is to require the global condition $H^2(\Sigma_t, \mathbf{R}) = 0$. This requires holonomies on Σ_t to vanish, which is equivalent to imposing condition (4) on the curvature tensor.

[16] More precisely, given a weak NCG connection and a timelike u^a, then there exists a unique ${}^u\Gamma$ and a 1-form A_a such that the connection can be decomposed as in (13). Conversely, for every weak NCG connection Γ, there exists locally a unit timelike, non-rotating, geodesic u^a such that $\Gamma = {}^u\Gamma$. See, e.g., Christian (1997, p. 4849) and references therein.

where w_a is an arbitrary 1-form, and $f \in C^\infty(M)$ is an arbitrary scalar function. They then formulate weak NCG as a theory given by a connection on a principle $U(1)$ bundle, call it P, over a classical spacetime satisfying conditions (1), (2), and (12). They construct P as a restriction of a Bargmann frame bundle $B(M)$ over M.[17] It turns out that the connection on $B(M)$ defines a family of connections on P that are in 1–1 correspondence with time-like vector fields u^a on M. They thus identify a "Bargmann gauge" as a choice of the pair (u^a, A_a) and identify the "gauge group" of weak NCG as the group Aut($B(M)$) of automorphisms of $B(M)$. This is given by the group Diff(M) of diffeomorphisms on M together with vertical automorphisms on the unit tangent bundle over P given by (17a) and the $U(1)$ phase factor transformations[18]

$$\chi \mapsto \chi' = \chi + f. \tag{17b}$$

In an earlier work (Duval & Künzle, 1978), it was shown that suitable conservation laws can be obtained if the general form of the matter Lagrangian depends on the fields (h^{ab}, t_a, u^a, A_a) and is invariant under Aut($B(M)$). The procedure is essentially an application of Noether's 2nd Theorem and follows the general relativistic case in which conservation of stress–energy is derived by requiring the general form of the matter Lagrangian to be invariant under Diff(M).[19] In the weak NCG case, invariance under (17a) and (17b) produces a matter current conservation equation, and invariance under Diff(M) produces a "stress–energy" conservation equation. In the latter case, however, the conservation equation obtained is not in the form of the vanishing of a divergence. In fact, as Duval and Künzle (1984, p. 340) concede, it is only called a "stress–energy" equation in analogy with the relativistic case, and for concrete weak NCG matter Lagrangians, the actual stress–energy tensor derived via Noether's 2nd Theorem is not Aut($B(M)$)-invariant. This is demonstrated in subsequent work, which established concrete matter Lagrangians for the coupling of the weak NCG gravitational field to a complex scalar field that obeys the Schrödinger equation (Duval & Künzle, 1984); for a non-relativistic analogue of

[17] The Bargmann group is the projective Galilei group, i.e., the Galilei group up to an arbitrary $U(1)$ phase (more precisely, it is the non-trivial central extension of the Galilei group). Lagrangians for Galilean massive particles are not, in general, invariant under (Gal), containing a gauge freedom given by the non-trivial exponents of (Gal) (see, e.g., Lévy-Leblond, 1971, pp. 254–257). Switching to the Bargmann group thus restores invariance. The $U(1)$ bundle P is constructed as the quotient $B_0(M)/G_0(M)$ of a "homogeneous" Bargmann bundle over M by a homogeneous Galilei bundle. The construction rests ultimately on the fact that $G_0(M)$ is uniquely determined by the pair (h^{ab}, t_a) on M satisfying (1) and (2) (see footnote 3).

[18] Technically, since the 1-form w_a in (17a) is defined modulo t_a, the vertical automorphisms must be factored with respect to the relation $w_a \sim w'_a$ *iff* $w'_a = w_a + \sigma t_a$, for arbitrary function σ.

[19] See, e.g., Wald (1984, p. 456). Such a law is sometimes referred to as a "strong" conservation law. It requires only that the gravitational field equations are satisfied, but is independent of both their explicit form and the explicit form of the matter field equations.

the Dirac–Maxwell theory (Künzle & Duval, 1984); and for a perfect fluid (Künzle & Nester, 1984).[20]

4.1.2. The status of weak NCG as a Gauge theory

Duval and Künzle (1984, p. 333) claim that their Bargmann frame bundle version of weak NCG "... achieves the status of a gauge theory about as much as general relativity". They furthermore state that, "We attempt to present Newtonian gravity as much as possible as a gauge theory of the Bargmann group. This cannot fully succeed, at least not in the narrow sense of a Yang–Mills-type gauge theory, just as general relativity is not simply the gauge theory of the Poincaré (or the Lorentz) group" (1984, p. 334).

Given these remarks, in what sense is Duval and Künzle's version of weak NCG a gauge theory? Since there are a number of senses of what it means to be a gauge theory, it is perhaps helpful to consider what Duval and Künzle's weak NCG is *not*. Two points seem relevant here. Note first that in the work reviewed above, while explicit *matter* Lagrangians have been constructed that, when extremized, produce appropriate conservation laws and equations of motion, no *gravitational* Lagrangian is given that produces all the relevant weak NCG field equations, and in particular, the generalized Poisson equation (10). This indicates immediately that this version of weak NCG cannot be formulated as a constrained Hamiltonian system; hence, at least according to one sense of gauge, it is not a gauge theory. Two qualifications are perhaps relevant here.

(i) In subsequent work, Duval and Künzle have extended their version of weak NCG to a theory given by a Bargmann frame bundle over a five-dimensional base manifold.[21] In this theory, they have shown that the Poisson equation can be obtained from a (singular) Lagrangian with a Lagrange multiplier interpreted as the mass density source of the gravitational field. Thus, the possibility exists for a constrained Hamiltonian analysis (which the authors do not give) and hence for treating this *five-dimensional* theory as a gauge theory. Briefly, the five-dimensional manifold $\mathscr{M}$ is equipped with a (five-dimensional) Lorentzian metric g and a vector field ζ. The quotient manifold $\mathscr{M}/\{\text{orbits of } \zeta\}$ then produces a four-dimensional Lorentzian manifold for spacelike ζ, or a four-dimensional classical spacetime satisfying (12) for null ζ, respectively. The construction is based on the fact that the Lorentz group SO(1,3) and homogeneous Galilei group are subgroups of (the identity component of) the

[20] To get a taste of these constructions, consider Duval and Künzle's (1984) derivation of an NCG-covariant Schrödinger equation. They start with the standard one-particle Schrödinger Lagrangian density $\mathscr{L} = \hbar^2/(2m)\delta^{ab}\partial_a\Phi\partial_b\bar{\Phi} + (i\hbar/2)(\Phi\partial_t\bar{\Phi} - \bar{\Phi}\partial_t\Phi)$ and impose minimal coupling in the form of the replacements $\delta^{ab} \to h^{ab}$, $\partial_a \to D_a$, and $\partial_t \to u^a D_a$, where $D_a \equiv \partial_a - im/\hbar A_a$ is the NCG-covariant derivative defined by the connection on P. The result is $\mathscr{L}_{\text{Sch}} = \{(\hbar^2/2m)h^{ab}D_a\Phi\overline{D_b\Phi} + (i\hbar/2)u^a(\Phi\overline{D_a\Phi} - \bar{\Phi}D_a\Phi)\}$, where Φ is now interpreted as a section of a vector bundle associated with P. Extremizing $\mathscr{L}_{\text{Sch}}$ with respect to the matter fields produces an NCG-covariant one-particle Schrödinger equation of the form originally derived by Kuchar (1980). Kuchar (1980) also provides a matter Lagrangian for a single massive classical particle which produces the appropriate equation of motion (11).

[21] For a summary, see Künzle and Duval (1984) and references therein.

de Sitter group SO(1,4) that leave invariant a spacelike or a null vector, respectively. In the null case, condition (12) is satisfied automatically due to the Riemannian nature of g (see footnote 12). To recover the Poisson equation, the authors adopt the Einstein–Hilbert action, add the null vector constraint $g(\zeta, \zeta) = 0$, and require $\delta \int_{\mathscr{M}} (R + \lambda g(\zeta, \zeta))\text{vol} = 0$, where R is the Ricci scalar on $\mathscr{M}$ and λ is a Lagrange multiplier. Upon extremization, they obtain $R = 0$ and $R^{ab} = -\lambda\zeta^a\zeta^b$. The latter projects to the four-dimensional manifold as $R_{ab} = 4\pi G t_a t_b$ if the Lagrange multiplier λ is interpreted as the mass density source $\lambda = -4\pi G\rho$.

(ii) Künzle and Nester (1984) cast weak NCG in a $(3+1)$-dimensional form in a manner similar to the ADM Hamiltonian formulation of general relativity. In particular, they indicate how the Poisson equation arises from a limit of constraint equations in the relativistic case. Instructively, these constraints are associated with non-rotating coordinates (more precisely, they stem from the maximal slicing and maximal distortion choices for the lapse and shift functions in the relativistic case). They are careful to note, however, that their $(3+1)$ formulation of weak NCG does not produce a Hamiltonian as in the relativistic case. Rather, their choices for the $(3+1)$ decomposition are informed by formulating the relativistic case in a way that allows a $c \to \infty$ limit to be consistently taken. (Mathematically, the "Hamiltonian" obtained from their $(3+1)$ decomposition cannot be obtained from a standard symplectic form on a cotangent bundle, as in the relativistic case.)

The second point is that Duval and Künzle's original Bargmann bundle formulation of weak NCG is similar to frame bundle formulations of general relativity, in which the base space M does not come prepackaged with absolute objects, and the frame bundle is (typically) the bundle of Poincaré frames. These formulations of general relativity can be given the status of gauge theories by "gauging" the Poincaré group in a manner similar to Yang–Mills theories. The result is what is generally referred to as *Poincaré Gauge theory* (PGT).[22] Here one starts with a matter Lagrangian that is invariant under "global" Poincaré transformations. These are then promoted to "local" transformations by requiring that they be dependent on spacetime coordinates.[23] Gauge potential fields are then introduced to maintain Poincaré invariance of the Lagrangian. These fields turn out to be the connection on the Poincaré frame bundle over M (rotational gauge) and the tetrad fields (translational gauge). The Einstein equations are then obtained by extremizing the Lagrangian with respect to the gauge potentials. It should be noted that PGT is not, strictly speaking, a Yang–Mills-type gauge theory. The algebra of constraints for PGT is open (it is not a Lie algebra), unlike the Yang–Mills case. At this point, it should be obvious that Duval and Künzle's weak NCG is not this type of gauge theory. They do not "gauge" the Bargmann group; hence, their theory should not be conceived as a non-relativistic version of PGT.

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[22] See Hammond (2002, pp. 612–615) for a quick review.

[23] See Earman (2002) for discussion on the terminological nuances of the terms in scare quotes, as well as Martin (2002) for discussion on the "logic" of the gauge argument.

One version of weak NCG that expressly follows the PGT lead is given by De Pietri et al. (1995). Here they "gauge" the Bargmann group and end up with 11 three-dimensional gauge potential fields given by Θ, h_{ij}, A_0, A_i (it turns out these are related to t_a, h^{ab}, and A_a in the four-dimensional formulation). They obtain an appropriate matter Lagrangian that reproduces the equations of motion, but to get a gravitational Lagrangian, they essentially employ the same tactic as Künzle and Nester (1984) by using a $c \to \infty$ limit procedure on the relativistic case. In particular, they identify the gravitational part of the weak NCG Lagrangian with the zeroth-order term of a $1/c^2$ expansion of the standard Einstein–Hilbert action of general relativity, motivated in part by Kuchar's (1980) method of obtaining a consistent NCG matter Lagrangian, and in part by the fact that the Bargmann group is the $c \to \infty$ contraction of the Poincaré group. They then perform a Hamiltonian analysis of their complete matter+gravitational NCG Lagrangian and indicate how the Poisson equation falls out of a combination of constraint equations. From Künzle and Nester's (1984) analysis, however, it appears that this theory is not yet in the form of a constrained Hamiltonian system.

What, then, is the status of Duval and Künzle's version of weak NCG as a gauge theory? The following conclusions can be drawn:

1. It is not a gauge theory in the sense of being a constrained Hamiltonian system. In this sense, it is unlike general relativity, which does admit constrained Hamiltonian formulations. In this sense, it is also not a Yang–Mills theory to the extent that a Yang–Mills theory can be defined as a certain type of constrained Hamiltonian system in which the algebra of constraints is closed.
2. It is not a gauge theory in a more looser sense of being a Yang–Mills theory; namely, a theory based on the gauging of a given symmetry group. In this sense, it is unlike general relativity, which admits formulations of this type (PGT-type theories). Note that, in this more looser sense, there is still a distinction between theories with closed constraint algebras (typical Yang–Mills theories) and theories with open constraint algebras (PGT-type theories).
3. It is a gauge theory in a very loose sense of being a theory associated with unphysical degrees of freedom (and being formulated in terms of fiber bundles). In this sense, it is like general relativity, which admits formulations simply in terms of a frame bundle over a base space.

It might be argued that (3) is too loose a notion for the concept of gauge. If this is the case, then Duval and Künzle's weak NCG is perhaps only suggestive of a gauge theory. General relativity, likewise, when formulated in terms of a frame bundle over a base space, is suggestive of a gauge theory. What Duval and Künzle make explicit in their formulation of weak NCG is the degrees of freedom of the weak NCG connection.

4.1.3. Weak NCG symmetries

Duval and Künzle's "gauge group" Aut($B(M)$), in addition to the vertical automorphisms (17a), (17b), also includes the base space automorphisms Diff(M). While this appears to motivate Duval and Künzle to consider weak NCG as "generally covariant" (Christian, 1997, p. 4852; follows suite), arguably Diff(M) should not be considered a symmetry of the theory. Certainly, including Diff(M) as a gauge symmetry cannot be motivated by an appeal to Noether's 2nd Theorem (as, perhaps, can be done in the case of general relativity): as mentioned above, the "stress–energy" tensor obtained by requiring invariance under Diff(M) of an appropriate weak NCG matter Lagrangian does not satisfy a "strong" conservation law. And, of course, including Diff(M) as a symmetry simply because the weak NCG objects (h_{ab}, t^a, ∇_a) are invariant under Diff(M) is ill-advised. This would conflate a trivial notion of general covariance (one that is satisfied by any theory formulated using tensors on manifolds) with a non-trivial symmetry principle (one that weak NCG does not satisfy and that general relativity does).

Given that the vertical automorphisms (17a), (17b) represent, if not the "gauge" structure of NCG, then at least the degrees of freedom in the weak NCG connection, what can we say about the spacetime symmetries of the theory? First, note that talk of spacetime symmetries should make sense in the context of NCG, in so far as NCG contains absolute objects that remain unaffected by the dynamical contents of spacetime. Some authors identify multiple candidates for such symmetries (see, e.g., Trautman, 1965, pp. 115–117; Duval, 1993; Christian, 1997, pp. 4852–4853). Duval (1993), for instance, lists as candidates three extensions of the Lie algebras $\mathfrak{leib}$, $\mathfrak{max}$, $\mathfrak{gal}$ of the Leibniz, Maxwell and Galilei groups. These candidates are associated with different choices of "Bargmann gauge" (u^a, A_a). For instance, the "standard flat" choice $u^a = (\partial/\partial t)^a$, $A_a = 0$ (and hence, implicitly, ϕ = constant) is invariant under transformations generated by the Bargmann algebra $\widetilde{\mathfrak{gal}}$ (the non-trivial central extension of $\mathfrak{gal}$), whereas the choice $u^a = (\partial/\partial t)^a$, $A_a = -\phi t_a$ is invariant under transformations generated by an extension $\widetilde{\mathfrak{max}}$ of $\mathfrak{max}$; and the choice $u^a = (\partial/\partial t)^a$, with arbitrary A_a, is invariant under transformations generated by an extension $\widetilde{\mathfrak{leib}}$ of $\mathfrak{leib}$.[24]

To see how these Lie algebras come about in a bit more detail, consider weak NCG as given by a structure (M, h^{ab}, t_a, u^a, A_a) that satisfies $t_a u^a = 1$ and conditions (1), (2), (10)–(13), where in the latter, $F_{ab} = 2\nabla_{[a}A_{b]}$. Condition (13) defines a weak NCG connection in terms of Duval and Künzle's "Bargmann gauge" (u^a, A_a). We have seen that such a connection is not unique: Any other "Bargmann gauge" (u'^a, A'_a) satisfying (17a) defines the same connection. Now note that, as far as the Poisson equation (8) is concerned, not all "Bargmann gauges" are created equal. Only for a subclass of gauges can (8) be recovered from (10). It is not hard to be convinced that this subclass, call it a "Poisson gauge" (v^a, ϕ), defines a weak NCG connection by the condition

$$\Gamma^a_{bc} = {}^v h^a_{bc} + h^{ad}\nabla_d \phi t_b t_c, \tag{18}$$

[24] The significance of the Bargmann group, and thus $\widetilde{\mathfrak{gal}}$, was indicated in footnote 17. The extensions $\widetilde{\mathfrak{leib}}$ and $\widetilde{\mathfrak{max}}$ are required to account for the additional degree of freedom in A_a (resp. ϕ) upon spacetime transformations. It turns out that $\widetilde{\mathfrak{leib}}$ leaves A_a determined up to an arbitrary scalar function; $\widetilde{\mathfrak{max}}$ leaves ϕ determined up to a function of time; and $\widetilde{\mathfrak{gal}}$ fixes ϕ up to a constant. Technically, $\widetilde{\mathfrak{leib}} = \widetilde{\mathfrak{leib}} \times C^\infty(M)$, $\widetilde{\mathfrak{max}} = \mathfrak{max} \times C^\infty(T)$, and $\widetilde{\mathfrak{gal}} = \mathfrak{gal} \times \mathbf{R}$, where "$\times$" denotes a semi-direct product, and $T = M/\{\text{orbits of } t_a\}$ (see Duval, 1993, pp. 2220–2221 for details).

(where ${}^{v}\Gamma$ is defined in analogy with ${}^{u}\Gamma$). Comparing (13) and (18), one obtains the relations between a general "Bargmann gauge" and a special "Poisson gauge" as (Christian, 1997, p. 4849; Duval, 1993, p. 2219)

$$v^a = u^a - h^{ab}A_b,$$
$$\phi = 1/2h^{ab}A_aA_b - u^aA_a. \tag{19}$$

Intuitively, transformations between members of a "Poisson gauge" should be those that leave the Poisson equation (8) covariant. As was seen above, a sufficient condition for this is (4), and this entails that the spacelike displacement vector $h^{ab}A_b$ is covariantly constant: $h^{ab}\nabla_b(h^{ac}A_c) = 0$; thus it can be given by $h^{ab}A_b = h^{ab}\partial_b f$, for arbitrary scalar f. Hence, the transformations that leave "Poisson gauges" invariant are given by

$$v^a \mapsto v'^a = v^a - h^{ab}\partial_b f,$$
$$\phi \mapsto \phi' = \phi - v^a\partial_a f. \tag{20}$$

This prompts Duval (1993) to consider weak NCG as given by structures of the form $(M, h^{ab}, t_a, u^a, v^a, \phi)$. On such a structure, the infinitesimal action of Aut($B(M)$) is the following:

$$\delta h = \mathfrak{L}_x h,$$
$$\delta t = \mathfrak{L}_x t,$$
$$\delta u = \mathfrak{L}_x u + h(y),$$
$$\delta v = \mathfrak{L}_x v + h(df),$$
$$\delta\phi = x(\phi) + v(f), \tag{21}$$

where x is a basis for the Lie algebra, y is an arbitrary 1-form on M, and f is an arbitrary scalar function (and indices have been suppressed for convenience). By setting one or more of these infinitesimal transformations to zero and solving for x, y, f, we recover the corresponding finite transformations on the objects h, t, u, v and ϕ. Duval (1993, pp. 2220–2221) now demonstrates that the conditions $\delta h = \delta t = \delta u = 0$ generate $\widetilde{\mathfrak{leib}}$, the conditions $\delta h = \delta t = \delta u = \delta v = 0$ generate $\widetilde{\mathfrak{max}}$, and the conditions $\delta h = \delta t = \delta u = \delta v = \delta\phi = 0$ generate $\widetilde{\mathfrak{gal}}$. From this we can infer, for instance, that u^a defines an "extended" Leibnizian frame (i.e., it is a member of a subclass of Leibnizian frames related by transformations generated by $\widetilde{\mathfrak{leib}}$), while v^a defines an "extended" Maxwellian frame. (Note that in Maxwellian coordinates, $v'^\alpha = v^\alpha + \ddot{a}^i h^{\alpha\beta}\partial_\beta x^i$, and thus $f = x^i\ddot{a}^i$. Hence, ϕ transforms as $\phi' = \phi - \partial f/\partial t = \phi - x^i\dddot{a}^i$ (see, e.g., Kuchar, 1980, p. 1288). Hence, an "extended" Maxwell transformation includes both a spacetime coordinate transformation and an accompanying ϕ-transformation. Thus, the dynamical symmetries of both Neo-Newt NG and Max NG are simply those generated by $\widetilde{\mathfrak{max}}$.)

The important question again is which symmetries should we associate with weak NCG? More precisely, which terms in (21) should we set to zero? Duval and others seem satisfied with simply listing the candidates. On the surface, such talk of multiple candidates for the symmetries of weak NCG is slightly misleading. The groups (and algebras) mentioned above represent very different symmetries. Certainly, the intrinsic structure posited by a given theory cannot exhibit *both* (Leib) and (Max) symmetries, for instance (Leibnizian spacetimes are rather different from Maxwellian spacetimes). Trautman (1965) indicates one view of the situation:

> A preferred coordinate system in a theory is one which puts some geometrical structure in the theory in a particularly simple form. The multiplicity of geometrical structures present in [geometricized] Newtonian theory thus enables us to have many different classes of preferred coordinate systems, some more useful than others. (Trautman, 1965, p. 116)

If putting a geometrical structure in a simple form makes a coordinate system privileged, then clearly there are multiple privileged coordinates in weak NCG, and hence multiple symmetries, insofar as there are many different geometrical objects in the theory. In this essay, however, a privileged coordinate system is one adapted to the intrinsic structure of the global spacetime, and not to individual geometrical object fields. Thus, it should make sense to say there is only one symmetry structure for weak NCG, as opposed to multiple candidates; namely, that one that is adapted to the structure of the background spacetime.

What then are the spacetime symmetries of weak NCG? Certainly, the absolute structure of weak NCG includes the metrics h^{ab} and t_a and condition (3) of spatial flatness; the latter since the generalized Poisson equation (10) entails spatial flatness (see, e.g., Malament, 1986, p. 188). Hence, weak NCG has as much structure as Leibnizian spacetime. Note further that weak NCG needs enough structure to support "extended" Leibnizian frames, in order to foliate spacetime with "Bargmann gauges" (u^a, A_a). But what weak NCG does not, strictly speaking, support is that particular subclass of "Bargmann gauge" (v^a, ϕ) that define "extended" Maxwellian frames. To pick out this subclass, additional assumptions need to be tacked on (viz., the "island universe" assumption, or the "no rotational holonomies" assumption). But it is now clear that weak NCG includes just a bit more absolute structure than Leibnizian spacetime, given specifically by condition (12), as well as the particular conditions, beyond spatial flatness, encoded in (10). These observations suggest that the symmetries of weak NCG be identified with the extended Leibniz algebra $\widetilde{\mathfrak{leib}}$ with basis x^a satisfying $\delta h = \delta t = \delta u = 0$ in (21). These are the symmetries that preserve (M, h^{ab}, t_a, u^a, A_a) subject to (1), (2), (10)–(13), and $u^a t_a = 1$. Again, these symmetries are a bit more constrained than those of Leibnizian spacetime (the symmetries of which are generated by $\mathfrak{leib}$). A weak NCG connection (14) is obtained from a Leibnizian connection (6) by the further conditions

$$R^i_m \dot{R}^m_j = h^{ik} F_{jk},$$
$$R^i_m \ddot{R}^m_j x^j + R^i_m \ddot{a}^m = 2h^{ik} F_{0k}, \tag{22}$$

due to the addition of (12), and subject to $-2\partial_i F^i_0 + F^{ij} F_{ij} = 4\pi G\rho$, due to the replacement of (3) with the stronger requirement (10). This suggests that, whereas in Leibnizian spacetime (in the absence of external forces), all rotations and (linear) accelerations are relative, in weak NCG spacetime, only certain types of rotation and (linear) acceleration are relative. In weak NCG spacetime, relative rotations are only those that can be given by a closed 2-form F_{ab}, and relative accelerations are only those induced by the mass density in conjunction with F_{ab}. Geometrically, a Leibnizian connection cannot distinguish between "straight", "curved", and "twisted" particle trajectories, in toto. A weak NCG connection fails to distinguish only a subclass of such trajectories.

The dynamical symmetries of weak NCG should leave the equation of motion (11) covariant in extended Leibnizian frames in which it takes the form $\ddot{x}^i + 2h^{ik}F_{0k} + h^{ik}F_{jk}\dot{x}^j = 0$. Evidently, these are transformations generated by $\widetilde{\mathfrak{leib}}$.

4.2. Strong NCG

Versions of strong NCG have appeared in Trautman (1965), Misner et al. (1973), Kuchar (1980), and Christian (1997, 2001). In the following, I will characterize its essential features and then assess Christian's (1997) version.

Strong NCG differs from weak NCG only in the addition of the rotation standard condition (4) on the curvature tensor. It can be characterized by dynamically possible models of the form $(M, h^{ab}, t_a, \nabla_a, \rho)$ that satisfy:

(1)	$h^{ab}t_b = 0$	(orthogonality),
(2)	$\nabla_c h^{ab} = 0 = \nabla_a t_b$	(compatibility),
(4)	$R^{ab}{}_{cd} = 0$	(Rotation standard),[25]
(12)	$R^{[a}{}_{[b}{}^{c]}{}_{d]} = 0$	(Curl-freeness),
(10)	$R^c{}_{abc} = R_{ab} = 4\pi G\rho t_a t_b$	(generalized Poisson equation),
(11)	$\xi^a \nabla_a \xi^b = 0$	(equation of motion).

Recall that the significance of adding condition (4) is that it allows recovery of the Poisson equation (8) without the need for imposing the "boundary condition" of asymptotic spatial flatness.

4.2.1. Christian's (1997) version of strong NCG

None of the versions of weak NCG reviewed in Section 4.1 are derived from a single four-dimensional Lagrangian. In particular, in none of these theories is the generalized Poisson equation (10) obtained by extremizing an appropriate four-dimensional action. Christian (1997) demonstrates that condition (4), which is sufficient to recover the Poisson equation (8), is *also* sufficient for the existence of a Lagrangian density for NCG. In particular, Christian is able to construct a Lagrangian density that is invariant under Duval and Künzle's Aut($B(M)$) and that reproduces all the field equations of strong NCG, including the generalized Poisson equation. Christian then recasts strong NCG as a $(3+1)$ constraint-free Hamiltonian system, and quantizes the theory in the reduced phase-space to obtain

[25] Trautman (1965, p. 107) writes $t_{[e}R^a{}_{b]cd} = 0$, which is equivalent to (4). In addition to (4) and (12), Misner et al. (1973, p. 300) also include the curvature constraint $R_b^{a\ cd} = 0$, which entails that arbitrary vector fields remain unchanged under parallel transport on spacelike hypersurfaces. This appears a bit redundant.

what amounts to a (Max)-invariant quantum field theory of Newtonian gravity. In the remainder of this section, I will review the main features of Christian's gravitational Lagrangian density and the $(3+1)$ Hamiltonian decomposition, stressing the role condition (4) plays in the derivation of the generalized Poisson equation.

4.2.2. *A Lagrangian density for strong NCG*

The following exposition may be made more perspicuous by a brief review of the Lagrangian formalism. In the Lagrangian formulation of a field theory, the field equations and equations of motion are derived from an action principle $\delta S = \delta \int \mathscr{L} d^4x = 0$, where the Lagrangian density $\mathscr{L} = \mathscr{L}(\varphi_i, \partial_\mu \varphi_i, x^\mu)$, $i = 1 \ldots N$, is a functional of N dynamical field variables $\varphi_i(x)$ and their first (and possibly higher-order) derivatives. The equations of motion take the form of the Euler–Lagrange equations $\partial \mathscr{L}/\partial \varphi_i - \partial_\mu \partial \mathscr{L}/\partial(\partial_\mu \varphi_i) = 0$. In some theories, the Hessian matrix of $\mathscr{L}$ is singular; hence, the dynamical variables are not all independent, but rather satisfy a set of constraint equations $\phi_m(\varphi_i, \partial_\mu \varphi_i) = 0$, $m = 1 \ldots M$. For such theories, both equations of motion and constraint equations can be derived from the modified action principle $\delta S' = \delta \int \mathscr{L}' d^4x \equiv \delta \int (\mathscr{L} - u^m \phi_m) d^4x = 0$, where u^m are arbitrary Lagrange multiplier fields. The equations of motion are obtained by extremizing $\mathscr{L}'$ with respect to the dynamical variables, while the constraint equations are obtained by extremizing $\mathscr{L}'$ with respect to the Lagrange multipliers u^m.[26]

Christian's strong NCG gravitational Lagrangian density $\mathscr{L}_{\text{grav}}$ is a functional of the dynamical field variables (u^a, A_a), up to second derivatives, and a set of parametrized kinematical variables ${}^{(s)}y$.[27] These kinematical variables are required to make $\mathscr{L}_{\text{grav}}$ manifestly invariant under the Diff(M) subgroup of Aut($B(M)$) in the manner of a parametrized field theory. They are given by maps ${}^{(s)}y : M \to M'$, from a "parametrized" manifold M to a "fixed" manifold M' containing absolute spacetime structure (in this case, M' is simply a classical spacetime). They thus allow the absolute structures on M' to be pulled back to M as dynamical fields.

In addition to the above variables, $\mathscr{L}_{\text{grav}}$ depends on a large set of Lagrange multiplier fields. By judiciously combining the resulting constraint equations, Christian is able to recover all the field equations of strong NCG. Below I indicate how, in particular, the generalized Poisson equation (10) is obtained, and how the relevant Lagrange multipliers are interpreted.

The field equations (1), (2), (4), (12), and the condition $u_a t^a = 1$, are all obtained as constraint equations by extremizing $\mathscr{L}_{\text{grav}}$ with respect to Lagrange multipliers.[28] In

[26] The transition to the Hamiltonian formalism involves defining the conjugate momenta $p_i \equiv \partial \mathscr{L}'/\partial \dot{\varphi}_i$, where $\dot{\varphi}_i \equiv \partial_t \varphi_i$, and the total Hamiltonian $H_{\text{T}} \equiv p_i \dot{\varphi}_i - \mathscr{L}'(\varphi_i, p_i(\varphi_i, \dot{\varphi}_i)) \equiv H_0 + u^m \phi_m$, where $H_0 = p_i \dot{\varphi}_i - \mathscr{L}$. (In general, some of the multipliers u^m may be determined via secondary constraints by requiring the ϕ_m to be conserved in the sense of $\{\phi_m, H_{\text{T}}\} = 0$. See, e.g., Henneaux & Teitelboim, 1992, pp. 13–14.)

[27] The following is a partial exposition of Christian (1997, pp. 4858–4867). I will ignore the matter field variables for simplicity, and refer the reader to Christian for the explicit form of $\mathscr{L}_{\text{grav}}$.

[28] A simple example of this method of deriving field equations for absolute spacetime objects as constraint equations of a singular Lagrangian density is given in Sorkin (2001).

particular, extremizing $\mathscr{L}_{\text{grav}}$ with respect to multiplier fields given by χ^{ab} and χ^{abc} produces (12) and (4), respectively. These latter two multipliers also play a role in the derivation of the generalized Poisson equation (10), which proceeds in two steps.

1. First, $\mathscr{L}_{\text{grav}}$ is extremized with respect to A_a (or, it turns out, equivalently with respect to u^a) to produce the equation of motion

$$\kappa \nabla_a \Theta^a + t_a \nabla_b \chi^{ab} + t_a \nabla_b \nabla_c \chi^{abc} = 4\pi G\rho, \tag{23}$$

   where κ is an arbitrary parameter and $\Theta^a \equiv h^{ab}\nabla_b\{1/2h^{cd}A_cA_d - A_cu^c\}$. It turns out that the multiplier χ^{ab} encodes the momentum conjugate to A_a, in so far as the momentum density conjugate to A_a is given by $\Pi^b = \wp t_a \chi^{ab}$ (where $\wp d^4x$ is the volume element).
2. The second step in the derivation of (10) involves extremizing $\mathscr{L}_{\text{grav}}$ with respect to yet another Lagrange multiplier χ, yielding the constraint equation

$$1/\wp \nabla_a \Pi^a - (1-\kappa)\nabla_a \Theta^a + \lambda \Lambda_N - \Lambda_0 = 0, \tag{24}$$

   where Λ_0 and λ are arbitrary scalars, and $\Lambda_N \equiv t_a \nabla_b \nabla_c \chi^{abc}$. As will be seen below in the Hamiltonian formulation, (24) is analogous to the momentum constraint in general relativity, and the multiplier χ can be interpreted as the $U(1)$ phase factor in (17b). Substituting (24) into (23) then yields

$$\nabla_a \Theta^a + \Lambda = 4\pi G\rho, \tag{25}$$

   where $\Lambda \equiv \Lambda_0 + (1-\lambda)\Lambda_N$. This reproduces the Poisson equation (8) with cosmological constant Λ just when $\Theta \equiv 1/2h^{ab}A_aA_b - A_cu^c$ can be interpreted as the Newtonian gravitational potential ϕ. This is justified by recalling from (19) that Θ is in the form of the potential ϕ in an arbitrary "Bargmann gauge", and that A_a in this formula can be interpreted as the NCG 1-form, given the recovery of (1), (2) and (12). Note further that this establishes the physical interpretation of the multiplier χ^{abc} as encoding a contribution Λ_N to the cosmological constant.[29]

From these two steps, the generalized Poisson equation (10) follows quickly. First, we know that the form for the Ricci tensor obtained from recovery of the field equations (1), (2), (4), and (12) is

$$R_{cd} = h^{ab}\nabla_a \nabla_b \phi t_c t_d \tag{26}$$

for some scalar ϕ identified as the Newtonian potential. The generalized Poisson equation (10) with cosmological constant is thus recovered from (25)

[29] Christian (1997, p. 4861) notes that if we set the free parameter $\lambda = \Lambda_0/\Lambda_N$, then $\Lambda = \Lambda_N$. In this way, recalling the definition of Λ_N, condition (4) (derived from the multiplier χ^{abc}) can be related, at least formally, to the cosmological constant. This is consistent with the fact that, if asymptotic flatness is imposed, then both condition (4) and the cosmological constant become redundant. Christian also notes that if one desires to make condition (4) independent of the cosmological constant, then one can simply set $\lambda = 1$, and thus have $\Lambda = \Lambda_0$.

and (26) in the form

$$R_{ab} + \Lambda t_a t_b = 4\pi G \rho t_a t_b. \tag{27}$$

The essential features of Christian's $\mathscr{L}_{\text{grav}}$ can be summarized by the following:

A. The strong NCG field equations (1), (2), (4), (12) appear as constraint equations, derived by extremizing $\mathscr{L}_{\text{grav}}$ with respect to Lagrange multipliers.
B. The generalized Poisson equation (10) is derived from extremizing $\mathscr{L}_{\text{grav}}$ with respect to the dynamical field variable A_a, and then applying constraint equations: Extremizing with respect to A_a yields an equation (23) dependent on multipliers. Solving for these multipliers, which involves making use of the momentum constraint equation (24), then reduces (23) to (10).

4.2.3. *Strong NCG as a constraint-free Hamiltonian system*

The Hamiltonian formalism of a field theory requires a $(3+1)$ split of spacetime into three-dimensional Cauchy surfaces Σ. It proceeds with the specification of Cauchy data in the form of dynamical variables that describe the instantaneous configuration of the fields on Σ, as well as their conjugate momenta. The equations describing the evolution of this data come in the form of Hamilton's equations of motion. In the context of constrained Hamiltonian systems, constraint equations on Σ are required for data to evolve uniquely. In the ADM constrained Hamiltonian formulation of general relativity, for example, the Cauchy surfaces are obtained as the level surfaces Σ_τ of a timelike field $\tau^a = (\partial/\partial\tau)^a$ and the Cauchy data consists of the 3-metric and the extrinsic curvature on Σ_τ. This data satisfies two constraint equations: the momentum constraint, which generates spatial diffeomorphisms on Σ_τ, and the Hamiltonian constraint, which generates "time" evolution (in terms of the chosen time function τ).[30] The latter implies that gauge-invariant quantities are constants of motion, which leads to the well-known problem of time.[31] It stems specifically from the freedom involved in making the initial $(3+1)$ split; in particular, in choosing a time function to label the 3-spaces.

In the NCG case, on the other hand, there is a natural choice of $(3+1)$ decomposition and a natural choice of time function; namely, the preferred foliation Σ_t adapted to t_a. The 3-metric on a given slice is given by the spatial projection ${}^u h_{ab}$ with respect to a unit timelike field u^a (footnote 13). With respect to u^a, any quantity can be decomposed into tangential and normal components. For the Cauchy data, Christian takes (u^a, A_a) and their conjugate momenta $({}^u\Pi_a, \Pi^a)$, as well as the kinematical variable θ^a (with conjugate momentum π_a), which is a parametrization of the global time function.[32] It turns out the normal components of the momenta

[30] In the ADM formulation, the vector field τ^a is decomposed as $\tau^a = Nn^a + N^a$, where n^a is normal to the Σ_τ and N^a and N are the "shift" vector and "lapse" function. These latter appear as Lagrange multipliers in the ADM singular Lagrangian, associated respectively with the momentum and the Hamiltonian constraints. For details, see Wald (1984, pp. 463–465).

[31] See, e.g., Earman (2002) for a quick review.

[32] Technically, $\theta^a : \Sigma_t \to M'$ is an embedding of the 3-spaces Σ_t in the fixed manifold M'. It allows the global time function on M' to be pulled back to the parametrized manifold M. Its "u^a-derivative" is given by $\dot{\theta}^a = u^a \nabla_a \theta^a = \partial\theta^a/\partial t$, and dictates the transition from one leaf of the foliation to another. It can be

${}^u\Pi_a, \Pi^a$ vanish, so only the tangential components of u^a and A_a contribute to unique Cauchy evolution. Moreover, these momenta are not independent, being related via

$$ {}^u\Pi^a = -{}^u h_{ab}\Pi^b, \tag{28}$$

which reflects the redundancy in specifying the pair (u^a, A_a) to determine an NCG connection.

By including the global time function as a canonical variable, Christian is setting the stage for a time-parametrized Hamiltonian system (see, e.g., Henneaux & Teitelboim, 1992, p. 103). In any theory with action $S[\varphi^a(t), p_a(t)] = \int (p_a(\mathrm{d}\varphi^a/\mathrm{d}t) - H_0)\,\mathrm{d}t$, the time variable t can be parametrized. This is done by introducing a canonical variable $\varphi^0 \equiv t$ with conjugate momentum p_0, and replacing S with $S'[\varphi^0(\tau), p_0(\tau); \varphi^a(\tau), p_a(\tau); u^0(\tau)] = \int (p_0\dot{\varphi}^0 + p_a\dot{\varphi}^a - u^0(p_0 + H_0))\,\mathrm{d}\tau$, where the dot represents the derivative with respect to the parameter τ, and u^0 is an arbitrary Lagrange multiplier. Extremizing S' with respect to p_0 and u^0 yields the equation of motion $\dot{t} - u^0 = 0$, and the "Hamiltonian" constraint equation $p_0 + H_0 = 0$. Substituting $p_0 = -H_0$ and $u^0 = \dot{t}$ into S' yields S; hence, the motion derived from S' is identical to that derived from S. Note further that no (first-class) Hamiltonian occurs in S', in the sense that the total Hamiltonian for S' consists solely of the constraint term $u^0(p_0 + H_0)$ (see footnote 26). In general relativity, since there is no global time function to begin with, the action is already time-parametrized, and the vanishing of the Hamiltonian reflects this. The task of "de-parametrizing" the theory involves solving the, in general, complicated Hamiltonian constraint. In NCG, since there is a natural global time function, the task of de-parametrizing a parametrized version of the theory should not be so difficult. This is reflected in Christian's approach.

Christian's Hamiltonian density is given in the time-parametrized form $\aleph^a H_a \equiv ({}^u\Pi_a\dot{u}^a + \Pi^a\dot{A}_a + \pi_a\dot{\theta}^a) - \mathscr{L}_{\text{grav}}$, where $\aleph^a$ is a multiplier field and $H_a \equiv \pi_a + H'_a$, where H'_a is a functional of the Cauchy data. One obtains a set of six equations of motion and 10 constraint equations. The equations of motion include (23), condition (12), and $\dot{\theta}^a - \aleph^a = 0$.[33] The constraint equations naturally include the "Hamiltonian constraint"

$$\pi_a + H'_a = 0. \tag{29}$$

Unlike its general relativistic counterpart, (29) is linear in the momenta conjugate to the "time" function $\dot{\theta}^a$, and hence can be solved. (In general relativity, the Hamiltonian constraint is quadratic in the momenta, which prevents, in general, a solution. Hence, a fully reduced phase space for general relativity cannot in general be constructed.) Additional constraint equations include:

(a) the classical spacetime structure equations (1), (2);
(b) the condition $u^a t_a = 1$;

(*footnote continued*)
decomposed into normal and tangential components as $\dot{\theta}^a = {}^\perp\dot{\theta}u^a + {}^\parallel\dot{\theta}^a$, where ${}^\perp\dot{\theta} \equiv \dot{\theta}^a t_a$ corresponds to the lapse function of the ADM formulation, and ${}^\parallel\dot{\theta}^a \equiv \delta^a_b\dot{\theta}^b$ is the shift (footnote 30).

[33] The equations of motion also include the condition $\dot{H}_a = 0$, indicating the Hamiltonian constraint (29) below is preserved in time.

(c) Eq. (23);
(d) condition (4);
(e) Eq. (24);
(f) the constraints on momenta (28).

Constraint (a) can be eliminated by working with a non-Diff(M) invariant background spacetime structure. (Formally, the functions $t_\alpha, h^{\alpha\beta}$ are constant on the phase space. This means, as far as the dynamics is concerned, we do not have to work with the parametrized manifold M; rather, we can work on the "fixed" manifold M'.) Constraint (b) is redundant, given already by the natural foliation Σ_t adapted to u^a; and constraint (c) is also redundant, since it already appears as an equation of motion. Thus, at this point, Christian has a constrained Hamiltonian formulation of strong NCG containing the three constraints (d)–(f) on the Cauchy data.[34]

It turns out that constraints (d) and (e) can be eliminated simultaneously. Enforcing condition (4) on the Cauchy data entails a modification of the constraint equation (24) that reduces it to the equation of motion (23) in the limit $\kappa \to 0$. It is instructive to compare this process with the analogous one in general relativity. By eliminating the Θ^a-term in (23) and (24), the NCG "momentum" constraint (24) can be rewritten as

$$1/\wp \nabla_a \Pi^a + \Lambda_N - (1-\kappa)4\pi G\rho - \kappa\Lambda = 0. \tag{30}$$

In general relativity, the momentum constraint can be satisfied by taking equivalence classes of 3-metrics up to spatial diffeomorphism on Σ_τ (the resulting partially reduced phase space is referred to as "superspace"). In the NCG case, Christian takes equivalence classes of A_a's up to $A_a \mapsto A_a + \nabla_a f$, which entails the momenta satisfy $1/\wp \nabla_a \Pi^a - 4\pi G\rho = 0$. This is not yet (30). However, further constraining the A_a's to satisfy condition (4) entails that the momenta satisfy $1/\wp \nabla_a \Pi^a + \Lambda_N - 4\pi G\rho = 0$, and this is identical to (30) in the limit $\kappa \to 0$. Hence, condition (4) guarantees that the NCG "momentum" constraint is satisfied for a particular choice of the free parameter κ.[35]

At this point in the reduction process, all constraints have been eliminated with the exception of (f). To eliminate (f), Christian further restricts the Cauchy data by defining a new set given by (v^a, p_a), where $v^a = u^a - h^{ab}A_b$ and $p_a = \Pi^u{}_a$. His constraint-free Hamiltonian density then takes the form $\aleph^a H_a = p_a \dot{v}^a + \pi_a \dot{\theta}^a - \mathscr{L}_{\text{grav}}$. The reduced phase space is the cotangent bundle T^*Z over an infinite-dimensional configuration space Z evaluated on Σ_t, where

$$Z = \{v^a | t_a v^a = 1,\ \nabla_{[a}\nabla_{b]} v^c = 0,\ v^a \sim v^a - h^{ab}\nabla_b f\}. \tag{31}$$

[34] Note that condition (4) is locally equivalent to $\nabla_{[a}\nabla_{b]}A_c - h^u{}_{cd}\nabla_{[a}\nabla_{b]}u^d = 0$ (see, e.g., Christian, 1997, p. 4851).

[35] Thus, the constraint $1/\wp \nabla_a \Pi^a + \Lambda_N - 4\pi G\rho = 0$ generates the gauge transformations (17a) in the case of no boosts (i.e., $w_a = 0$); and the constraints (24), in the limit $\kappa \to 0$, and (4) *together* generate the gauge transformations (17a), (17b), for "Maxwell boosts" (i.e., $w_a = \partial_a f$, for some f). This justifies identifying the Lagrange multiplier χ in Christian's $\mathscr{L}_{\text{grav}}$ as the $U(1)$ phase factor (see comments below Eq. (24)).

Intuitively, the v^a constitute extended Maxwellian frames that pick out an "extended" Maxwellian connection determined by the Poisson equation.

4.2.4. Strong NCG symmetries

Christian's explicit formulation of strong NCG as a constraint-free Hamiltonian system indicates that strong NCG is a gauge theory in the sense of Earman (2002). Evidently, the gauge group is generated by the extended Maxwell algebra $\widetilde{\mathfrak{max}}$, obtained by setting $\delta h = \delta t = \delta u = \delta v = 0$ in (21). The reduced phase space (31) consists simply of extended Maxwellian frames v^a. In particular, note that the "momentum" constraint (24) is obtained by extremizing $\mathscr{L}_{\text{grav}}$ with respect to the multiplier χ associated with the arbitrary $U(1)$ phase factor. Hence, in Christian's construction of the reduced phase space, constraint (d) picks out the Maxwell algebra $\mathfrak{max}$, constraint (e) extends the Maxwell algebra to $\widetilde{\mathfrak{max}}$ *via* χ, and constraint (f) eliminates the redundancy of specifying both u^a and A_a initially as dynamically variables.

In terms of absolute and dynamical structure, the spacetime symmetries of strong NCG are generated by the extended Maxwell algebra $\widetilde{\mathfrak{max}}$, in so far as the absolute objects of the theory can be identified as the metrics h^{ab}, t_a, and the family of extended Maxwellian frames v^a. These symmetries are a bit more constrained than those of Maxwellian spacetime (the symmetries of which are generated by $\mathfrak{max}$). A strong NCG connection is obtained from a Maxwellian connection (7) by the further condition

$$R^i_m \ddot{a}^m = h^{ij} \partial\phi / \partial x^j, \tag{32}$$

subject to $h^{ij}\partial_i\partial_j\phi = 4\pi G\rho$. This suggests that, whereas in Maxwellian spacetime (in the absence of external forces), all linear accelerations are relative, in strong NCG spacetime, only a certain type of linear acceleration is relative. In strong NCG spacetime, relative accelerations are only those induced by the mass density in conjunction with ϕ. Geometrically, a Maxwellian connection cannot distinguish between "straight", and "curved" particle trajectories, in toto. A strong NCG connection fails to distinguish between only a subclass of such trajectories.

The dynamical symmetries of strong NCG should leave the equation of motion (11) covariant in extended Maxwellian frames in which it takes the form $\ddot{x}^i + h^{ij}\partial\phi/\partial x^j = 0$. These are transformations generated by $\widetilde{\mathfrak{max}}$ (i.e., $\mathfrak{max}$ plus the ϕ-transformation $\phi \mapsto \phi - x^i \ddot{a}^i + \varphi(t)$).

5. Conclusion

I now would like to return to the issue raised in the introduction of empirical indistinguishability. Some authors have claimed that the standard way of formulating Newtonian gravity (i.e., Neo-Newt NG) and the curved spacetime formulation share the same empirical commitments but subscribe to different ontologies; hence, they constitute a non-trivial example of empirically indistinguishable theories.[36] It is now evident that, without further ado, this claim is ambiguous, in so far as it fails to distinguish between weak NCG and strong NCG. Before attempting a bit of disambiguation, perhaps it is best to first explain what is at stake.

Empirical indistinguishability plays a central role in one form of underdetermination argument in the debate over scientific realism. Briefly, some anti-realists attempt to use empirical indistinguishability to drive a wedge between semantic realism (the realist's desire to read successful theories literally) and epistemic realism (the realist's contention that there can be good reasons to believe the theoretical claims of successful theories). A conventionalist, for instance, argues that empirical indistinguishability conjoined with epistemic realism entails semantic anti-realism: If T and T' are distinct empirically indistinguishable theories, then, to the extent that any reason to believe one is also a reason to believe the other, we cannot read both of them literally. A constructive empiricist, on the other hand, argues that empirical indistinguishability conjoined with semantic realism entails epistemic anti-realism: T and T', read literally, make different, possibly conflicting, theoretical claims, with respect to which we cannot be epistemic realists.

The task for such anti-realists then is to identify non-trivial examples of empirically indistinguishable theories. In particular, an anti-realist may look to Neo-Newt NG and NCG as such an example. On the other hand, a realist might respond by claiming that Neo-Newt NG and NCG are really a trivial example of empirical indistinguishability, in so far as they are the same theory. Such a realist might suggest that the empirical content of Neo-Newt NG is only recoverable from NCG under conditions that effectively reduce the ontology of NCG to the ontology of the standard formulation. Hence, NCG is just Neo-Newt NG in disguise. I will now attempt to describe the contexts in which *both* of these anti-realist and realist claims are correct.

Note, first, that the different versions of Newtonian gravity canvassed above can be distinguished in terms of their symmetries:

Theory	Spacetime symmetries	Dynamical symmetries
Neo-Newt NG	$\mathfrak{gal}$	$\widetilde{\mathfrak{max}}$
Neo-Newt NG *w*/b.c.	$\mathfrak{gal}$	$\mathfrak{gal}$ and $\phi \mapsto \phi + \varphi(t)$
Max NG	$\mathfrak{max}$	$\widetilde{\mathfrak{max}}$
Weak NCG	$\widetilde{\mathfrak{leib}}$	$\widetilde{\mathfrak{leib}}$
Weak NCG *w*/b.c.	$\mathfrak{gal}$	$\mathfrak{gal}$ and $\phi \mapsto \phi + \varphi(t)$
Strong NCG	$\widetilde{\mathfrak{max}}$	$\widetilde{\mathfrak{max}}$

where "b.c." denotes "boundary condition" in the form of the "island universe" assumption. Now suppose the anti-realist makes the following claim:

(A) If two theories agree on their dynamical symmetries, then they are empirically indistinguishable.

[36] See, e.g., Earman (1993, p. 31) and Friedman (1983, p. 121, footnote 15).

One way to make (A) plausible is in terms of two further claims: (i) two theories are empirically indistinguishable just when they share the same set of observables; and (ii) the observables of a theory are the invariants of its dynamical symmetries. If the anti-realist can mount arguments in support of these claims, then she has found two cases of empirical indistinguishability in the context of Newtonian theories of gravity; namely,

(a) Neo-Newt NG *w*/b.c. and Weak NCG *w*/b.c.;
(b) Neo-Newt NG, Max NG, and Strong NCG.

A realist may now take issue with case (a), claiming that, to the extent that both theories have the same spacetime symmetries, and hence posit the same absolute objects, they are really the same theory with the same ontological commitments. (Of course, "pathological" interpretations that would distinguish between the two are always possible, but perhaps disingenuous on the part of the anti-realist). The realist may claim that the "island universe" assumption imposed on weak NCG effectively reduces its ontology to that of Neo-Newt NG *w*/b.c.

However, to the extent that the realist labels case (a) as a trivial example of empirical indistinguishability in this fashion, she will have to admit that case (b) is a non-trivial example; namely, that Neo-Newt NG, Max NG, and Strong NCG, while agreeing on their observables, posit different absolute objects and hence constitute different theories with different ontologies. (For instance, a typical semantic realist, who desires to read Max NG and Strong NCG literally, will admit ϕ into the ontology of the former, but not the latter.)

Note, finally, that this is not to say that case (b) provides fool-proof ammunition for the anti-realist in her attack on scientific realism. In the light of case (b), the realist has at least two options available:

(i) Case (b) can be given a semantic gloss. For example, a structural realist may take heart with the above classification of Newtonian gravitation theories in terms of their symmetries and claim: dynamical structure is real, and not the contents of "individuals-based" ontologies.[37] A structural realist interpretation of Neo-Newt NG, Max NG and Strong NCG will then view all three as the same theory.
(ii) Case (b) can be given an epistemic gloss: Typical semantic realists who adopt "individuals-based" ontologies can appropriate a suitable epistemic component for their realism; one based on, for instance, epistemic criteria like simplicity, explanatory and/or unifying power, etc. Case (b) can then be addressed by attempting to adjudicate between Neo-Newt NG, Max NG, and Strong NCG in terms of these criteria.

Hence, arguably, even in the light of non-trivial examples of empirically indistinguishable theories, the debate over scientific realism is far from settled.

[37] See, e.g., Ladyman (1998) for a statement of such structural realism.

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