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The emergence of spacetime in condensed matter approaches to quantum gravity

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ABSTRACT

Condensed matter approaches to quantum gravity suggest that spacetime emerges in the low-energy sector of a fundamental condensate. This essay investigates what could be meant by this claim. In particular, I offer an account of low-energy emergence that is appropriate to effective field theories in general, and consider the extent to which it underwrites claims about the emergence of spacetime in effective field theories of condensed matter systems of the type that are relevant to quantum gravity.

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1. Introduction

Condensed matter approaches to quantum gravity seek to recover general relativity and the Standard Model as low-energy effective field theories (EFTs) of a condensed matter system (Smolin, 2003, p. 57). Such approaches differ on the type of condensed matter system chosen, whether it be a Bose–Einstein condensate (Barceló, Liberati & Visser, 2005; Laughlin, 2003; Liberati, Visser, & Weinfurtner, 2006), a fermionic superfluid (Dziarmaga, 2002; Volovik, 2003), or a quantum Hall liquid (Zhang & Hu, 2001). What they have in common are informal references to notions of emergence associated with fields, particles, and spacetime. In their review of models of analogue gravity, Barceló et al. (2005) speak of "emergent gravitational features in condensed matter systems" (p. 59), and "emergent spacetime symmetries" (p. 62); Dziarmaga (2002, p. 274) describes how "… an effective electrodynamics emerges from an underlying fermionic condensed matter system"; Volovik (2003) in the preface to his text on low-energy properties of superfluid helium, lists "emergent relativistic quantum field theory and gravity" and "emergent nontrivial spacetimes" as topics to be discussed within; and Zhang and Hu (2001, p. 825) speak of the "emergence of relativity" at the edge of 2-dim and 4-dim quantum Hall liquids.

This essay addresses the question of what could be meant by the claim that spacetime emerges in the low-energy sector of a condensed matter system. This will require articulating a notion of emergence that is appropriate in the context of an EFT. Such a notion is arguably distinct from typical notions of emergence associated with spontaneous symmetry breaking in condensed matter systems, and should rather be based on the relation between an effective field theory and the high-energy theory from which it is obtained. It will also require investigating the sense in which the fundamental condensate can and cannot be said to be non-spatiotemporal. Under one reading, for instance, the type of spatiotemporal emergence associated with the condensed matter program might be viewed as one in which relativistic spacetime structure emerges from the more fundamental non-relativistic spacetime structure of the condensate (irrespective of whether this structure is interpreted substantivally or relationally). Under another reading, spatiotemporal emergence might be viewed as the emergence of relativistic spacetime structure from a non-spatiotemporal fundamental condensate. Making this distinction clear is important in assessing how the notion of an emergent spacetime in the condensed matter approach to quantum gravity compares with notions of emergent spacetime associated with other approaches.

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The plan of the essay is as follows. Section 2 reviews the notion of an effective field theory and then considers two examples of the condensed matter approach to quantum gravity, one based on superfluid Helium 3-*A* (Volovik, 2003), and the other on a 4-dim quantum Hall liquid (Zhang & Hu, 2001; Sparling, 2002). These examples exemplify two distinct versions of the condensed matter approach: one in which the concept of universality, associated with a renormalizable EFT, plays an essential role, and another in which universality is not an explicit criterion. Section 3 considers the extent to which these examples can be interpreted in terms of emergent spacetime, including the sense in which a fundamental condensate can and cannot be said to be spatiotemporal. Finally, Section 4 identifies a notion of emergence appropriate to EFTs in general, and the condensed matter approach to quantum gravity in particular.

2. The condensed matter approach to quantum gravity

The goal of the condensed matter approach is to recover general relativity (GR) and the Standard Model in the low-energy sector of a condensed matter system. This is an approach to quantum gravity insofar as the latter is taken to encompass attempts to reconcile GR with quantum theory. Here the reconciliation takes the form of a common origin in the condensate, with low-energy excitations of its ground state mimicing the metric, gauge, and matter fields that appear in GR and the Standard Model. To describe these low-energy excitations, one employs the techniques of EFTs. An EFT associated with a high-energy theory describes low-energy excitations of the system described by the theory. One thus seeks an appropriate condensate described by a ‘‘high-energy’’ theory, and from the latter constructs an EFT from which GR and the Standard Model can be recovered.

2.1. How to construct an effective field theory

When a high-energy theory is known, an EFT is constructed by eliminating degrees of freedom associated with energies above some characteristic scale. The practical goal is to obtain a theory in which calculations are more tractable (whether due to less degrees of freedom, or a more simple dynamics). The construction of an effective theory that accomplishes this is not just a matter of ignoring the high-energy degrees of freedom, for they may be functionally related to the low-energy degrees of freedom in non-trivial ways. One method of extracting the low-energy degrees of freedom involves two steps:[1]

(I) First, the high-energy degrees of freedom are identified and integrated out of the Lagrangian density. Suppose the latter $\mathcal{L}[\phi]$ is a functional of a field variable ϕ. Suppose further that, with respect to a cutoff Λ, low- and high-energy degrees of freedom have been identified so that the field can be decomposed schematically as $\phi = \phi_H + \phi_L$ (where, for instance, ϕ_H and ϕ_L are associated with momenta greater than and less than Λ, respectively). Formally, one then constructs the generating functional Z associated with $\mathcal{L}[\phi_H, \phi_L]$ and performs the path integral over ϕ_H:

$$Z = \int \mathcal{D}\phi_L \mathcal{D}\phi_H e^{i\int d^D x \mathcal{L}[\phi_L, \phi_H]} = \int \mathcal{D}\phi_L e^{i\int d^D x \mathcal{L}_{eff}[\phi_L]} \quad (1)$$

where D is the dimension of the spacetime, and the effective Lagrangian density $\mathcal{L}_{eff}[\phi_L]$, defining the EFT, depends only on the low-energy degrees of freedom.

(II) Typically, (1) is not tractable, and even when it is, it may result in an effective Langrangian density that contains non-local terms (in the sense of depending on more than one spacetime point). These problems are jointly addressed by constructing a low-energy expansion of the effective Langrangian density

$$\mathcal{L}_{eff} = \mathcal{L}_0 + \sum_i g_i \mathcal{O}_i \quad (2)$$

where $\mathcal{L}_0$ is the free Lagrangian density, the sum runs over all local operators $\mathcal{O}_i$ allowed by the symmetries of $\mathcal{L}$, and the g_i are coupling constants.

One can perform a dimensional analysis on the terms in (2), which sorts them into three types: relevant, irrelevant, and marginal, depending on whether they increase, decrease, or remain constant, respectively, as the energy is lowered. One can show that for theories in 4-dim spacetimes, $\mathcal{L}_{eff}$ contains a finite number of relevant and marginal terms, and while in principle, there may be an infinite number of irrelevant terms, such terms are suppressed at low-energies E by powers of E/Λ (Bain, 2012, p. 5).

This method for constructing an EFT had its origins in the development of renormalization group (RG) techniques which relate high-energy theories to theories with cutoffs. Such techniques are particularly concerned with the relation between a *renormalizable* high-energy theory and a low-energy theory with a cutoff. One might thus ask whether the relation between an EFT and its high-energy theory is the same as this relation. It turns out that the answer is no; in particular, an EFT need not be renormalizable, nor need it be obtained from a renormalizable high-energy theory. This distinction is important, insofar as it serves to distinguish between two versions of the condensed matter approach to quantum gravity, or so I shall now argue.

2.2. Comparison with renormalization group techniques

In the RG approach to renormalization, the intent is to analyze the behavior of a theory at different energy scales. A scale-dependent cutoff $\Lambda(s)$ is thus used for an initial distinction between low- and high-energy degrees of freedom. With respect to an initial energy Λ_0, the high-energy degrees of freedom are integrated out of the theory. The cutoff is then lowered to $\Lambda(s) = s\Lambda_0$, $(s < 1)$, and the parameters of the theory are rescaled to restore the cutoff back to Λ_0. Successive iterations of this procedure generate a flow in the parameter space of the theory. Scale-dependent parameters can then be classified in terms of how they behave as the cutoff is *increased*: relevant (shrinking), irrelevant (growing), or marginal (constant).[2] A theory is said to be renormalizable if it contains no irrelevant parameters. Such a theory is cutoff independent, insofar as its parameters become independent of $\Lambda(s)$ in the high-energy limit $s \to \infty$. Moreover, for such theories with a finite number of relevant and marginal terms, successive rescalings do not affect the form of the original Lagrangian density: any new terms that are generated by an iteration of the RG transformations can be absorbed into a redefinition of the finite number of parameters. A non-renormalizable theory, on the other hand, is one in which there are (scale-dependent) irrelevant parameters. Such parameters cannot be ignored at high energies and thus contribute to ultraviolet divergent integrals.

A renormalizable theory is further characterized by a *fixed point*: a point that is invariant under RG transformations (at a fixed point, not only does the form of the Lagrangian density

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[1] The following exposition is based on the review in Polchinski (1993).

[2] In this context, relevant terms are called ‘‘super renormalizable’’, irrelevant terms are called ‘‘non-renormalizable’’, and marginal terms are called ‘‘renormalizable’’.

remain unchanged, so do the values of the parameters). A fixed point defines a universality class, insofar as the theory associated with the fixed point is independent of the details of the high-energy theory from which it emanates. There can be more than one such high-energy theory: any two formally distinct high-energy theories that flow to the same fixed point differ only in irrelevant terms. Thus for a given fixed point, there is associated a renormalizable high-energy theory T (a theory with no irrelevant terms), and (in principle infinitely) many non-renormalizable high-energy theories (theories that differ from each other and T only in irrelevant terms).

One can now discern two distinct versions of the condensed matter approach to quantum gravity. Both versions seek to construct EFTs of a fundamental condensate that mimic GR and the Standard Model, but differ (implicitly) on whether such EFTs are renormalizable. One version focuses on EFTs that can be identified with fixed points in an RG flow (thus such EFTs are renormalizable). The goal of this version is to identify the appropriate condensate whose EFT belongs to the same universality class as the Standard Model, with the hope that GR can be recovered, too. This approach is exemplified in Section 2.2 below by Volovik's (2003) topological analysis of the momentum space structure of superfluid Helium 3-*A*, which demonstrates that the latter belongs to the same universality class as a sector of the Standard Model. A second version of the condensed matter approach to quantum gravity employs the EFT procedure indiscriminantly to construct a (not necessarily renormalizable) EFT from a fundamental condensate. Universality plays no motivating role in this approach, which is exemplified below by the attempt to recover GR and the Standard Model from the edge of a 4-dim quantum Hall liquid.[3]

2.3. Superfluid Helium 3-A

Volovik (2003) demonstrates that an EFT for superfluid Helium 3-*A* (^{3}He-*A* hereafter) belongs to the same universality class as the sector of the Standard Model below electroweak symmetry breaking (the sector that describes massless chiral fermions). This then suggests a method by which GR and the Standard Model in its totality might be recovered.

The ground state of superfluid ^{3}He-*A* is believed to be a Bose condensate consisting of (bosonic) pairs of ^{3}He atoms. These pairs are similar to the electron Cooper pairs described by the Bardeen–Cooper–Schrieffer (BCS) theory of conventional superconductors. The non-relativistic second-quantized Hamiltonian that describes superfluid ^{3}He-*A* is thus a modified form of the standard BCS Hamiltonian and takes the following schematic form:

$$H_{3HeA} = \chi^\dagger \sigma^b g_b(\mathbf{p})\chi, \quad b = 1,2,3, \tag{3}$$

where the χs are 2-spinors that encode creation and annihilation operators for ^{3}He atoms, σ^b are Pauli matrices, and $g_b(\mathbf{p})$ are functions of 3-momentum that encode the kinetic energy and interaction potential for ^{3}He-*A* Cooper pairs.[4] The energy spectrum associated with (3) vanishes at two points, call them $\mathbf{p}^{(a)}$ (a=1, 2), referred to as "Fermi points". In 4-momentum space, these Fermi points are topologically stable insofar as they define singularities at $p_\mu^{(a)}=(0,\ \mathbf{p}^{(a)})$ in the one-particle Feynman propagator $G=(ip_0-\sigma^b g_b(\mathbf{p}))^{-1}$, that are insensitive to small perturbations.[5] This means that the form of the energy spectrum remains unchanged when the form of the Hamiltonian is (slightly) perturbed. In the language of RG techniques, this means that the theory associated with this energy spectrum defines a fixed point in an RG flow, and thus a universality class.

To uncover the form of the theory of this fixed point, one expands the inverse propagator G^{-1} in terms of deviations $p_\mu - p_\mu^{(a)}$ from the Fermi points $p_\mu^{(a)}$

$$G^{-1} = \sigma^b e_b^\mu (p_\mu - p_\mu^{(a)}), \quad b = 0,1,2,3 \tag{4}$$

where the tetrad field e_b^μ encodes the linear approximations of the $g_b(\mathbf{p})$.[6] The energy spectrum is given by the poles in the propagator, and hence takes the relativistic form,

$$g^{\mu\nu}(p_\mu - p_\mu^{(a)})(p_\nu - p_\nu^{(a)}) = 0 \tag{5}$$

where $g^{\mu\nu}=\eta^{ab}e_a^\mu e_b^\nu$, and η^{ab} is formally identical to the Minkowski metric.[7] The low-energy parameters p_μ and $g^{\mu\nu}$ change the positions of the Fermi points, and the slope of the curve of the energy spectrum in momentum space, respectively. Formally, they describe a potential field in a curved spacetime. To make this explicit, note that the Lagrangian density corresponding to (5) can be written as

$$\mathcal{L}_{eff} = \overline{\Psi}\gamma^\mu(\partial_\mu - q_{(a)}A_\mu)\Psi \tag{6}$$

where $\gamma^\mu = g^{\mu\nu}(\sigma_\nu \otimes \sigma_3)$ are Dirac γ-matrices, the Ψs are 4-spinors, and $q_{(a)}A_\mu = p_\mu^{(a)}$. This describes massless relativistic fermions coupled to a potential A_μ in a curved spacetime with Lorentzian metric $g_{\mu\nu}$. A term describing the gauge field associated with A_μ can be obtained in a low-energy expansion of (6) (Volovik, 2003, pp. 109–110). This follows the procedure in Section 2.1: A_μ is decomposed into low- and high-energy degrees of freedom relative to a cutoff, the high-energy modes are integrated out of $\mathcal{L}_{eff}$, and the result is expanded about the free Lagrangian density. To second order this produces a term identical to the Maxwell–Lagrangian density in a curved spacetime $\mathcal{L}_{Max} = (4\beta)^{-1}\sqrt{-g}g^{\mu\nu}g^{\alpha\sigma}F_{\mu\alpha}F_{\nu\sigma}$, where $F_{\mu\nu}$ is the Maxwell gauge field associated with A_μ, and β is a constant.[8] Combining this with (6), the effective Lagrangian density for superfluid Helium 3-*A* is then formally identical to the Lagrangian density for massless (3+1)-dim quantum electrodynamics:

$$\mathcal{L}_{eff} = \overline{\Psi}\gamma^\mu(\partial_\mu - q_{(a)}A_\mu)\Psi + \mathcal{L}_{Max} \tag{7}$$

At this point, we've recovered massless quantum electrodynamics as a low-energy EFT of superfluid ^{3}He-*A*. This is remarkable in and of itself, insofar as it demonstrates how a relativistic quantum field theory can be recovered in the form of an effective Lagrangian density associated with a *non-relativistic* theory. However, massless quantum electrodynamics is just a fragment of the Standard Model, which, in addition to the $U(1)$ Maxwell gauge field, includes $SU(2)$ and $SU(3)$ gauge fields associated with the weak and strong forces, respectively. Moreover, these gauge fields are supposed to mesh with each other in non-trivial ways (technically, they are required to belong to representations of the

[3] Another example of the second version is the acoustic spacetime program (see, e.g., Barceló et al., 2005). This program attempts to model general relativistic spacetimes by low-energy EFTs of Bose–Einstein condensates and superfluid Helium 4 among others. Whether or not universality is an appropriate criterion for the condensed matter approach is open to debate. On the one hand, the RQFTs that comprise the Standard Model are renormalizable, and hence can be associated with universality classes. On the other hand, GR cannot be formulated as a renormalizable RQFT, and hence cannot be said to possess a universality class.

[4] See Volovik (2003, p. 82) for the explicit form of the sum $\sigma^b g_b(\mathbf{p})$. The following exposition draws on Volovik (2003, pp. 94–101) and Dreyer (2009, pp. 100–102). The "*A*" designates a phase of superfluid Helium 3 characterized by a particular spin and orbital angular momentum configuration.

[5] The topological invariant associated with this stability is the winding number of the phase of G.

[6] This amounts to constructing an EFT of (3) by expanding the Hamiltonian about the ground state. The expansion is taken around both Fermi points.

[7] This formal appearance of a relativistic energy spectrum in the low-energy sector of a non-relativistic fermionic condensed matter system is not unusual (Bain, 2008, pp. 302–303).

[8] For details, see Dziarmaga (2002).

product group $SU(3)\otimes SU(2)\otimes U(1)$). This raises the question of whether more aspects of the Standard Model can be recovered from (7).

Volovik (2003, pp. 114–115) has demonstrated how the analysis that produced (7) can be extended to include an $SU(2)$ gauge field. Briefly, in addition to their charge, the fermion fields in (6) are characterized by a two-valued degree of freedom arising from the relation between their spin and the spin of the underlying ^{3}He atoms. Volovik interprets this as an $SU(2)$ isospin symmetry and represents it by coupling Ψ to a field W^i_μ identified as an $SU(2)$ potential field (analogous to the potential for the weak force). Performing a low-energy expansion of this modified Lagrangian density with respect to W^i_μ then produces an $SU(2)$ gauge field term to second order. The general moral for Volovik is that discrete degeneracies in the Fermi point structure of the energy spectrum induce local symmetries in the low-energy sector of the condensate (2003, p. 116). For a discrete two-fold symmetry, we obtain a low-energy $SU(2)$ symmetry; and in principle for larger discrete N-fold symmetries, we could obtain larger $SU(N)$ symmetries.

Unfortunantly, superfluid ^{3}He-*A* does not have such larger N-fold symmetries, thus not all aspects of the Standard Model can be recovered from it. Formally, one cannot recover additional terms in (7) that might be interpreted as encoding the electroweak and quantum chromodynamics sectors of the Standard Model. Moreover, a term that appropriately encodes general relativity cannot be recovered either (for discussion see Bain 2008, pp. 311–312). While this explicitly indicates that superfluid ^{3}He-*A* cannot play the role of the fundamental condensate, it leaves open the question of whether such a fundamental condensate can be identified.

2.4. Quantum Hall liquids

A 2-dim quantum Hall (QH) liquid is a condensed matter system that displays the quantum Hall effect. This occurs when electrons moving in a 2-dim conductor are deflected towards the edge in the presence of a strong external magnetic field perpendicular to the conductor. When the force due to the magnetic field is balanced by the force due to the induced electric field, the transverse conductivity is given by

$$\sigma = \nu(e^2/h) \tag{8}$$

where ν=(# of electrons)/(# of states per energy level) is the filling factor.[9] The Integer Quantum Hall Effect (IQHE) is characterized by integer values of ν, and the Fractional Quantum Hall Effect (FQHE) is characterized by values of ν given by simple fractions. The IQHE can be explained by reference to the discrete spacing between the energy levels: at integer values of ν, the first ν energy levels are full, and the system is incompressible (in the sense that no further electrons can be added without a cost in energy). The FQHE can be explained as "…the IQHE of composite fermions" (Schakel, 2008, p. 343). If an even number of magnetic fluxes are attached to each electron, this cancels a part of the external magnetic field; just enough to change the filling factor back to an integer value.[10] An incompressible condensed matter system consisting of composite fermions behaving in this manner constitutes a 2-dim QH liquid.

Incompressibility signals the fact that there can be no low-energy excitations in the bulk of a QH liquid. There can, however, be such excitations at the edge. Wen (1990) identified the low-energy degrees of freedom at the edge as encoded in a density function $\rho(x)$, and modeled perturbations in the density as edge deformations (*i.e.*, surface waves). The EFT that describes these deformations is given by an effective Lagrangian for (1+1)-dim relativistic massless spin-0 bosons

$$\mathcal{L}_{eff-edge} = (1/8\pi)\{(\partial_t\phi)^2 - (\partial_x\phi)^2\} \tag{9}$$

where $\rho(x) = (1/2\pi)\partial_x\phi(x)$.[11]

Zhang and Hu (2001) extended this result to (3+1)-dim by constructing a 4-dim QH liquid (for details, see Bain, 2008, p. 320). Their main result was to establish that the low-energy sector of the edge of a 4-dim QH liquid contains stable bosonic states that satisfy the (3+1)-dim zero rest mass field equations for all (integral) helicities. These include, for instance, the massless Klein–Gordon equation, the source-free Maxwell equations, and the linearized vacuum Einstein equations.

This recovery of (3+1)-dim zero rest mass field theories does not produce the complete versions of GR and the Standard Model, and it is here that twistor theory has been introduced. Twistors are elements of a complex 4-dim vector space $\mathbb{T}$ which carries matrix representations of $SU(2, 2)$. $\mathbb{T}$ admits a Hermitian 2-form $\Sigma_{\alpha\beta}$ by means of which the subspace $\mathbb{N} = \{Z_\alpha \in \mathbb{T} : \Sigma_{\alpha\beta}Z_\alpha Z_\beta = 0\}$ of null twistors is defined. Sparling (2002) demonstrated that the edge of Zhang and Hu's 4-dim QH liquid can be identified with $\mathbb{N}$. A main result of twistor theory then shows that deformations of (projective) twistor space (in the form of elements of the first cohomology group) are solutions to the zero rest mass field equations of all helicities (Ward & Wells, 1990, Chap 7.2). Thus, according to Sparling (2002, p. 25), "…if we think of the edge-states as in some sense providing deformations of the boundary, they are associated with… first cohomology and thus with massless particles…".

We thus arrive at the following version of the condensed matter approach to quantum gravity: the fundamental condensate is taken to be a 4-dim QH liquid, and the task is to recover GR and the Standard Model from the twistors that arise in the low-energy sector of its edge. Note that there is no reference to universality in this approach.[12]

3. Two ways spacetime might emerge from a fundamental condensate.

Taken literally, the examples reviewed in Section 2 suggest that we award fundamental ontological status to a condensate. In the superfluid ^{3}He-*A* example, low-energy excitations of the condensate in its ground state take the form (ideally) of the matter, gauge, and metric fields of general relativity and the Standard Model. This example, arguably, is a background dependent approach to quantum gravity, insofar as the spatiotemporal structure of the fundamental condensate is fixed *a priori* (regardless of whether it is subsequently interpreted in relationalist or substantivalist terms). Moreover, this structure is non-relativistic:

[9] $\nu = n_e/n_B$, where n_e is the electron density (# per unit area) and $n_B = B/\Phi_0$ is the flux density, where B is the external magnetic field strength and $\Phi_0 = h/e$ is the quantum of flux. In the lowest energy (or "Landau") level, there is one electron state for each flux quantum; a flux quantum in this context can be heuristically thought of as the amount of flux that penetrates the area occupied by an electron. Thus the flux density n_B measures the degeneracy per unit area of each Landau level.

[10] Formally this is achieved by coupling the fermions to a Chern-Simons field (see (12) below). An alternative theoretical description of the FQHE is in terms of composite bosons and appeals to the mechanism of BEC formation to explain the FQHE (Zhang, 1992).

[11] Eq. (9) is formally equivalent to a (1+1)-dim relativistic Lagrangian density describing massless fermions ψ (Bain, 2008, p. 320) *via* the prescription $\psi(x) = \eta e^{i(1/\nu)\phi(x)}$ (Wen 1990, p. 840).

[12] For limitations of this approach, see Bain (2008, pp. 323–324).

the initial "high-energy" Lagrangian density that describes the fundamental condensate is Galilei-invariant. We might thus claim:

E1. Relativistic spatiotemporal structure emerges in the low-energy sector of a non-relativistic fundamental condensate.

Note that E1 is neutral as to whether the emergent spatiotemporal structure should be interpreted relationally or substantivally. Moreover, the fundamental reality that underwrites E1 is that of a condensate that possesses non-relativistic spatiotemporal structure. Thus E1 does not suggest that relativistic spacetime structure emerges from a more fundamental non-spatiotemporal reality.

Now consider the QH liquid example. In this example, low-energy excitations of the edge of the condensate take the form of twistors. Spacetime and its contents are then reconstructed from twistors. This example is not a background dependent approach to quantum gravity, at least under one sense of the term. Technically, the theory of a QH liquid is a topological quantum field theory involving a Chern–Simons gauge field.[13] In such a theory, a spacetime metric does not appear. Thus, to the extent that background dependence of a theory entails invariance of the theory under the symmetries associated with a particular spatiotemporal structure as encoded in a metric, the theory of a QH liquid is not background dependent. Intuitively, there is no prior metrical structure associated with the theory (although there *is* topological and differentiable structure).

This suggests that the QH liquid example underwrites a slightly different claim about emergent spacetime. In the first instance, this claim is,

E2. Relativistic spatiotemporal structure emerges in the low-energy sector of the edge of a 4-dim QH liquid.

Note, again, that, like E1, E2 is neutral as to whether the emergent spatiotemporal structure should be interpreted relationally or substantivally. E2 also suggests that relativistic spatiotemporal structure emerges directly from a more fundamental twistor reality, which itself might be said to arise from a condensate that does not possess metrical structure. This suggests that there are two ways in which E2 can be interpreted as claiming that relativistic spatiotemporal structure emerges from a more fundamental non-spatiotemporal reality:

(a) Claim that relativistic spatiotemporal structure emerges from a more fundamental non-spatiotemporal twistor reality. One can show that the points of conformally flat Lorentzian spacetimes can be constructed from structures defined on twistor space. In particular, the "twistor group" $SU(2,2)$ is the 2-fold double covering group of the conformal group $C(1,3)$ of Minkowski spacetime. This allows a map to be defined between (projective) null twistor space and (compactified) Minkowski spacetime, which can be extended to conformally flat Lorentzian spacetimes (Ward & Wells 1990, Chaps 1, 9). Insofar as twistors can be said to encode the conformal structure of spacetime, viewing twistors as non-spatiotemporal requires viewing conformal structure as non-spatiotemporal.
(b) Claim that relativistic spatiotemporal structure is encoded in twistors, which emerge from the edge of a fundamental non-spatiotemporal QH liquid. Since the latter is described by a topological quantum field theory, to view it as non-spatiotemporal requires viewing topological and differentiable structure as non-spatiotemporal.

Note that interpretation (a) refers to the emergence of spacetime independently of reference to a fundamental condensate. Indeed, the claim in (a) is typical of advocates of twistor theory in general, irrespective of its particular application to QH liquids. This suggests that in the context of the condensed matter approach to quantum gravity, in which reference to a fundamental condensate is essential, the more appropriate interpretation of E2 is (b).[14]

Now insofar as metaphysical notions of laws and causation assume spatiotemporal structure, interpretation (b) might initially be taken to imply that they are not fundamental; but this would be a mistake without further clarification. One would first have to clarify the type of spatiotemporal structure that underwrites such notions: for instance, is it metrical, conformal, or topological and differentiable? If laws are represented by topologically well-behaved differential equations, then one might assume that topological and differentiable structure suffices to underwrite them. In this case, interpretation (b) would allow that laws are fundamental. If a notion of causation minimally requires conformal structure (in order to support a light-cone structure on spacetime, say), then interpretation (b) would entail that causation is not fundamental. Finally, if one's metaphysical intuitions (for whatever reasons) require spatial and temporal distances (as encoded in metrical structure) to prefigure laws of nature and a notion of causation, then (and perhaps only then) interpretation (b) would entail that these notions are not fundamental.[15]

A final comment should perhaps be made concerning the extent to which claims E1 and E2 generalize beyond the specific examples that motivate them. In addition to superfluid ^{3}He-*A*, E1 is also supported by examples from the acoustic spacetime program (see footnote 3 above). These examples are also characterized by Galilei-invariant "high-energy" Lagrangian densities and relativistic effective Lagrangian densities (see Bain, 2008, pp. 304–309, for discussion). They differ from superfluid ^{3}He-*A* to the extent that they belong to the second version of the condensed matter approach identified in Section 2.1 above, whereas superfluid ^{3}He-*A* belongs to the first version (recall that these versions split on whether to adopt universality as a guiding principle). E2, on the other hand, is specific to a particular example of the condensed matter approach, and one might thus be concerned about its significance. Towards assuaging such concerns, the following comments seem relevant. First, the 4-dim QH liquid example combines several disparate approaches to quantum gravity; namely the condensed matter approach, approaches based on topological quantum field theories, and the twistor theory approach. Second, it touches on such guiding principles in quantum gravity research as holography (which, informally, places an emphasis on the edge states of a system as encoding its bulk properties), and duality (which manifests itself in this context in terms of correspondences between twistor-theoretic constructions and constructions in spacetime). Finally, and perhaps more importantly, it has become part of a recent large and growing body of literature on the theoretical foundations and implications of twistor-theoretic methods for calculating scattering amplitudes in quantum field theories. This resurgence of interest in twistor theory, which had lain fairly dormant for many years, is due to Witten (2004) who identified

[13] For the Chern–Simons theory of a 2-dim QH liquid, see Schakel (2008, p. 347) and Zhang (1992). For the Chern–Simons theory of a 4-dim QH liquid, see Bernevig, Chern, Hu, Toumbas, & Zhang (2002).

[14] This assumes that twistor theorists who are willing to take the QH liquid example seriously should take seriously the claim that twistors, while being more fundamental than spatiotemporal structure, are less fundamental than the QH liquid.

[15] This would not preclude making claims about the fundamental level; it would just preclude making law-like and/or causal claims.

previously unknown connections between twistor theory and string theory.[16]

4. A concept of emergence for EFTs.

I'd now like consider an appropriate notion of emergence for claims E1 and E2. The fact that E1 and E2 are claims about the emergence of spatiotemporal structure from a fundamental condensate suggests that they can be characterized by a common notion of emergence. This is further suggested by the fact that in both cases, the spatiotemporal structure is associated with an EFT that describes the low-energy sector of the fundamental condensate. This is not to say that E1 and E2 can only be characterized by a common notion of emergence, nor is it to say that the notion articulated below is the only one common to both claims. The following discussion only seeks to understand one way emergence might be understood in the context of E1 and E2. It will be prefigured by the following constraints:

(i) My concern is to identify a notion of emergence that is compatible with E1 and E2; I will not be concerned with whether this notion is appropriate in other contexts.

(ii) I will assume that an appropriate notion of emergence for E1 and E2 should be informed by the relation between an EFT and its high-energy theory. This is clear in E1, insofar as, in the superfluid Helium 3-*A* example, the emergent spatiotemporal structure is associated with an EFT constructed directly from the "high-energy" theory of the condensate. It is less clear, perhaps, in E2, insofar as, in the QH liquid example, there is an extra layer of twistor-theoretic structure between the condensate and the emergent spatiotemporal structure. This might suggest that the notion of emergence that underwrites E2 should be informed by the procedure by which spatiotemporal structure is recovered in twistor theory. But this would be to adopt interpretation (a) at the end of Section 3, which does not seem appropriate for E2. If instead we adopt interpretation (b), then emergence is descriptive of the process by which twistors arise in the edge of a QH liquid, and this process is associated with an EFT.

(iii) Finally, to say that spacetime emerges in the low-energy sector of a condensate could mean that spatiotemporal structure *ontologically emerges* in the latter, either in the form of spatiotemporal entities (*e.g.*, spacetime points), or spatiotemporal properties. Alternatively, it could mean that spacetime is characterized by the EFT of the low-energy sector of a condensate, and the relation between this EFT and the theory of the condensate is characterized by emergence. The former notion construes emergence as descriptive of the ontology (entities, properties) associated with a physical system with respect to another. The latter notion construes emergence as a relation between theories. In the following, I will use the intertheoretic relation between an EFT and its high-energy theory to inform an ontological notion of emergence appropriate for EFTs.

4.1. The EFT intertheoretic relation.

Assuming that the appropriate notion of emergence in E1 and E2 should be informed by the relation between an EFT and its high-energy theory, the task now is to make that relation explicit. Recall from Section 2.1 that the construction of an EFT involves two steps: (I) one first identifies and then integrates out high-energy degrees of freedom from a high-energy Lagrangian density; and (II) when appropriate, one then expands the resulting effective Lagrangian density in terms of local operators. The relation defined by this procedure has one important characteristic in the context of a discussion on emergence; namely, its *relata* are *distinct* theories. To see this, consider the following consequences of steps (I) and (II):

(1) The effective Lagrangian density $\mathcal{L}_{eff}$ typically is formally distinct from the high-energy Lagrangian density $\mathcal{L}$.

(2) The degrees of freedom of $\mathcal{L}_{eff}$ are associated with states that are formally distinct from those associated with the degrees of freedom of $\mathcal{L}$. This suggests they admit distinct ontological interpretations.

Both of the examples reviewed in Section 2 display these features. Consider the example of superfluid Helium 3-*A*. Above a critical temperature T_c, the system consists of Helium 3 atoms described by a non-relativistic Lagrangian density corresponding to the Hamiltonian (3):

$$\mathcal{L} = \chi^{\dagger}\{i\partial_t - (\partial_i^2/2m + \mu)\}\sigma_3\chi + \mathcal{L}_{int}[\chi^{\dagger}, \chi, \Delta, \Delta^*] \quad i = 1,2,3 \tag{10}$$

where $\mathcal{L}_{int}$ describes the particular interaction, encoded in the order parameter Δ, that produces a Helium 3-*A* Cooper pair (see, *e.g.*, Schakel, 1998, p. 27). At T_c a phase transition occurs, accompanied by a spontaneously broken symmetry, and the system enters the superfluid phase. To derive the properties of this phase, one can construct an EFT obtained from (10) by integrating out the fermion degrees of freedom.[17] One obtains a low-energy (relative to T_c) theory that describes hydrodynamical sound waves:

$$\mathcal{L}_{eff} = -n\{\partial_t\varphi + (1/2m)(\partial_i\varphi)^2\} + \rho\{\partial_t\varphi + (1/2m)(\partial_t\varphi)^2\}^2 \tag{11}$$

where n and ρ are the fermion number density and density of states, and $\varphi(x)$ is the phase of the order parameter (Schakel 1998, p. 28). If the temperature is lowered further, *i.e.,* as we enter the low-energy sector close to the ground state, the superfluid can be described in terms of the relativistic Lagrangian density (7).

The EFTs described by (11) and (7) differ in terms of their respective energy regimes: (11) is defined with respect to energies comparable to T_c, whereas (7) is defined with respect to energies comparable to the ground state. Both are formally distinct from the "high-energy" theory described by (10). More importantly, both are *dynamically* distinct from (10). This is clear in the case of (7), which describes a relativistic dynamics as opposed to the non-relativistic dynamics encoded in (10). While (11) is also non-relativistic, its dynamics involve the time evolution of bosonic degrees of freedom, compared with the dynamics of fermionic degrees of freedom encoded in (10).[18] Moreover, the degrees of freedom described in (11) and (7) refer to bosonic hydrodynamic waves on the one hand, and a spacetime metric and an electromagnetic field on the other, whereas the degrees of freedom of (10) refer to Helium 3 atoms. All of this suggests (if it wasn't already apparent) that (10), (11) and (7) describe different theories.

[16] Witten (2004) demonstrated that $N = 4$ supersymmetric Yang–Mills theory is equivalent to a certain type of string theory defined on twistor space. This allows scattering amplitudes of the former, which are notoriously difficult to calculate using Feynmann diagrams, to be calculated using much simpler twistor-theoretic techniques. This has led, in recent years, to increased contact between phenomenological particle physicists, string theorists, and twistor advocates. For a recent sample of this literature, including the significance of the 4-dim QH liquid example, see Heckman & Verlinde (2012).

[17] This notion of an EFT, pursued, for instance, in Schakel (1998, 2008), is at slight odds with the notion described by Polchinski (1993, p. 9). The latter includes a constraint of "naturalness" which precludes EFTs for systems that possess energy gaps, such as the BCS-inspired approach to Helium 3-*A* that produces (11) (as well as the QH bulk liquid system described by (13)).

[18] Formally, the Euler–Lagrange equations of motion associated with (10), (11) and (7) are all distinct.

Now consider the QH liquid example. In the 2-dim case, the "high-energy" theory describes electrons in the presence of external electric and magnetic fields. In the composite fermion version of the FQHE, these electrons are coupled to a Chern–Simons field, and the Lagrangian density that describes this is given by

$$\mathcal{L} = i\psi^\dagger\{\partial_t - ie(A_0 - a_0)\}\psi - (1/2m)\psi^\dagger\{\partial_i + ie(A_i + a_i)\}^2\psi + \mu\psi^\dagger\psi + \vartheta\varepsilon^{\sigma\nu\lambda}a_\sigma\partial_\nu a_\lambda \quad (12)$$

where A_0, A_i ($i = 1, 2$) are external electric and magnetic fields, a_σ ($\sigma = 0, 1, 2$) is the Chern–Simons field, and the coefficient ϑ in the last "Chern–Simons" term is chosen to guarantee that the composite electrons have an even number of fluxes (Schakel, 2008, p. 349). To derive the properties of the QH liquid, one constructs a low-energy EFT of (12) by integrating out the fermion degrees of freedom. The result is a pure Chern–Simons effective Lagrangian density

$$\mathcal{L}_{eff} = \vartheta\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \vartheta'\varepsilon^{\mu\nu\lambda}(A_\mu + a_\mu)\partial_\nu(A_\lambda + a_\lambda) \quad (13)$$

where the coefficient ϑ' describes a Chern–Simons term associated with the IQHE (Schakel, 2008, p. 349). If we now consider the low-energy edge of the QH liquid, we obtain the EFT (9). Again, the "high-energy" theory (12) is formally and dynamically distinct from (13) and (9). In this case, (12) (*sans* the CS term) is a non-relativistic QFT, (13) is a topological QFT, and (9) is a relativistic QFT. Moreover, (12) describes non-relativistic composite electrons, (13) describes two Chern–Simons fields (a_μ and $(A_\mu + a_\mu)$), and (9) describes relativistic bosons. Note, finally, that these descriptions all have (fairly complex) extensions to 4-dim.

These examples suggest that an EFT is *dynamically independent* of its high-energy theory, in so far as a specification of the equations of motion of the high-energy theory (together with pertinent initial and/or boundary conditions) will fail to specify solutions to the equations of motion of the EFT. To the extent that the laws of a theory are encoded in its equations of motion, this entails that the laws of an EFT are not deducible consequences of the laws of its high-energy theory.

The examples also suggest that an EFT can be interpreted as being *ontologically distinct* from its high-energy theory, but perhaps not *ontologically independent* of the latter. An EFT is ontologically distinct from its high-energy theory insofar as its degrees of freedom can be associated with physical systems (the metric and gauge fields of (7), say) that are distinct from the physical systems (the Helium 3 atoms of (10), say) that can be associated with the degrees of freedom of its high-energy theory. Again, one aspect of this ontological distinctness involves different dynamics: one reason we can interpret the metric and gauge fields of (7) as being ontologically distinct from the Helium 3 atoms of (10) is that the former figure into and obey different equations of motion than the latter. But insofar as the degrees of freedom of an EFT are the low-energy degrees of freedom of the high-energy theory, the physical systems associated with an EFT are not completely independent of those of its high-energy theory. The metric and gauge fields of (7), for example, do not "float free" of the Helium 3 atoms of (10); rather, they might be interpreted as low-energy "ripples" in the condensate whose microphysical constituents remain Helium 3 atoms.

4.2. Emergence in EFTs.

The considerations in Section 4.1 suggest that EFTs satisfy the following *desiderata* for a notion of emergence:

(i) Emergence should involve *microphysicalism*, in the sense that the emergent system should ultimately be composed of microphysical systems that comprise the fundamental system and that obey the fundamental system's laws.
(ii) Emergence should involve *novelty*, in the sense that the properties of the emergent system should not be deducible from the properties of the fundamental system alone.

These *desiderata* are underwritten in an EFT by the elimination of degrees of freedom in its construction. Thus the properties of a system described by an effective Langrangian density $\mathcal{L}_{eff}$ can be said to emerge from a fundamental system described by a high-energy Langrangian density $\mathcal{L}$ in the following sense:

(a) High-energy degrees of freedom are integrated out of $\mathcal{L}$. This secures microphysicalism insofar as it entails that the degrees of freedom of $\mathcal{L}_{eff}$ are exactly the low-energy degrees of freedom of $\mathcal{L}$.
(b) This elimination of degrees of freedom also secures novelty in so far as the solution $\mathcal{L}_{eff}$ of the path integral (1) is dynamically distinct from $\mathcal{L}$, and is a functional of field variables that do not appear in $\mathcal{L}$. Dynamical distinctness entails a failure of law-like deducibility from $\mathcal{L}$ of the properties described by $\mathcal{L}_{eff}$, and a difference in field variables suggests the properties and entities described by $\mathcal{L}_{eff}$ and $\mathcal{L}$ are ontologically distinct. Note that both dynamical and ontological distinctness also characterize $\mathcal{L}_{eff}$ in the typical scenarios in which either the path integral (1) is not exactly solvable, or produces a non-local solution, or if there is no high-energy Lagrangian to begin with. In these scenarios, $\mathcal{L}_{eff}$ is obtained by means of the local operator expansion (2).[19]

The sense in which relativistic spatiotemporal structure emerges in the low-energy sector of a fundamental condensate can now be given by the following:

(a) Relativistic spatiotemporal entities or properties are composed of the microphysical entities or properties of a fundamental condensate (*microphysicalism*).
(b) Relativistic spatiotemporal entities or properties cannot be deduced from the entities or properties of the fundamental condensate alone (*novelty*).

This account of emergence in EFTs is similar to that given by Mainwood (2006, p. 20) who takes (i) and (ii) to be partially descriptive of a view associated with prominent condensed matter physicists (*e.g.*, Anderson, 1972, Laughlin & Pines, 2000). According to Mainwood, the mechanisms these authors identify as underwriting (i) and (ii) are spontaneous symmetry breaking and universality (Mainwood, 2006, pp. 107, 116). Neither of these is generally applicable to EFTs, as the QH liquid example indicates. Mainwood (2006, p. 284) further suggests that a nontrivial notion of emergence requires the specification of a *physical* mechanism to underwrite (i) and (ii), and it might seem that the elimination of degrees of freedom in an EFT is a *formal*, as opposed to a physical, mechanism. I take it, however, that the elimination of degrees of freedom associated with an EFT represents the experimental fact that the low-energy sector of the system described by a theory is physically (*i.e.*, dynamically) distinct from the high-energy sector. A notion of emergence applies in this context not simply because we are puzzled by

[19] How can the elimination of degrees of freedom underwrite *both* microphysicalism and novelty? This is a peculiar characteristic of the process of constructing of an EFT. It results from the imposition of a constraint (an energy cut-off) directly on a Lagrangian density, as opposed to an equation of motion. A consequence of this is a formally distinct effective Lagrangian density with a distinct set of equations of motion and a distinct set of dynamical variables.

the behavior of the low-energy sector, but because it is physically (as well as theoretically) distinct from the high-energy sector.

Wilson (2010) similarly identifies the elimination of degrees of freedom (DOF) as an essential characteristic of a notion of emergence. For Wilson, DOF elimination plays two roles. First it secures the physical acceptability of an emergent entity by securing the lawlike deducibility of the entity's behavior from its composing parts (2010, p. 295), and such physical acceptability partially underwrites physicalism.[20] Second, according to Wilson, DOF elimination entails that an emergent entity is characterized by different law-governed properties and behavior than those of its composing parts, and this suggests that the former cannot be reduced to the latter (2010, p. 301). This failure of ontological reduction might charitably be associated with a notion of novelty (although Wilson's explicit goal is simply to establish peaceful coexistence between physicalism and non-reductivism). This might suggest similarity with the above account of emergence in EFTs. However, the type of DOF elimination involved in the construction of an EFT is distinct from Wilson's notion in two major respects. First, DOF elimination in an EFT is typically characterized by a *failure* of lawlike deducibility: the lawlike behavior of entities described by an EFT cannot, in general, be deduced from the lawlike behavior of the entities described by its high-energy theory. This failure, I suggested above, is what underwrites a notion of novelty. Second, in the presence of such failure, physicalism is preserved, insofar as, in DOF elimination in an EFT, the degrees of freedom of the EFT are exactly the low-energy degrees of freedom of the high-energy theory.

5. Conclusion

According to the condensed matter approach to quantum gravity, relativistic spatiotemporal structure can be said to emerge in the low-energy sector of a condensed matter system in one of two ways:

(A) It can emerge from a fundamental condensate with non-relativistic metrical structure, such as superfluid Helium 3-*A*.
(B) It can emerge from a fundamental condensate with topological and differentiable, but not metrical, structure, such as a 4-dim quantum Hall liquid.

If metrical structure is a necessary characteristic of spatiotemporal structure, then Process A is one in which relativistic spatiotemporal structure emerges from a more fundamental non-relativistic spatiotemporal reality; whereas Process B is one in which relativistic spatiotemporal structure emerges from a more fundamental non-spatiotemporal reality.

Finally, the sort of emergence associated with these claims is characterized by the elimination of high-energy degrees of freedom from a theory that describes the fundamental condensate. This elimination results in an EFT that can be interpreted as describing *novel* entities or properties in the sense of being dynamically independent of, and thus not deducible from, the entities or properties associated with the condensate. These novel entities or properties, however, can be said to ultimately be composed of the entities or properties that are constituent of the condensate, insofar as the degrees of freedom exhibited by the former are exactly the low-energy degrees of freedom exhibited by the latter.

[20] Two brief remarks: first, Wilson is concerned with what she calls "weak ontological emergence", which is taken to be compatible with physicalism, as opposed to strong ontological emergence, which is not. Second, for Wilson, physicalism in the context of weak ontological emergence is also underwritten by the claim that "…the law-governed properties and behavior of [an emergent entity] are completely determined by the law-governed properties and behavior of the [composing entities]…" (2010, pg. 280). Fleshing out the sense of Wilson's notion of determinism is a task for another essay.

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