Einstein Algebras and the Hole Argument

JONATHAN BAIN

Department of Humanities and Social Sciences
Polytechnic University
6 Metrotech Center
Brooklyn, NY 11201

(718) 260-3688 office
jbain@duke.poly.edu

word count: 5498

Einstein Algebras and the Hole Argument

ABSTRACT. Einstein algebras have been suggested (Earman 1989) and rejected (Rynasiewicz 1992) as a way to avoid the hole argument against spacetime substantivalism. In this article, I debate their merits and faults. In particular, I suggest that a gauge-invariant interpretation of Einstein algebras that avoids the hole argument can be associated with one approach to quantizing gravity, and, for this reason, is at least as well motivated as sophisticated substantivalist and relationalist interpretations of the standard tensor formalism.

1. Introduction

The hole argument is an argument against a particular interpretation of general relativity (GR) in a particular formalism. The formalism is that of tensor analysis on differential manifolds, and the suspect interpretation is based, in part, on the claim that the points of a differential manifold represent real spacetime points. The fact that GR can be presented in formalisms in which manifold points do not explicitly appear naively suggests that spacetime realism might be better articulated in such formalisms. If, however, the latter are expressively equivalent to the tensor formalism, then perhaps nothing is gained in adopting them as a means of avoiding the hole argument. Of course an alternative formalism may be expressively equivalent to the tensor formalism with respect to the hole argument, in so far as it reproduces the relevant manifold structures on which the hole argument turns, but may be expressively inequivalent with respect to other considerations. In particular, alternative formulations of GR may suggest alternative approaches to quantizing gravity.

In this article, I will attempt to show how a gauge-invariant interpretation of the Einstein algebra formulation of GR can be motivated by indicating how it follows naturally from one approach to quantum gravity. I will suggest that this not only provides a means to justify Earman's 1989 algebraic substantivalist interpretation of Einstein algebras, but also indicates how such an interpretation is at least as well-motivated as gauge-invariant interpretations of the tensor formalism. In Section 2, I describe the hole argument as an argument against a manifold substantivalist interpretation of GR in the tensor formalism. In Section 3, I review how GR can be formulated in the Einstein algebra formalism, how Earman's 1989 algebraic substantivalism claims to avoid the hole argument, and Rynasiewicz's 1992 critique of this claim. Finally, in Section 4, I indicate how Earman's interpretation can be motivated by an appeal to Heller and Sasin's 1999 non-commutative approach to quantum gravity.

2. The Hole Argument and Manifold Substantivalism

The original victim of Earman and Norton's 1987 hole argument was manifold substantivalism. This interpretation of GR is motivated by the formalism of tensor analysis on differential manifolds (TADM) in which general relativity is usually presented. A TADM model (M, g) of GR consists of a differential manifold M (with various attendant properties) on which is defined a metric field g that satisfies the Einstein equations. A differential manifold M is a collection of points with the same differential and topological properties. Hence the points of M cannot be distinguished by these properties. Manifold substantivalism is a literal (in a very literal sense) interpretation of such manifolds. It makes the following two claims:

(1) Substantivalism: Manifold points represent real spacetime points.

(2) Denial of Leibniz Equivalence: Diffeomorphically related TADM models of GR represent distinct physically possible worlds.

This interpretation of GR in the tensor formalism then falls victim to the hole argument: One first notes that solutions to the Einstein equations are unique up to diffeomorphism. A "hole" diffeomorphism is then constructed that is the identity outside a region of the manifold and smoothly differs from the identity inside the region. Two TADM models of GR related by a hole diffeomorphism agree everywhere outside the hole and differ inside. In the context of GR, it is then claimed that this constitutes a violation of determinism: specification of the conditions outside the hole via the Einstein equations fails to uniquely specify the conditions inside the hole. In a bit more detail, let m and m' be TADM models of GR that share a common manifold M, and let S be a Cauchy surface on M. Let "$=_L$" be the relation of Leibniz Equivalence (i.e., "$=_L$" holds between two models m, m' of a theory in a given formalism just when m and m' represent the same physically possible world). Then determinism requires,

$$(m|_S =_L m'|_S) \Rightarrow (m =_L m') \qquad (\mathrm{Det})_{\mathrm{TADM}}$$

In words: If the restrictions $m|_S$ and $m'|_{S'}$ represent the same physically possible world, then m and m' represent the same physically possible world. In so far as the manifold substantivalist denies Leibniz Equivalence for diffeomorphic TADM models (i.e., she claims that m $\neq_L$ m' for any m, m' related by a diffeomorphism), she violates $(\mathrm{Det})_{\mathrm{TADM}}$ when m' = h*m, where h is a hole diffeomorphism.

Informally, then, the hole argument can be given by the following:

$$(1 \,\&\, 2) \Rightarrow \sim(\mathrm{Det})_{\mathrm{TADM}}$$

Three distinct options for avoiding the hole argument have appeared in the literature:

(A) Reject (1) and (2).

(B) Sophisticated Substantivalism: Uphold (1) and reject (2).

(C) Reject the implication from Denial of Leibniz Equivalence to Indeterminism.

Usually (A) is identified with relationalism and a denial of the claim that spacetime exists independently of its material contents.[1]

In general, Option (B) observes that the hole argument requires the view that TADM models m and m' related by a hole diffeomorphism are *ontologically distinct*, hence they represent *different possible worlds*. Option (B)'ers dispute this implication by engaging in metaphysical excursions into the notions of identity and possibility. Ontologically intrepid B'ers claim (Bi) Realism with respect to manifold points need not entail that m and m' are ontologically distinct (e.g., Hoefer 1996). Modally intrepid B'ers claim (Bii) If m and m' are ontologically distinct, it need not follow that they represent different possible worlds (e.g., Maudlin 1990; Butterfield 1989; Brighouse 1994).

Alternatively, one may maintain one's manifold substantivalism by rejecting the implication from Denial of Leibniz Equivalence to Indeterminism *a la* Option (C). In particular, some authors have argued (at least implicitly) that determinism is a formal property of a theory and does not depend on how the theory is interpreted. Thus to the extent that $(\mathrm{Det})_{\mathrm{TADM}}$ is interpretation-dependent, its denial should not be identified with Indeterminism with a capital I (Leeds 1995; Mundy 1992). Another version of Option (C) claims that the denial of Leibniz Equivalence merely entails a purely semantic underdetermination of co-intended interpretations of a formal language, and has nothing to do with physical indeterminism (Liu 1996; Rynasiewicz 1992). This version views the hole argument as a variant of Quine- and Putnam-style inscrutability of reference arguments.

But what about interpretations of GR formulated in formalisms in which points do not explicitly appear? Perhaps realism with respect to spacetime can be better motivated in such formalisms.

3. Earman's Algebraic Substantivalism

Geroch 1972 has demonstrated how GR can be reformulated in terms of what he called Einstein algebras. Note first that the following chain of successive structures appears in the tensor formalism:

set of points → topology → maximal atlas → differential manifold

Simply put, the Einstein algebra formalism is based on the alternative chain,

commutative ring → differential structure → differential manifold

Specifically, the Einstein algebra formalism takes advantage of an alternative to the standard definition of a differential manifold as a set of points imbued locally with topological and differential properties. The manifold substantivalist's gloss of this definition awards ontological status to the point set. The alternate definition emphasizes the differential structure, as opposed to the points of M on which such structure is predicated. It is motivated by the following considerations: The set of all real-valued C^∞ functions on a differential manifold M forms a commutative ring $C^\infty(M)$ under pointwise addition and multiplication. Let $C^c(M) \subset C^\infty(M)$ be the subring of constant functions on M. A derivation on the pair $(C^\infty(M), C^c(M))$ is a map $X : C^\infty(M) \to C^\infty(M)$ such that $X(af + bg) = aXf + bXg$ and $X(fg) = fX(g) + X(f)g$, and $X(a) = 0$, for any $f, g \in C^\infty(M)$, $a, b \in C^c(M)$. The set $\mathcal{D}(M)$ of all such derivations on $(C^\infty(M), C^c(M))$ forms a module over $C^\infty(M)$ and can be identified with the set of smooth contravariant vector fields on M. A metric g can now be defined as an isomorphism between the module $\mathcal{D}(M)$ and its dual $\mathcal{D}^*(M)$. Tensor fields may be defined as multi-linear maps on copies of $\mathcal{D}(M)$ and $\mathcal{D}^*(M)$, and a covariant derivative can be defined with its associated Riemann tensor. Thus all the essential objects of the TADM formalism necessary to construct a model of GR may be constructed from a series of purely algebraic definitions based ultimately on the ring $C^\infty(M)$. All Geroch's 1972 observation at this point is that the manifold only appears initially in the definition of $C^\infty(M)$. This suggests viewing C^∞ and C^c as algebraic structures in their own right, with M as simply a point set that induces a representation of them.[2] Formally, Geroch 1972 defined an Einstein algebra $\mathcal{A}$ as a tuple $(\mathcal{I}, \mathcal{R}, g)$, where $\mathcal{I}$ is a commutative ring, $\mathcal{R}$ is a subring of $\mathcal{I}$ isomorphic with the real numbers, and g is an isomorphism from the space of derivations on $(\mathcal{I}, \mathcal{R})$ to its algebraic dual such that the associated Ricci tensor vanishes (and a contraction property is satisfied).

Earman 1989 has observed that the ring C(X) of real-valued continuous functions on a topological space X determines X up to homeomorphism (given X satisfies certain conditions).[3] In general, $C^\infty(M)$ determines M up to diffeomorphism. This suggests adopting a one-many representation map between Einstein algebras and TADM models: any given algebra represents a diffeomorphism-equivalence class of TADM models. Earman describes this tactic as a substantivalist one:

> While the Leibniz [i.e., Einstein] algebras provide a solution to the problem of characterizing the structure common to a Leibniz-equivalence class of [TADM] models, and the solution eschews substantivalism in the form of space-time points, the solution is nevertheless substantival, only at a deeper level (Earman 1989, 192-193).

According to Earman, this "substantival" interpretation of the Einstein algebra formalism avoids the hole argument, given that we restrict the meaning of determinism to equivalence classes of TADM models. In particular, for a given Einstein algebra $\mathcal{A}$, let $R(\mathcal{A})$ be the collection of all TADM representations m of $\mathcal{A}$. Determinism can then be defined in the following manner (after Earman 1989, 218, footnote 17): Let S(m) be a Cauchy surface in the TADM model $m \in R(\mathcal{A})$. Let $R(\mathcal{A})|_S$ be the restriction of the members of $R(\mathcal{A})$ to those times up to and including the Cauchy surfaces S(m) for each $m \in R(\mathcal{A})$. Then GR is deemed deterministic just when,

$$(R(\mathcal{A})|_S =_L R(\mathcal{A}')|_{S'}) \Rightarrow (R(\mathcal{A}) =_L R(\mathcal{A}'))$$

In effect, Earman's algebraic substantivalist satisfies the above under the threat of the hole argument by indirectly upholding Leibniz equivalence for Einstein algebras: If homomorphically related Einstein algebras were not Leibniz equivalent, then determinism as defined above would fail for algebras related by "hole homomorphisms". Moreover, the algebraic substantivalist interpretation is eliminativist in motivation: it attempts to do without manifold points, constructing them when needed from a more ontologically primitive algebra.

Critique

Rynasiewicz 1992 has stressed the significance of the reconstruction of the point-set of a topological space X from the ring C(X). In particular, there is a bijective (and hence invertible) map between the maximal ideals of C(X) and the points of X (see footnote 3). A differential manifold M can then be reconstructed by imposing a differential structure (i.e., a maximal atlas) on X.[4] Rynasiewicz concludes that Earman's one-many representation map between Einstein algebras and TADM models is not justified -- to every Einstein algebra there corresponds a unique TADM model, and *vice versa.* In particular, to every hole-diffeomorphism on manifolds, there corresponds a hole-homomorphism on algebras. Explicitly, suppose ψ is a structure-preserving map by means of which a TADM model m of GR represents an Einstein algebra $\mathcal{A}$. Since ψ is invertible at the level of the point structure of m, to every hole diffeomorphism h on TADM models, there corresponds a hole homomorphism ϕ on Einstein algebras, given by $\phi = \psi^{-1} \circ h \circ \psi$.

Hence, according to Rynasiewicz, the hole argument translates into the algebraic formalism, and nothing is gained by adopting the latter. In particular, the eliminativist motivation for algebraic substantivalism cannot be upheld without further ado, in so far as an Einstein algebra reproduces the diffeomorphism "redundancy" of a differential manifold. Formally, then, "...Leibnizian theories [i.e., Einstein algebras] are subject to radical local indeterminism to the same extent as substantivalist theories [TADM models of GR]" (Rynasiewicz 1992, 582).

Einstein algebras, then, do not provide a formal solution to the hole argument. I think, however, that it is still left open as to whether they can provide an interpretive solution. Minimally, the expressive equivalence described above indicates that there is an analog, in the algebraic formalism, of the manifold substantivalist interpretation of TADM; and that there are analogs in the algebraic formalism of various other substantivalist interpretations of TADM. But I do not think we should conclude from this either that interpretive issues in GR should be restricted to the tensor formalism, or even that the algebraic formalism is expressively equivalent to the tensor formalism in all respects relevant to the interpretive enterprise.

4. Einstein Algebras and Interpretations of General Relativity

In this section, I would like to consider what possible interpretations of Einstein algebras could be like, and how they might differ, in particular, from interpretations of GR in the tensor formalism.

Interpretations of Einstein algebras can be classified initially in terms of how they stand on Leibniz equivalence in the algebraic context:

> <u>Leibniz Equivalence (algebraic formalism):</u> Homomorphically related Einstein algebras represent the same possible world.

Call an interpretation of an Einstein algebra that minimally denies Leibniz equivalence, a literal interpretation. Call an interpretation that minimally adopts Leibniz equivalence, a gauge-invariant interpretation. (This distinction is made much more precise by Belot and Earman 2001 in the Hamiltonian formulation of GR.) Obviously, Earman's algebraic substantivalism is a gauge-invariant interpretation of Einstein algebras.

Function Literalism

Consider first a literal interpretation of an Einstein algebra. Evidently, given the 1-1 correspondence between Einstein algebra homomorphisms and manifold diffeomorphisms, literal interpretations of Einstein algebras will suffer from indeterminism (modulo Option C to the hole argument). But what might a literal interpretation of an Einstein algebra amount to? One possibility is to attempt to carry over as much as possible of the manifold substantivalist interpretation of TADM models of GR to Einstein algebras. The correlates of manifold points in the algebraic formalism are the maximal ideals of the algebra. What then does it mean to be a realist with respect to these maximal ideals? Note that a differential manifold can be reconstructed from a concrete representation of an Einstein algebra as an algebra of real-valued C^{∞} functions (the Einstein algebra itself is an abstract algebra). The maximal ideals of such an algebra essentially are (certain types of) subsets of these functions. Taking a cue from Demaret, Heller and Lambert (1997, 146), a function literalist might interpret these functions as a system of scalar fields and include them, as opposed to points, in her ontology.

Gauge-Invariant Interpretations

Depending on one's attitude towards indeterminism, one might wish to adopt a gauge-invariant interpretation of Einstein algebras. Note that in the tensor formalism, such gauge-invariant interpretations are identified as relationalist or sophisticated substantivalist (Option B in Section 2 above). Two questions arise in this context: Why adopt a gauge-invariant interpretation of Einstein algebras, as opposed to relationalism or sophisticated substantivalism in the tensor formalism?, and What would the ontological commitments of a gauge-invariant interpretation of Einstein algebras be?

By way of addressing these questions, consider first some problems associated with gauge-invariant interpretations of the tensor formalism (see. e.g., Belot and Earman 2001):[5]

(1) They do not give an intrinsic characterization of the reduced phase space of GR. (They characterize it "extrinsically" (or "parasitically") in terms of the gauge orbits of the full phase space.)

(2) The only known gauge-invariant observables of GR are non-local.

(3) There is no dynamical evolution of gauge-invariant quantities in GR.

Note first that Problem (1) is meant to apply only to those gauge-invariant interpretations of TADM models of GR that are "internal" to classical GR; i.e., that do not rely on motivation from sources like research programmes in quantum gravity. Most of the standard relationalist and sophisticated substantivalist interpretations in the philosophical literature are of this nature. Some claim that even this limited applicability is poorly motivated (e.g., Pooley 2001). The intuition seems to be that gauge-invariant interpretations always come bundled with other claims, and these latter can be appealed to in justifying the appropriation (and re-interpretation) of the mathematics that underlies literal interpretations (the structure of the full phase space, in the Hamiltonian formalism, for instance). Regardless of where one stands on this issue, certainly problems (2) and (3) are a challenge to any gauge-invariant interpretation of GR. At the least, it could be argued, a gauge-invariant interpretation should give an account of (2) and (3). In any event, certainly those interpretations that can give such accounts are at least as well-motivated, if not better motivated, than those that do not. It turns out that there is an approach to quantum gravity due to Heller and Sasin 1999 that derives from the Einstein algebra formalism and that does give accounts of (2) and (3) above. This suggests that a gauge-invariant interpretation of Einstein algebras can be motivated by appeal to this approach, and that it is at least just as well-motivated as gauge-invariant interpretations of the tensor formalism.

The Heller/Sasin theory originated in extensions of Einstein algebras to spacetimes with singularities. A non-singular general relativistic spacetime can be represented by a differential manifold M, or an Einstein algebra $C^\infty(M)$. To represent certain types of curvature singularities in the tensor formalism requires additional structures on M. In particular, the b-boundary construction collects singularities in a space $\partial_b M$ and attaches it as a boundary to M to create a differential manifold with boundary $M' = M \cup \partial_b M$. In the algebraic formalism, one can now consider an algebraic object of the schematic form $C^\infty(M')$, consisting of real-valued C^∞ functions on M'. Originally, this object was identified as a sheaf of (commutative) Einstein algebras over M' (Heller and Sasin 1995). Heller and Sasin 1996 demonstrated that such an object can also be analyzed as a non-commutative Einstein algebra of complex-valued C^∞ functions over a more general structure (in particular, the semi-direct product $OM \times O(1, 3)$, of the Cauchy completed frame bundle OM over M' and the structure group O(1, 3)). The success of this analysis suggested a way to unite general relativity with quantum mechanics.

The theory presented in Heller and Sasin 1999 takes as the fundamental object an "Einstein C*-algebra" $\mathcal{E}$. $\mathcal{E}$ is constructed from the non-commutative algebra $\mathcal{A}$ of complex-valued C^∞ functions with compact support on a transformation groupoid $G = E \times \Gamma$. Here E is the total space of a fiber bundle (with structure group Γ) over a spacetime manifold, and G is a Lie group acting on E.[6] In a manner similar in outline to Geroch's procedure, an algebraic analog of the Einstein equations can be constructed from the derivations defined on $\mathcal{A}$. On the strength of a theorem due to Connes, $\mathcal{A}$ can be realized as the algebra of bounded operators on the Hilbert space of square-integrable functions defined on a fiber G_q of G, $q \in E$, and can then be completed with respect to an appropriate norm to form a C*-algebra that Heller and Sasin refer to as an Einstein C*-algebra $\mathcal{E}$.[7] Their theory of quantum gravity is then based on the following postulates (Heller and Sasin 1999, pg. 1630):

Postulate 1: A quantum gravitational system is represented by an Einstein C*-algebra $\mathcal{E}$, and its observables by self-adjoint elements of $\mathcal{E}$.

Postulate 2: The states of $\mathcal{E}$ are defined as normalized positive linear functionals f on $\mathcal{E}$.

Postulate 3: If a is an observable and ϕ is a state, then $\phi(a)$ is the expectation value of the observable a when the system is in the state ϕ.

Postulate 4: The dynamical equation of the system, described by $\mathcal{E}$, is

$$i\hbar\pi_q(v(a)) = [\pi_q(a), F]$$

where $\pi_q : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is the Connes representation of $\mathcal{A}$, $a \in \mathcal{A}$, v is a derivation on $\mathcal{A}$ that is a solution to the algebraic Einstein equations, and $F : \mathcal{H} \to \mathcal{H}$ is an operator that reproduces a Hamiltonian operator in the commutative "limit".

The first three postulates are familiar from algebraic quantum field theory. The fourth defines a dynamics that unites GR and quantum theory in the sense of reproducing the Einstein equations and the Shrodinger equation in the appropriate "commutative limit".[8]

The Heller/Sasin theory is more of a scheme for quantum gravity than a complete theory, in so far as the structure group Γ is left unspecified, as well as the operator F. As such a scheme, it can be characterized in the following way:

(a) It does not have a fundamental diffeomorphism gauge symmetry. The easiest way to see this is by noting that non-commutative algebras in general do not have maximal ideals, hence Einstein C*-algebras do not have the diffeomorphism redundancy that Einstein algebras defined over differential manifolds have.

(b) It provides a story about non-local observables in GR. The observables of the Heller/Sasin theory are non-local in the following sense: In the Connes representation, the wave functions ψ of the quantum sector of the theory are defined on fibers over the "base space" E of the groupoid G. When they are projected down to $M = E/\Gamma$, they do not necessarily distinguish between the points of M.

(c) It provides a story about a-temporal dynamics in GR. In the non-commutative regime, there is no explicit time parameter -- the dynamics imposed on the Einstein C*-algebra $\mathcal{E}$ is written explicitly in terms of its derivations. To recover a time parameter for the commutative limit, Heller and Sasin 1998 propose identifying an "internal" time flow with the one-parameter group of modular automorphisms of the von Neumann algebra generated by $\mathcal{E}$ (after the work of Connes and Rovelli 1994). They then demonstrate that their dynamics can be written in terms of this group.

These characteristics can be seen as solutions to the three problems afflicting gauge-invariant interpretations of GR. So far as (1) is concerned, the Heller/Sasin theory suggests that the diffeomorphism gauge of GR is not a fundamental symmetry. Hence there is no need to provide an intrinsic characterization of the reduced phase space of GR. This suggests that, if the problems associated with gauge-invariant interpretations of GR are to be taken seriously, then a gauge-invariant interpretation of Einstein algebras is at least as well-motivated as gauge-invariant interpretations of the tensor formalism. (One might argue that it is better motivated than those interpretations that take solutions to these problems for granted.)

This leaves open the question of what a gauge-invariant interpretation of Einstein algebras motivated by the Heller/Sasin theory would look like. Note, first, that it is a gauge-invariant interpretation of Einstein algebras as defined by Geroch, those algebras that are expressively equivalent to TADM models of GR in the context of the hole argument (i.e., those Einstein algebras that have well-defined maximal ideals). The fundamental object of the Heller/Sasin theory is an Einstein C*-algebra which, when appropriately restricted, reproduces a Geroch Einstein algebra. This suggests that a gauge-invariant interpretation of Geroch Einstein algebras can be based on a literal interpretation of Einstein C*-algebras. In other words, it would make the following two claims:

1. Homomorphically related Geroch Einstein algebras represent the same physically possible world in virtue of being appropriate commutative restrictions of a single non-commutative Einstein C*-algebra.

2. Homomorphically related Einstein C*-algebras represent different physically possible worlds.

This leaves open the question of what a literal interpretation of an Einstein C*-algebra would look like. Here one has at one's disposal the various interpretive options open to interpreters of algebraic quantum field theory. Issues include how to delimit the set of physically relevant observables and the set of physically relevant states.[9] Such issues are best left to future discussion.

5. Conclusion

Belot and Earman (2001) have indicated how the interpretive issues raised by the hole argument are intimately linked to research programmes in quantum gravity: The way one chooses to interpret classical GR will influence the approach one takes to the quantized theory, and *vice versa* -- approaches to quantum gravity can be classified in terms of how they interpret the classical theory. In this article, I've indicated how the Heller/Sasin theory suggests a gauge-invariant interpretation of Einstein algebras of the sort initially proposed by Earman. The fundamental object of the Heller/Sasin theory is a non-commutative Einstein C*-algebra $\mathcal{E}$. Restricting $\mathcal{E}$ to an appropriate commutative subalgebra produces a (Geroch) Einstein algebra isomorphic to a TADM model of general relativity. The diffeomorphism redundancy of (Geroch) Einstein algebras is seen to be a property of the commutative restriction of $\mathcal{E}$, and not a property of $\mathcal{E}$ itself. In this sense, it is an artefact of looking only at a part of the full theory, and hence should not be construed literally.

Acknowledgements

Thanks to Jeremy Butterfield for insightful comments on an earlier draft. Any remaining mistakes are, of course, those of the author.

References

Belot, G. and J. Earman (2001), 'Pre-Socratic Quantum Gravity', in C. Callander and N. Huggett (eds.), *Physics Meets Philosophy on the Planck Scale*, Cambridge: Cambridge University Press, 213-255.

Brighouse, C. (1994), 'Spacetime and Holes', in D. Hull, M. Forbes and R. M. Burian (eds.) *PSA 1994*, Vol. 1, East Lansing: Philosophy of Science Association, 117-125.

Butterfield, J. (1989), 'The Hole Truth', *British Journal for the Philosophy of Science* **40**, 1-28.

Connes, A. (1994), *Noncommutative Geometry,* San Diego: Academic Press.

Connes, A. and C. Rovelli (1994), 'Von Neumann Algebra Automorphisms and Time-Thermodynamics Relation in Generally Covariant Quantum Theories', *Classical and Quantum Gravity* **11,** 2899-2917.

Demaret, J., M. Heller, and D. Lambert (1997), 'Local and Global Properties of the World', *Foundations of Science* **2**, 137-176.

Earman, J. (1989), *World Enough and Spacetime*, Cambridge: MIT Press.

Earman, J. and J. D. Norton (1987), 'What Price Spacetime Substantivalism? The Hole Story', *British Journal for Philosophy of Science* **38**, 515-525.

Geroch, R. (1972), 'Einstein Algebras', *Communications of Mathematical Physics* **26**, 271-275.

Heller, M. and W. Sasin (1995), 'Sheaves of Einstein Algebras', *International Journal of Theoretical Physics* **34**, 387-398.

Heller, M. and W. Sasin (1996), 'Noncommutative Structure of Singularities in General Relativity', *Journal of Mathematical Physics* **37**, 5664-5671.

Heller, M. and W. Sasin (1998), 'Emergence of Time', *Physics Letters A* **250**, 48-54.

Heller, M. and W. Sasin (1999), 'Noncommutative Unification of General Relativity and Quantum Mechanics', *International Journal for Theoretical Physics* **38**, 1619-1642.

Hoefer, C. (1996), 'The Metaphysics of Space-Time Substantivalism', *Journal of Philosophy* **43**, 5- 27.

Leeds, S. (1995), 'Holes and Determinism: Another Look', *Philosophy of Science* **62**, 425-437.

Liu, C. (1996), 'Realism and Spacetime: Of Arguments against Metaphysical Realism and Manifold Realism', *Philosophia Naturalis* **33**, 243-263.

Maudlin, T. (1990), 'Substances and Spacetime: What Aristotle Would Have Said to Einstein', *Studies in History and Philosophy of Science* **21**, 531-561.

Mundy, B. (1992), 'Space-Time and Isomorphism', in D. Hull, M. Forbes and K. Okruhlik (eds.) *PSA 1992,* Vol. 1, East Lansing: Philosophy of Science Association, 515-527.

Pooley, O. (2001), Relationism Rehabilitated? II: Relativity', preprint PITT-PHIL-SCI00000221.

Rynasiewicz, R. (1992), 'Rings, Holes And Substantivalism: On the Program of Leibniz Algebras', *Philosophy of Science* **59**, 572-589.

[1]Pooley 2001 suggests two forms of relationalism: an *anti-substantivalist* relationalist rejects (1), whereas an *anti-haecceitist* relationalist rejects (2). On the other hand, Belot and Earman 2001 suggest that sophisticated substantivalism (their term) is really relationalism in disguise and that "straightforward substantivalism" be associated simply with (2). It seems to me that spacetime substantivalism should include both a realist component (some form of realist claim about spacetime) and a component that identifies relevant degrees of freedom associated with a spacetime theory. Pooley's point may be taken to be that these components are not necessarily dependent on each other.

[2]Such a representation is given by the Gelfand representation. Any abstract linear algebra $\mathcal{A}$ (over a field $\mathbb{K}$) admits a Gelfand representation defined by $\rho : \mathcal{A} \rightarrow \mathbb{K}^{\mathcal{A}^*}$, $\rho(x)(\phi) = \phi(x)$, where $x \in \mathcal{A}$, $\phi \in \mathcal{A}^*$, and $\mathcal{A}^*$ is the algebraic dual of $\mathcal{A}$ (i.e., the set of homomorphisms $\phi : \mathcal{A} \rightarrow \mathbb{K}$) and $\mathbb{K}^{\mathcal{A}^*}$ is the algebra of $\mathbb{K}$-valued functions on $\mathcal{A}^*$. Intuitively, the Gelfand representation turns the abstract object $\mathcal{A}$ into a "concrete" algebra of functionals on a space $\mathcal{A}^*$.

[3]The maximal ideals of an abstract algebra $\mathcal{A}$ (if they exist) are in 1-1 correspondence with the elements of $\mathcal{A}^*$ (the "characters" of $\mathcal{A}$). Hence, if $\mathcal{A}$ has maximal ideals, the points of the space $\mathcal{A}^*$ can be reconstructed by means of the Gelfand representation of $\mathcal{A}$ (see footnote 2). (A maximal ideal of $\mathcal{A}$ is the largest proper subset of $\mathcal{A}$ closed under (left or right) multiplication by any element of $\mathcal{A}$.) In particular, the points of a topological space X can be reconstructed from the maximal ideals of the ring C(X). Concretely, one shows that any maximal ideal of C(X) consists of all functions that vanish at a given point of X.

[4]Note that there are (at least) two ways to view the reconstruction of points of a differential manifold. One can reconstruct the points of a topological space X from the maximal ideals of C(X), and then impose a differential structure on X to obtain a differential manifold. Alternatively, one can directly reconstruct the points of M from the maximal ideals of $C^{\infty}(M)$. See, e.g., Demaret, Heller and Lambert (1997, 163).

[5]By a gauge-invariant interpretation of the tensor formalism, I mean one that denies Leibniz Equivalence for diffeomorphically related TADM models of GR. Belot and Earman 2001 articulate the following problems in the Hamiltonian formulation of GR. These problems, however, are not specific to any particular formalism.

[6]Intuitively, a groupoid differs from a group by having its binary operation only partially defined on its elements, and by having (possibly) more than one unit element. A transformation groupoid G is the semi-direct product of a "base space" E, consisting of the unit elements of G, and a group Γ that acts on E (see Connes 1994, pp. 99-100, for more precise definitions). Heller and Sasin identify the quotient space $M = E/\Gamma = (G/\Gamma)/\Gamma$ as a spacetime manifold.

[7]See, e.g., Connes (1994, pg. 102). The "Connes representation" $\pi_q : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, $\mathcal{H} = L^2(G_q)$, is given by $(\pi_q(a)\psi)(\gamma) = \int_{G_q} a_q(\gamma_1)\psi(\gamma_2)$, where $a \in \mathcal{A}$, $\psi \in \mathcal{H}$, and $\gamma, \gamma_1, \gamma_2$ are elements of G such that $\gamma = \gamma_1 \circ \gamma_2$, and the integral is taken with respect to the left invariant Haar measure. The completion of $\mathcal{A}$ with respect to the norm $||a|| = \sup_{q \in E} ||\pi_q(a)||$ produces a C*-algebra.

[8]The commutative limit involves a restriction of $\mathcal{A}$ to a subalgebra of its (commutative) center. The operator $F : \mathcal{H} \rightarrow \mathcal{H}$ is left unspecified with the condition that, in the commutative limit, it becomes a Hamiltonian operator. (This allows recovery of the standard evolution of operators in the Heisenberg picture.)

[9]The interpretational problems associated with unitarily inequivalent representations in algebraic quantum field theory do not arise for interpretations of Einstein C*-algebras, insofar as the latter do not admit an intrinsic Hamiltonian time evolution. This suggests that the study of dynamics on Einstein C*-algebras might indicate how the problem of time in interpretations of GR can be linked with interpretational problems in quantum field theory arising from Haag's Theorem. (Note that this is true for the dynamics of any generally covariant quantum field theory, and hence is not unique to Einstein C*-algebras.)