Condensed Matter Physics and the Nature of Spacetime

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Abstract

In this essay, I consider what condensed matter physics has to say about the nature of spacetime. In particular, I consider the extent to which spacetime can be modeled as a quantum liquid, with matter and force fields described by effective field theories of the low-energy excitations of the liquid. After a brief review of effective field theories in 2-dim highly-correlated condensed matter systems, I evaluate analogies in the recent physics literature between spacetime and superfluid Helium, and proposals that suggest spacetime is an emergent phenomenon arising from the edge states of a 4-dim Quantum Hall liquid. *Keywords*: spacetime, effective field theory, condensed matter, quantum gravity

1 Introduction

In the philosophy of spacetime literature not much attention has been given to concepts of spacetime arising from condensed matter physics. This essay attempts to address this. I look at analogies between spacetime and a quantum liquid that have arisen from effective field theoretical approaches to highly

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correlated many- body quantum systems. Such approaches have suggested to some authors that spacetime can be modeled by a quantum liquid with its contents (matter and force fields) described by effective field theories (EFTs, hereafter) of the low-energy excitations of this liquid. While directly relevant to ongoing debates over the ontological status of spacetime, this programme also has other consequences that should interest philosophers of physics. It suggests, for instance, a particular attitude towards quantum gravity, as well as an anti-reductionist attitude towards the nature of symmetries in quantum field theory. A secondary goal of this essay is to address some of these issues. While discussion of these topics has appeared in the philosophical and historical literature (*e.g.*, Cat 1999), the gory theoretical details have not been made explicit. Moreover, while the topic of EFTs in the philosophy of quantum field theory literature has also been given some attention (*e.g.*, Castellini 2002, Hartmann 2001, Huggett and Weingard 1995), surprisingly little has been said about how EFTs arise in condensed matter systems.

The plan of the paper is as follows. Section 2 sets the stage by reviewing some simple (2+1)-dim EFTs arising in condensed matter systems. Following the presentation in Zhang (2004), these include (2+1)-dim quantum electrodynamics (QED3) in superfluid Helium 4 films and in high temperature superconductors, and the (2+1)-dim Chern-Simons gauge theory of the 2-dim Quantum Hall Effect (QHE). One extraordinary property of these EFTs is the emergence of a Lorentz-invariant theory in the low-energy sector of a non-relativistic theory (in the case of the QHE, the relativistic theory emerges at the edges). Section 3 considers the extent to which this carries over to (3+1)-dim. It first looks at claims that the superfluids Helium 3 and Helium 4 provide models for (3+1)-dim spacetime. In particular, a growing body of literature indicates how such models can describe aspects of black hole physics *via* "acoustic" analogues (*e.g.*, Barceló *et al.* 2005), and Volovik (2001, 2003) has claimed that such models solve the cosmological constant problem. Section 3 concludes with a look at recent work (Sparling 2002) that links twistor theory with Zhang and Hu's (2001) extension of the QHE to 4-dim. This work suggests that spacetime is an emergent phenomenon that arises from the edge states of a 4-dim quantum Hall liquid. The conclusion summarizes these results and offers commentary on the relevance of condensed matter analogues of spacetime to both philosophy of spacetime and philosophy of quantum field theory.

2 Effective Field Theories in Condensed Matter Systems

The condensed matter systems to be discussed below are highly-correlated quantum many-body systems; that is, many-body systems that display macroscopic quantum effects. Examples include superfluids, superconductors, Bose condensates, and quantum Hall liquids. In general, an effective field theory of such a system describes the dynamics of the states with energy close to zero; *i.e.*, the "zero modes" of the system. These zero modes can take the following forms (Volovik 2003, pg. 4):

(i) Bosonic collective modes of the ground state of the system.

(ii) Fermionic excitations of the system above its ground state, referred to as "quasiparticles".

(iii) Topological defects of the system, primarily in the form of vortices.

In the remainder of this section, I will flesh out this general description by means of concrete examples, concentrating on EFTs that arise in 2-dim condensed matter systems. This review will then inform the discussion in Section 3 of concepts of spacetime arising from higher dimensional systems. In the following, two methods will be used to take the low-energy limit of a theory. The first expands the Lagrangian density about small fluctuations in the field variables above the ground state and then integrates out the high-energy fluctuations. An example of this will be the EFT for superfluid Helium 4. Alternatively, the theory's energy spectrum can be linearized about its zero points (the points where it vanishes), and the corresponding low-energy Hamiltonian can then be determined. Examples of this will be the EFTs for high temperature superconductors and superfluid Helium 3. (Note that neither of these methods involves taking a mathematical limit; rather, they are approximation schemes.)

Finally, a few words on the notion of "emergence" may be appropriate. The examples below suggest to some authors that novel phenomena (fields, particles, spacetime, *etc.*) "emerge" in the low-energy limit of certain condensed matter systems. As will be seen, this sense of emergence consists of both formal and interpretive elements. The formal element involves a relation between the emergent structure and the "host" structure; in the present context, this takes the form of a low-energy limit (*viz.*, low-energy approximation). The interpretive element involves adopting an interpretation under which the emergent structure is not merely a definitional extension of the host; in other words, the emergent structure is interpreted to be ontologically distinct from its host. For instance, the claim that QED_3 emerges from a superfluid ^4He film is based on *both* the derivation of the QED_3 Lagrangian from the low-energy limit of the ^4He Lagrangian, *and* an interpretation (a "duality representation") in which the superfluid velocity and density are interpreted as electric and magnetic fields, respectively.

2.1 Superfluid Helium Films

At low temperatures, liquid Helium forms a superfluid characterized by dissipationless flow and quantized vortices. This superfluid comes in two forms, depending on the isotope of Helium involved: In addition to two electrons, ^4He

contains two nucleons and hence is a boson; whereas ^3He contains three nucleons and hence is a fermion. The superfluids formed by these isotopes occur at different temperatures, but it is believed that both consist of Bose condensates (in the case of ^3He the condensate is formed from pairs of ^3He atoms). In this section I review how QED3 arises as an EFT in 2-dimensional ^4He superfluid films (^3He will make its appearance in Section 3.1 below).[1]

The phase transition between the normal liquid and superfluid states can be described by an order parameter that vanishes above a critical temperature T_c and becomes finite below T_c. The ^4He order parameter takes the form of a "macroscopic" wave function $\varphi_0 = \sqrt{\rho_0}e^{i\theta}$ describing the coherent ground state of a Bose condensate with density ρ_0 and coherent phase θ. An appropriate Lagrangian density describes *non-relativistic* neutral bosons (*viz.*, ^4He atoms) interacting *via* a quartic potential with coupling constant g^2,

$$\mathcal{L}_{^4He} = i\varphi^\dagger\partial_i\varphi - \frac{1}{2m}\partial_i\varphi^\dagger\partial_i\varphi + \mu\varphi^\dagger\varphi - g^2(\varphi^\dagger\varphi)^2. \tag{1}$$

Here m is the mass of a ^4He atom, and the term involving the chemical potential μ enforces particle number conservation. (1) is invariant under a global $U(1)$ symmetry with ground state energy density given by $\Omega_0(\mu) = -\mu|\varphi_0|^2 + g^2|\varphi_0|^4$. Minimizing Ω_0 determines the ground state: For $\mu < 0$, the ground state vanishes, while for $\mu > 0$, it is degenerate, given by $\varphi_0 = \sqrt{\mu/2g^2}e^{i\theta}$. The transition at $\mu = 0$ can be described qualitatively as a spontaneous breaking of the $U(1)$ symmetry in which one of the degenerate ground states is chosen, thereby allowing the bosons to condense.

It is perspicacious to rewrite (1) in a form in which the dynamical variables are the particle density ρ and the phase θ. This form makes the hydrodynamical properties of the superfluid explicit. If we let $\varphi = \sqrt{\rho}e^{i\theta}$, (1) becomes,

$$\mathcal{L}_{^4He} = -\rho\partial_t\theta - \frac{\rho}{2m}(\partial_i\theta)^2 - \frac{1}{8m\rho}(\partial_i\rho)^2 - g^2(\rho-\rho_0)^2 \tag{2}$$

(dropping a total divergence and a constant term). The second term represents the kinetic energy density in which the superfluid velocity v_i is identified as $v_i = (1/m)\partial_i\theta$. The fact that the macroscopic wave function is unique up to phase entails that the superflow in a multiply-connected domain is quantized,

$$\oint \vec{v}\cdot d\vec{x} = \frac{1}{m}\oint \vec{\partial}\theta\cdot d\vec{x} = \frac{1}{m}2\pi q, \tag{3}$$

around a closed path encircling a "hole", where q is an integer. Such holes may be interpreted as *vortices* – points where the real and imaginary parts of φ_0 vanish. Then (3) entails that the superflow about a vortex is quantized.[2]

[1] The following draws on pedagogical treatments in Wen (2004) and Zee (2003). Throughout this essay, unless otherwise noted, units are chosen such that $\hbar = 1$.

[2] More precisely, vortices are soliton solutions to the equations of motion of (1) characterized by $\varphi = f(r)e^{i\theta}$, with boundary conditions $f(0) = 0$ and $f(r) \to \varphi_0$, as $r \to \infty$. Intuitively, these conditions describe a localized wave with finite energy that does not dissipate over time.

The task now is to find the low-energy limit of (2) and demonstrate that the resulting EFT restricted to 2-dim is QED$_3$.[3] This seems initially plausible, given the similarity between (3) and Gauss' Law. This similarity can be cashed out in terms of a "duality" transformation under which vortices become the sources of a gauge field formally identical to the Maxwell field. One starts by expanding (2) in small fluctuations in ρ and θ about their ground state values: Formally, we let $\rho = \rho_0 + \delta\rho$, $\theta = \theta_0 + \delta\theta$. Discarding 2nd order terms, one obtains $\mathcal{L}_{^4He} = \mathcal{L}_0[\rho_0, \theta_0] + \mathcal{L}'_{^4He}[\delta\rho, \delta\theta]$, where the first term reproduces (2), and the second term describes fluctuation contributions. To obtain a low-energy EFT, the high-energy contributions to $\mathcal{L}'_{^4He}$ must be eliminated. Note that since the ground state is a function only of the phase, low-energy excitations take the form of phase fluctuations. High energy excitations will thus depend on density fluctuations. To eliminate these, extremize $\mathcal{L}'_{^4He}$ with respect to $\delta\rho$, solve for $\delta\rho$ in the resulting equation of motion, and then substitute back into $\mathcal{L}'_{^4He}$. In 2-dim the result is,

$$\mathcal{L}'_{^4He} = \frac{1}{4g^2}(\partial_t\theta)^2 - \frac{\rho_0}{2m}(\partial_i\theta)^2, \qquad i = 1, 2, \tag{4}$$

with $\delta\theta$ replaced by θ for the sake of notation. In this derivation, the first term has been simplified from $\rho_0\partial_t[4g^2\rho_0 - (1/2m)\partial_i^2]^{-1}\partial_t\theta$. This is justified in the low- energy limit for momenta k much smaller than $\sqrt{8g^2\rho_0 m}$ (Zee 2003, pg. 258). As will be seen, the $(1/2m)\partial_i^2$ term becomes important in the formalism for acoustic black holes in Section 3.1. below, where it is the source of curvature in the acoustic metric. Formally, (4) describes a scalar field propagating at speed $c^2 = 2g^2\rho_0/m$. For units in which $c = 1$, it can be re-written as

$$\mathcal{L}'_{^4He} = \frac{1}{4g^2}\eta^{\mu\nu}\partial_\mu\theta\partial_\nu\theta, \qquad \mu, \nu = 0, 1, 2, \tag{5}$$

where $\eta^{\mu\nu}$ is the (2+1)-dim Minkowski metric. (5) is manifestly Lorentz invariant; in fact, it is formally identical to the Lagrangian density for a massless scalar field propagating in (2+1)-dim Minkowski spacetime. To obtain QED$_3$, note that in 2 dimensions, the magnetic field is a scalar, whereas the electric field is a 2-vector. This motivates the following identifications:[4]

(a) $B \equiv \partial_0\theta = -2g^2(\rho - \rho_0)$

(b) $E_i \equiv (1/m)\epsilon_{ij}\partial_j\theta = \epsilon_{ij}v_j$

in which the magnetic field is identified with the density, and the electric field with the superfluid velocity (here ϵ_{ij} is the skew volume 2-form). Substituting into (5), one obtains the Lagrangian density for sourceless QED$_3$

[3]The following draws on Wen (2004, pp. 82-83; 259-264) and Zee (2003, pp. 257-258; 314-316).

[4]To derive the equality in (a), extremize (2) with respect to ρ to obtain the equation of motion $\partial_0\theta + (1/2m)(\partial_i\theta)^2 + 2g^2(\rho - \rho_0) = 0$, and then discard the $(1/2m)\partial_i^2$ term.

$$\mathcal{L}_{QED_3} = \frac{1}{4g^2}\eta^{\mu\sigma}\eta^{\nu\lambda}F_{\sigma\lambda}F_{\mu\nu} \qquad \mu,\nu = 0,1,2, \tag{6}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with the potential A_μ defined by $E_i = \partial_0 A_i - \partial_i A_0$, $B = \partial_1 A_2 - \partial_2 A_1$. Now note that (3) entails that the density for "elementary" vortices ($q = \pm 1$) is given by $(1/2\pi)\vec{\partial} \times \vec{\theta}$. This can be identified as the 0th component of a vortex current $j_v^\mu = (1/2\pi)\epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda\theta$, where $\epsilon^{\mu\nu\lambda}$ is the skew volume 3-form.[5] This vortex current is the dual of the electromagnetic current, in so far as adding a source term $A_\mu j_v^\mu$ to (6) and extremizing with respect to A_μ produces the sourced Maxwell equations.

Remarks: To recap, we started with the Lagrangian density (1) for a superfluid ^4He film and found that, in the low-energy limit, it becomes the Lagrangian density for (2+1)-dim quantum electrodynamics. Under a literal interpretation of the dual representation (6), low energy excitations of such a film take the form of electric and magnetic fields, the former being given by the superfluid velocity, and the latter being given by the superfluid density. Moreover, topological defects (*i.e.*, elementary vortices) take the form of charge-carrying electrons. This is a first example of a Lorentz-invariant EFT that emerges in the low-energy limit of a non-relativistic (*i.e.*, Galilei- invariant) theory (1). While this is by itself remarkable, one might initially question whether (5) has any additional physical relevance (massless scalar fields after all are limited in their physical applications). The dual representation (6) lays this question to rest. Note, however, that this particular dual representation only works in (2+1)-dim, in which the magnetic field can be identified as the scalar particle density.

2.2 2-dim Superconductors

QED3 also arises in some accounts of high temperature (high-T_c, hereafter) cuprate superconductors. Such materials consist of a 2-dim layer of copper and oxygen atoms and are characterized by an orbital symmetry referred to as "*d*-wave" symmetry. Due to this symmetry, the standard BCS Hamiltonian becomes a relativistic (2+1)-dim Dirac Hamiltonian in the low energy limit. QED3 then arises in some attempts to describe the phase transition to *d*- wave superconductivity. In this section, after a brief review of BCS theory, I will indicate how this occurs. The following review will also inform the discussion of superfluid Helium 3 in section 3.1 below.

Superconductors are experimentally characterized by dissipationless current flow and the expulsion of external magnetic fields (the Meissner Effect). The phase transition to the superconducting state is accompanied by the appearance of an energy gap at the Fermi surface in momentum space that separates occupied states from unoccupied states. The Bardeen-Cooper-Schrieffer (BCS) theory explains these effects in terms of a Bose condensate consisting of pairs of electrons

[5] Note that the form of j_v^μ contracts over skew and symmetric indices; however it is not identically zero, since for vortices, θ is not a globally defined function.

with opposite spin and momenta (Cooper pairs). The order parameter that describes this condensate is called the BCS *gap parameter* Δ and characterizes the energy gap in the superconducting phase.

To flesh this out in just a bit more detail, consider the second-quantized many-body momentum space BCS Hamiltonian

$$\begin{aligned} H_{BCS} &= \sum_{k,\alpha}(\varepsilon_k - \mu)c^\dagger_{k\alpha}c_{k\alpha} + \sum_k(\Delta c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow} + \Delta^* c_{-k\downarrow}c_{k\uparrow}) \\ &= \sum_{k,\alpha,\beta} \chi^\dagger_{\alpha\beta}(\vec{k})\left((\varepsilon_k - \mu)\sigma_3 + \Delta\sigma_+ + \Delta^*\sigma_-\right)\chi_{\alpha\beta}(\vec{k}). \end{aligned} \tag{7}$$

In the first line, the c's are creation/annihilation (particle/hole) operators for electrons with momentum $\vec{k}$ and spin α. The first term describes the kinetic energy in terms of the energy ε_k of an effective electron in a crystal lattice. The second term describes an effective interaction in which a pair of electrons with opposite spins and momenta (a Cooper pair) interact *via* the exchange of a lattice phonon.[6] In the second line, the notation has been simplified by introducing Pauli matrices σ, and recasting the particle/hole operators as $SU(2)$ 2-spinors $\chi^\dagger_{\alpha\beta}(\vec{k}) = (c^\dagger_{k\alpha}, -i\sigma_2 c_{-k\beta})$.[7] Finally the BCS *gap parameter* Δ is defined by,

$$\Delta = \lambda\sum_k \langle c_{-k\downarrow}c_{k\uparrow}\rangle. \tag{8}$$

Δ takes the form of the expectation value for a Cooper pair annihilation operator, and hence is a measure of the presence of Cooper pairs. It can also be interpreted as a gap in the energy spectrum for quasiparticle excitations above the Cooper pair condensate. This is motivated by diagonalizing (7) *via* a Bolgoliubov transformation to obtain $H_{BCS} = \sum_k \psi^\dagger(\vec{k})E(\vec{k})\sigma_1\psi(\vec{k})$, where $\psi^\dagger(\vec{k}) = (b^\dagger_{k\uparrow}, b_{-k\downarrow})$, with the b operators being linear combinations of the c's, and $E^2(\vec{k}) = (\varepsilon_k - \mu)^2 - |\Delta|^2$. The b's are interpreted as creation/annihilation operators for quasiparticles with energy E, with Δ representing a constant energy gap between the Fermi surface and the lowest allowable quasiparticle energy

[6] The first line is actually a mean field approximation to a Hamiltonian of the general form $H_{BCS} = \sum_{k,\alpha}\varepsilon_k c^\dagger_{k\alpha}c_{k\alpha} + \lambda\sum_{k,k'} c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}c_{-k'\downarrow}c_{k'\uparrow}$, where λ is a coupling constant. The mean field approximation involves replacing the interaction term with its expectation value with respect to a suitably chosen expression for the ground state and applying Wick's Theorem. See Annett (2004, pg. 140).

[7] The Pauli spin matrices are given by $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$. Technically, in (7) they act in the particle/hole configuration space, whereas in the expression for the 2-spinor, σ_2 acts in the spin space. In the latter expression, this serves to flip the spin of the second component. Summing over all spin indices α, β then reproduces the interaction term with opposite spins.

state. Under this interpretation, above T_c the gap parameter vanishes and ε_k is the energy to create or annihilate an (effective) electron. Below T_c, the gap parameter is finite, indicating the presence of Cooper pairs that form a Bose condensate, and E is the energy to create/annihilate a quasiparticle excitation of the condensate. Note finally that below T_c, the energy is never zero due to the gap; hence a low-energy approximation about zero points cannot be taken. This can be done, however, for high-T_c superconductors for which the gap parameter has non-trivial symmetries.

To move to high-T_c superconductors, one may first note that a Cooper pair wavefunction can be expressed as the product of a k-dependent orbital part and a spin part: schematically, $\Psi_{\alpha\beta}(\vec{k}) = \phi_{\alpha\beta}\Psi_k$, where α, β are spin indices, with the spin part in either a spin singlet state ($S = 0$) , or a spin triplet state ($S = 1$). The gap parameter inherits this structure, $\Delta_{\alpha\beta}(\vec{k}) = \phi_{\alpha\beta}\Delta_k$. Conventional superconductors are described by spin singlet Cooper pairs with $\Delta_k = \Delta$ in (8) above. This is referred to as "s- wave" pairing symmetry, since the orbital part is constant and hence can be described by an $l = 0$, or "s-orbital", spherical harmonic function. Cooper pairs for high-T_c superconductors are also characterized by spin singlets, but the general consensus is that they have a "d-wave" symmetry, with their orbital parts described by $l = 2$ (d-orbital) spherical harmonics. For example, the d_{xy}-gap parameter is given by[8]

$$\Delta_k = \Delta \sin k_x \sin k_y. \tag{9}$$

Unlike the s-wave gap parameter (8), the d-wave gap parameter (9) is a function of $\vec{k}$ and vanishes at the four points $\pm(k_F, 0)$, $\pm(0, k_F)$, where $k_F = \sqrt{2m\mu}$ is the Fermi momentum. About these "Fermi" points, quasiparticle excitations can take place at arbitrarily low energies. The energy spectrum can now be linearized about these points and the corresponding low-energy Hamiltonian constructed.

One may start by expanding the 2-spinors in (7) about the Fermi points:[9]

$$\chi(\vec{x}) = e^{i\vec{k}_1\cdot\vec{x}}\tilde{\chi}_1(\vec{x}) + e^{i\vec{k}_2\cdot\vec{x}}\sigma_2\tilde{\chi}_2(\vec{x}) + e^{i\vec{k}_3\cdot\vec{x}}\tilde{\chi}_3(\vec{x}) + e^{i\vec{k}_4\cdot\vec{x}}\sigma_2\tilde{\chi}_4(\vec{x}). \tag{10}$$

Substituting back into the Fourier transform of (7) and neglecting second order terms, one obtains the d-wave effective Hamiltonian (Franz, *et al.* 2002, pg. 12),

[8]See, *e.g.*, Annett (2004, pg. 162). Current evidence suggests a $d_{x^2-y^2}$ symmetry, under which $\Delta_k = \Delta(\cos k_x - \cos k_y)/2$, with nodes at $\pm(\pi/2, \pi/2)$ and $\pm(-\pi/2, \pi/2)$ (Annett 2004, pp. 164-166). The d_{xy} gap parameter is obtained by a rotation, and more readily facilitates taking the low-energy limit. The EFTs for both are identical.

[9]The following is based on Franz *et al.* (2002). See also Herbut (2002), and Zhang (2004), who take the low-energy limit for $d_{x^2-y^2}$ symmetry. The Fourier transformed (configuration space) spinor is given by $\chi(\vec{x}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}<\Lambda} e^{i\vec{k}\cdot\vec{x}}\chi(\vec{k})$, for momentum cut-off Λ and phase space volume V. In (10) the Fermi points have been labeled 1, 2, 3, 4.

$$H'_{d-w} = \sum_{j=1,2} \tilde{\chi}_j^\dagger(\vec{x})(iv_F\partial_x\sigma_3 + iv_\Delta\partial_y\sigma_1)\tilde{\chi}_j(\vec{x}) + \sum_{j=3,4}(x \leftrightarrow y) \tag{11}$$

where $v_F = \partial\varepsilon_k/\partial k|_{k=k_F}$ is the Fermi velocity and $v_\Delta = \partial\Delta_k/\partial k|_{k=k_F}$ is the "gap velocity".[10] This is a (2+1)-dim Hamiltonian for massless Dirac fields. To see this, note that the corresponding Lagrangian density can be written as,

$$\mathcal{L}'_{d-w} = \bar{\Psi}_1(\gamma_0\partial_t + \gamma_1 v_F\partial_x + \gamma_2 v_\Delta\partial_y)\Psi_1 + (1 \leftrightarrow 2; x \leftrightarrow y) \tag{12}$$

where the γ matrices are given by $\gamma_0 = \sigma_3\otimes\sigma_2$, $\gamma_1 = \sigma_3\otimes\sigma_1$, $\gamma_2 = -\sigma_3\otimes\sigma_3$, and the 2-spinors have been combined into 4-spinors $\bar{\Psi}_1 = \Psi_1^\dagger\gamma_0 = (\tilde{\chi}_1^\dagger, \tilde{\chi}_2^\dagger)\gamma_0$, $\bar{\Psi}_2 = (\tilde{\chi}_3^\dagger, \tilde{\chi}_4^\dagger)\gamma_0$ (Franz *et al.* 2002, pg. 13). So-defined, the γ matrices generate a Clifford algebra, and the 4-spinors are irreducible representations of this algebra; in other words, the γ matrices are Dirac γ matrices, and the 4-spinors are Dirac 4-spinors. $\mathcal{L}'_{d-w}$ can now be compared with the general form $\bar{\Psi}\gamma_\mu\partial_\mu\Psi$ which describes a massless Dirac field.

Remarks: While it is "anisotropic", in so far as, in general, $v_F \neq v_\Delta$, the Lagrangian density (12) nevertheless displays a gapless relativistic energy dispersion relation $E^2(\vec{k}) = v_F^2 k_x^2 + |v_\Delta|^2 k_y^2$ (similarly for x, y exchanged). Again, it is a further example of a Lorentz-invariant EFT that emerges in the low-energy limit of a non- relativistic theory (7). In particular, under the intended interpretation, the low-energy quasiparticle excitations of a d-wave superconductor are identified as relativistic massless Dirac fields.

The move to QED3 occurs by coupling the effective Dirac fermions of (12) to a gauge field generated by vortex excitations. This is done in a manner similar in principle to the duality transformation of the superfluid Lagrangian in Section 2.1. There the phase degree of freedom of the complex order parameter was responsible for the low-energy excitations, as well as the vortex source current. This source current depended on the fact that the phase θ was multi-valued on its domain. The BSC gap parameter is also a complex scalar, $\Delta = |\Delta|e^{i\phi}$, and we can similarly identify vortex solutions with the multi- valued part of its phase ϕ. To couple (12) to such vortices, the procedure is first to re-write ϕ in terms of smooth and multi-valued (vortex) components $\phi = \phi_s + \phi_v$, substitute back into (12), and then integrate out the smooth component.[11] The result is formally identical to QED3 and takes the form (Franz *et al.* 2002, pg. 13),

[10] The second term in (11) is identical to the first with x- and y- indices interchanged. To get a feel for the approximations, note first that the kinetic energy in (7) is that for electrons hopping on a 2-dim lattice, and is given explicitly by $\varepsilon_k = -2t(\cos k_x + \cos k_y)$, where $t = 1/m_{eff}$, for m_{eff} the effective mass of an electron (see, *e.g.*, Zhang 2004, pg. 673). The Fourier transform of ε_k can then be approximated by $t(\partial_x^2 + \partial_y^2)$, hence acting on the first term in (10), it produces $(iv_F\partial_x)e^{ik_Fx}\tilde{\chi}_1(\vec{x})$, where $v_F = k_F/m_{eff}$. The Fourier transform of the gap parameter (9) can be approximated by $\Delta i\partial_x\partial_y$. Acting on the first term in (10), it produces $(iv_\Delta\partial_y)e^{ik_Fx}\tilde{\chi}_1(\vec{x})$, where $v_\Delta = -\Delta k_F$.

[11] The non-trivial details are given in Herbut (2002) and Franz *et al.* (2002). This procedure could have been used above in Section 2.1. See, *e.g.*, Zee (2003), pp. 314-316.

$$\mathcal{L} = \mathcal{L}''_{d-w} + \frac{1}{2}K_\mu(\partial \times b)^2_\mu. \tag{13}$$

Here, $\mathcal{L}''_{d-w}$ is obtained from (12) by minimal coupling $\partial_\mu \to \partial_\mu + ib_\mu$, where b_μ is a $U(1)$ gauge potential dependent on ϕ, and K_μ is a constant. The second term in (13) is the Maxwell term written in the transverse gauge $(\partial \cdot b)_i = 0$.

Remarks: It should be noted that this QED3 theory of high-T_c superconductivity is one approach among several in an active field of research. While the physics of the phase transition for conventional superconductors is well understood, that for high-T_c superconductors is not. Above the critical temperature, the normal state for a high-T_c superconductor is not a conductor, but an antiferromagnetic (AF) insulator. Moreover, there is a "pseudogap" in the phase diagram that separates this AF phase from the d-wave superconducting phase, and much current research is directed towards discovering the physics of this pseudogap. The approach represented by (13) starts with the d-wave superconducting state and works backwards to recover the normal AF state. This approach views the transition between these states as a symmetry-breaking of the effective QED3 theory, analogous to chiral symmetry breaking in the Standard Model.[12]

2.3 2-dim Quantum Hall Liquids

The 2-dim Quantum Hall Effect (QHE) provides a final example of a condensed matter system described by a (2+1)-dim EFT. The set-up consists of current flowing in a 2-dim conductor in the presence of an external magnetic field perpendicular to its surface. The classical Hall Effect occurs as the electrons in the current are deflected towards the edge by the magnetic field, thus inducing a transverse voltage. Suppose the conductor lies in the xy plane, the magnetic field B^{ext} is in the negative z-direction, and the current density $J_x = n_e e v_x$ is in the x-direction, where n_e is the electron density.[13] In the steady state, the force due to the magnetic field is balanced by the force due to the induced electric field, $eE_y = ev_x B^{ext}$, and the *Hall resistivity* is given by $\rho_H \equiv \rho_{xy} = E_y/J_x = B^{ext}/n_e e$. The QHE occurs in the presence of a strong magnetic field, in which the Hall resistivity becomes quantized in units of the ratio of fundamental constants h/e^2,

$$\rho_H = (h/e^2)\nu^{-1}. \tag{14}$$

Here ν is the *filling factor* defined by $\nu = n_e/n_B$, where $n_B = B^{ext}/\Phi_0$ is

[12] This is Herbut's (2002) take. Similar strategies are given in Balents *et al.* (1998) and Franz *et al.* (2002).

[13] In this section particle densities are labeled by "n", as opposed to "ρ", in keeping with the literature, and to make the notation less cumbersome. Note, further, that the resistivity below is labeled by "ρ" and is not to be confused with particle density.

the flux density, with $\Phi_0 \equiv h/e$ the flux quantum.[14] The Integer Quantum Hall Effect (IQHE) is characterized by integer values of ν, and the Fractional Quantum Hall Effect (FQHE) is characterized by values of ν given by odd-denominator fractions. Two properties experimentally characterize the system at such quantized filling factors: the diagonal resistivity $\rho_{xx} = \rho_{yy}$ vanishes, and the system is incompressible.

For filling factors $\nu = 1/p$, for p an odd integer, these properties can be explained by an effective Chern-Simons gauge theory of a Bose condensate in which the particle content is given by "composite bosons" consisting of electrons with p quanta of magnetic flux attached to them. Mathematically, this is achieved by coupling the electrons to a Chern-Simons gauge field. This effectively turns the electrons into bosons. When the magnetic field due to the attached fluxes exactly balances the external magnetic field, the bosons can condense and the resulting Bose condensate is identical to a superconductor. The properties of the latter then explain the properties of the QHE.

To see how this comes about, one can start with a 2-dim Lagrangian density for nonrelativistic electrons in the presence of an electromagnetic field

$$\mathcal{L}_{QHE} = \psi^\dagger(i\partial_t - eA_0^{ext})\psi - \frac{1}{2m}\psi^\dagger(-i\partial_i + eA_i^{ext})^2\psi + V[\psi^\dagger\psi] + \mathcal{L}_{EM}, \quad (15)$$

where A_0^{ext}, A_i^{ext} are potentials for the magnetic and electric fields, V is an interaction potential, and $\mathcal{L}_{EM}$ is the electromagnetic Lagrangian density. At low energies, this fermion system can be shown to be equivalent to a boson system described by[15]

$$\mathcal{L}_{QHE} = \varphi^\dagger(i\partial_t - eA_0)\varphi - \frac{1}{2m}\varphi^\dagger(-i\partial_i + eA_i)^2\varphi + V[\varphi^\dagger\varphi] + \mathcal{L}_{CS}. \qquad (16)$$

Here $A_0 \equiv a_0 + A_0^{ext}$, $A_i \equiv a_i + A_i^{ext}$, where $(a_0, a_i) \equiv a_\mu$ is a Chern-Simons (CS, hereafter) potential field described by the term $\mathcal{L}_{CS} = (p/4\pi)\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$, with p an odd integer. At low energies, this term dominates the $\mathcal{L}_{EM}$ term in (15). Now note that extremizing (16) with respect to the CS field yields the equation of motion,

$$(e/2\pi p)\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda = J^\mu \qquad \mu, \nu, \lambda = 0, 1, 2, \qquad (17)$$

where J^0 is the particle density[16] , and J^i is the current density. To see what this entails, consider a boson at rest at $x = 0$, with $J^0 = \delta(x)$, $J^i = 0$. Upon

[14] In the lowest energy (or "Landau") level, there is one electron state for each flux quantum; a flux quantum in this context can be heuristically thought of as the amount of flux that penetrates the area occupied by an electron. Thus the flux density n_B measures the degeneracy per unit area of each Landau level. Hence, the filling factor determines the ratio of the number of electrons to the number of states in a Landau level.

[15] See, *e.g.*, Zhang (1992, pg. 32).

[16] We assume the particle density n_e is the same for both the electron system and the boson system, and that the bosons have the same electromagnetic charges.

integration, (17) yields the flux of the CS field through the area containing the boson, $\oint \vec{a}\cdot d\vec{x} = p\Phi_0$ (in units in which $\hbar = 1$). This motivates an interpretation of the CS field as attaching to each boson a magnetic flux given by an odd number p of flux quanta Φ_0. The effect is to modify the exchange statistics of the bosons: When two are exchanged, they pick up an Aharonov-Bohm phase equal to $e^{ie\int_0^\pi \vec{a}\cdot d\vec{x}} = -1$, which mimics the Fermi-Dirac statistics of the original electrons. Note that this statistical transmutation only works in 2-dim: In 1-dim, point particles cannot be exchanged, and for dim 3, any closed, continuous exchange path taken by two point particles can be continuously deformed into a point; hence such paths cannot be distinguished by means of winding numbers.

Thus, as advertised, the electrons of (15) have been turned into composite bosons by attaching an odd number of flux quanta to them. To see how this accounts for the FQHE, note that the combined external and Chern-Simons magnetic field felt by the composite bosons is given by $\vec{\partial}\times(\vec{a}+\vec{A}^{ext}) = p\Phi_0 n_e - B^{ext}$, which vanishes when $\Phi_0 n_e/B^{ext} = \nu = 1/p$. Thus at filling factors $\nu = 1/p$, the composite bosons feel no net magnetic field and hence can form a condensate at $T = 0$. This QH condensate has the same properties as a superconductor; namely, dissipationless current flow and the Meissner effect. The former entails that the diagonal resistivity vanishes. The Meissner effect entails there is no net internal magnetic field in the QH condensate, and since the internal CS magnetic field is determined by the particle density *via* (17), this entails that the particle density is constant; *i.e.*, the QH condensate is incompressible.

The exact derivation of these results requires the techniques of Ginsberg-Landau theory, with the resulting Chern-Simons-Ginsberg-Landau (CSGL) theory mapping the incompressible QH phase of the original electrons onto the superconducting phase of the composite bosons. Note finally that this CSGL-EFT describes QH liquids only at filling factors $\nu = 1/p$, for odd p; however, it can be extended in a hierarchy scheme for other filling factors (see Zhang 1992, pg. 26, and references therein).

Remarks: The CSGL-EFT based on (16) differs in two majors respects from the EFTs considered in previous sections. First, the CSGL theory is a *topological* quantum field theory: explicitly, no metric occurs in (16) (indices are raised and lowered by $\epsilon^{\mu\nu\lambda}$). Second, since the QH liquid is incompressible, excitations have finite energy gaps, and this prevents the construction of a low-energy limit in the manner outlined above for superfluid films and high-T_c superconductors. Such a low-energy limit can, however, be constructed for the edge states of the liquid. Wen (1990) demonstrated that the low-energy excitations of the edge states are described by a (1+1)-dimensional EFT of chiral fermions. This edge EFT is a further example of an emergent relativistic EFT and will also play a role in the conception of spacetime discussed in Section 3.2 below.

Edge State EFT for 2-dim QH Liquids
Since the bulk QH liquid is incompressible, Wen (1990) assumed that an edge

excitation takes the form of a surface wave described by the density function $\rho(x) = n_e f(x)$, with $f(x)$ being the wave amplitude. This wave propagates according to the wave equation,

$$\partial_t \rho - v\partial_x \rho = 0 \tag{18}$$

with speed $v = E_y/B^{ext}$, and it's energy is given by the edge Hamiltonian,

$$H_{edge} = \frac{1}{2}eE_y \int dx f(x)\rho(x) = \pi p v \int dx \rho^2(x), \tag{19}$$

for odd integer p.[17] The formulas (18) and (19) can now be encoded in the Lagrangian density for a (1+1)-dim massless chiral fermion field,

$$\mathcal{L}_{edge} = i\psi^\dagger(x)(\partial_t - v\partial_x)\psi(x). \tag{20}$$

The encoding is accomplished by writing $\psi(x) \propto e^{ip\varphi(x)}$, which relates the fermion field to the density by means of a boson field φ, which is required to satisfy $\rho(x) = (1/2\pi)\partial_x\varphi$. This procedure of rewriting a fermion field in terms of a boson field is an instance of "bosonization". Key to this construction are the commutation relations for the density operator $[\rho_k, \rho_{k'}] = (1/2\pi)k\delta_{k+k'}$, which follow from (18), (19), and Hamilton's equations of motion. The physical system described by these commutation relations and the Hamiltonian (19) defines what is referred to as a (chiral) Luttinger liquid.

Remarks: Note first that (20) is a relativistic (1+1)-dim EFT, and not a topological theory like the bulk liquid EFT. However, it is a bit different from the relativistic EFTs in Sections 2.1 and 2.2 above. In those cases, a relativistic theory emerges near the Fermi surface of a non-relativistic system. For the edge state EFT (20), the relativistic theory emerges near the surface of a different sort of liquid; namely, a Luttinger liquid.[18] Moreover, the low-energy limit in this case corresponds to a hydrodynamical analysis in which it is initially assumed that excitations are gapless and obey a linear dispersion relation (this is implicit in (18)). In the previous cases of EFTs, the gapless energy condition was derived, and not put into the analysis by hand.

3 Spacetime and Quantum Liquids

The above examples of EFTs in condensed matter systems are (1+1)- and (2+1)-dim. The restriction to these dimensions is essential: QED_3 appears in ^4He films

[17] Here I follow Wen (2004, pp. 312-313). The increment in energy of an edge surface wave in the interval Δx may be given by $\delta H_{edge} = 1/2[V(f(x)) - V(0)]$ whence the first equality in (19) follows upon integration. The second equality follows from $f(x) = \rho(x)/n_e$, with $n_e = \nu n_B = eB^{ext}/2\pi p$, for $\nu = 1/p$.

[18] A Luttinger liquid differs from a Fermi liquid mathematically in the form of the electron propagator: Luttinger liquids are characterized by propagators with non-trivial exponents. Physically this entails that Luttinger liquids cannot be analyzed in terms of single particle occupancies, as can Fermi liquids. See Wen (2004, pp. 314-315) for discussion.

only because the magnetic field is a scalar in (2+1)-dim; massless Dirac fermions appear near the Fermi surface of high-T_c superconductors only because of the 2-dim d-wave symmetry of the gap parameter; and the CSGL theory of the QHE depends on the statistical transmutation of electrons into composite bosons, possible only in 2-dim. While these examples are instructive in demonstrating how a relativistic theory can emerge in the low-energy sector of a non-relativistic theory, they cannot be said to be instructive as to the ontological nature of spacetime, if we require the latter to be (3+1)-dim. If condensed matter EFTs are to be informative about the nature of spacetime, we should ask if there are (3+1)- dim examples. In this section, I review some results, starting with superfluid Helium and ending with the 4-dim QHE.

3.1 Spacetime and Superfluid Helium

In this subsection I review two ways in which superfluid Helium can provide information about the nature of spacetime. The first is by providing an analogue of (3+1)-dim general relativistic spacetimes, and the second is by providing an analogue of the (3+1)-dim vacuum of the Standard Model of particle physics.

(A) Helium 4 and General Relativity

Superfluid Helium analogues of general relativistic spacetimes can be motivated by recalling that the effective Lagrangian (5) for ^4He is identical to that for a massless scalar field in (3+1)-dim Minkowski spacetime (after reattaching a third spatial dimension). To move to curved spacetime requires reinserting a term left out in the derivation of (5). Recall that the result of linearizing (2) is given schematically by $\mathcal{L}_{^4He} = \mathcal{L}_0 + \mathcal{L}_1$, where $\mathcal{L}_0$ describes the ground state and $\mathcal{L}_1$ describes contributions from fluctuations. After integrating out the high-energy density fluctuations, $\mathcal{L}_1$ becomes

$$\mathcal{L}_1 = \frac{1}{4g^2}(\partial_0\theta + v_i\partial_i\theta)^2 - \frac{\rho_0}{2m}(\partial_i\theta)^2, \qquad i = 1, 2, 3 \tag{21}$$

with $\delta\theta$ replaced by θ for the sake of notation. In the derivation of (21), a term $(1/2m)\partial_i^2$ has been retained (see the remarks after (4)). This is responsible for the v_i term in $\mathcal{L}_1$, which is the only difference between it and (4) above. $\mathcal{L}_1$ can now be rewritten in the compact form,

$$\mathcal{L}_1 = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\theta\partial_\nu\theta, \qquad \mu, \nu = 0, 1, 2, 3 \tag{22}$$

for the curved metric given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\rho/cm)[-c^2dt^2 + \delta_{ij}(dx^i - v^i dt)(dx^j - v^j dt)] \tag{23}$$

where $\sqrt{-g} = \rho^2/m^2c$, and $c^2 = 2g^2\rho/m$. (22) describes a massless scalar field propagating in a (3+1)-dim curved spacetime with a low- energy "acoustic" metric $g_{\mu\nu}$.[19]

To make contact with general relativity, Volovik (2003, pg. 38) interprets $\mathcal{L}_{^4He}$ as comprised of a "gravitational" part $\mathcal{L}_0$ describing a background spacetime expressed in terms of the variables θ_0, ρ_0, with gravity being simulated by the superfluid velocity, and a "matter" part $\mathcal{L}_1$, expressed in terms of the variable θ. To obtain the "gravitational" equations of motion, one can proceed in analogy with general relativity by extremizing $\mathcal{L}_{^4He}$ with respect to θ_0, ρ_0. This results in a set of equations that are quite different in form from the Einstein equations (Volovik 2003, pg. 41). This indicates immediately that the *dynamics* of this EFT does not reproduce general relativity. However (23) can be used to reproduce aspects of general relativity that do not depend explicitly on the Einstein equations. In particular, acoustic spacetimes can be exploited to probe the physics of black holes and the nature of the cosmological constant.

(i) *Acoustic Black Holes.* Acoustic black holes are regions in the background fluid from which phonons (*i.e.*, low-energy excitations traveling at c) cannot escape. This can be made more precise with the definitions of acoustic versions of ergosphere, trapped region, and event horizon, among others. A growing body of literature seeks to exploit such formal similarities between relativistic black hole physics and acoustic "dumb" hole physics (see, *e.g.*, Barceló *et al.* 2005). The primary goal is to provide experimental settings in condensed matter systems for relativistic phenomena such as Hawking radiation that do not depend explicitly on the Einstein equations.

(ii) *The Cosmological Constant.* Volovik (2001, 2003) has argued that the analogy between superfluid Helium and general relativity provides a solution to the cosmological constant problem. The latter he takes as the conflict between the theoretically predicted value of the vacuum energy density in quantum field theory (QFT), and the observationally predicted value: The QFT theoretical estimate is 120 orders of magnitude greater than the observational estimate. Volovik sees this as a dilemma for the marriage of QFT with general relativity. If the vacuum energy density contributes to the gravitational field, then the discrepancy between theory and observation must be addressed. If the vacuum energy density is not gravitating, then the discrepancy can be explained away, but at the cost of the equivalence principle. Volovik's preferred solution is to grab both horns by claiming that both QFT and general relativity are EFTs that emerge in the low-energy sector of a quantum liquid.

[19] The metric (23) agrees with Barceló *et al.* (2001, pp. 1146- 1147). In the literature on acoustic black holes, the derivation of the acoustic metric generally starts with a classical irrotational, inviscid fluid described by a *mass* density function and a fluid velocity given by $v_i = -\partial_i\theta$ (*e.g.*, Barceló *et al.* 2005). To regain contact with (23), in which continuity is maintained with the notation in Section 2.1, set $m = 1$. Note finally that Volovik's (2003, pp. 36-39) derivation of the acoustic metric for superfluid ^4He rescales the low-energy phase fluctuation as $m\theta$, to allow fluctuations in the superfluid velocity to be given simply by $\partial_i\theta$ (in keeping with the literature). His action is obtained by setting $\theta = m\theta$ in (21) above.

(a) The first horn is grasped by claiming that QFTs are EFTs of a quantum liquid. As such, the vacuum energy density of a QFT does not represent the true "trans-Planckian" vacuum energy density, which must be calculated from the microscopic theory of the underlying quantum liquid. And simply put, at $T = 0$, the pressure of such a liquid is equal to the negative of its energy density, $P = -\rho_\Lambda$ (Volovik 2003, pg. 14, 26). This relation between pressure and vacuum energy density also arises in general relativity, if the vacuum energy density is identified with the cosmological constant term in the Einstein equations. However, in the case of a quantum liquid in equilibrium, the pressure is identically zero; hence, so is the vacuum energy density. Moreover, if the liquid is in the form of a droplet, the pressure is not zero, but scales as an inverse power of the droplet size, and this models the cosmological constant term in the Einstein equations, which scales as the inverse square of the size of the universe.

(b) The second horn is grasped simply by claiming that general relativity is an EFT. Thus, we should not expect the equivalence principle to hold at the "trans-Planckian" level, and hence we should not expect the true vacuum energy density to be gravitating.

Remarks: Two questions seem pertinent at this point: First, to what extent is (22) a low-energy EFT; and second, to what extent is (22), and acoustic spacetimes in general, an analogue of general relativity?

In regard to the first question, note that the velocity term in (21) responsible for the curvature in the acoustic metric is a higher order fluctuation term. As such, one might question its presence in a supposedly low-energy derivation. In fact, it was explicitly discarded as a high-energy term in the derivation of the low-energy EFT (5) (see the remarks after (4)). On the other hand, note that it is a 2nd order fluctuation only in the phase, and not the density. Hence the acoustic metric can be viewed as a low-energy limit that includes 2nd order fluctuations away from a given ground state (in the form of phase fluctuations), but still discards high energy fluctuations in the density. This might be seen as modeling fluctuations in curvature above the flat Minkowski spacetime associated with (5). This analogy cannot, however, be stretched too far, for the following two reasons. First, interpreting Minkowski spacetime as the kinematic "background" spacetime of the acoustic metric (23) is problematic. The ground state associated with (23) is the Lagrangian density (2) that describes a superfluid in a background (Galilei-invariant) Neo-Newtonian spacetime. Low energy fluctuations about this ground state obey the Lorentz symmetries associated with Minkowski spacetime, and higher-order fluctuations obey the symmetries of the curved metric (23). Hence, from a kinematic point of view, Neo-Newtonian spacetime is the background for the acoustic metric. Second, from a dynamical point of view, Minkowski spacetime cannot be derived from (5). While Lorentz invariance can be said to emerge from (5), Minkowski spacetime cannot be constructed simply from the dynamics of a relativistic massless scalar field. Of course, if the Einstein equations were derivable from the curved case, we could get Minkowski spacetime from (5) as the zero curvature solution.

The fact that the Einstein equations are not derivable from $\mathcal{L}_{^4He}$ raises the question of how effectively acoustic spacetimes model general relativity. The general impression given by the literature is that acoustic spacetimes account for the "kinematics" of general relativity, but not the dynamics:

> ... the features of general relativity that one typically captures in an "analogue model" are the *kinematic* features that have to do with how fields (classical or quantum) are defined on curved spacetime, and the *sine qua non* of any analogue model is the existence of some "effective metric" that captures the notion of the curved spacetimes that arise in general relativity. (Barceló, *et al.* 2005, pg. 10.)

> The acoustic analogue for black-hole physics accurately reflects half of general relativity – the kinematics due to the fact that general relativity takes place in a Lorentzian spacetime. The aspect of general relativity that does not carry over to the acoustic model is the dynamics – the Einstein equations. Thus the acoustic model provides a very concrete and specific model for separating the kinematic aspects of general relativity from the dynamic aspects. (Visser 1998, pg. 1790.)

While it is undeniable that acoustic black holes offer much in the way of analyzing real black holes, one might question the extent to which the acoustic metric reflects the kinematics of general relativity. The latter is a bit hard to pin down, simply because what normally counts as the kinematics of a field theory (*i.e.*, those variables that describe the field in the absence of external forces), is dynamic in general relativity. Moreover, identifying the kinematics of general relativity with curved spacetimes in general seems inappropriate, since not all curved spacetimes satisfy the Einstein equations; hence not all curved spacetimes are physically relevant to the theory. These observations simply point to the fact that diffeomorphism invariance is essential for modeling general relativity, and the low-energy EFT (21) is not generally covariant. (The acoustic metric takes the form of an ADM metric which explicitly splits space from time.) So while Lorentz invariance does emerge from (5), diffeomorphism invariance does not. At this point it might be instructive to compare acoustic spacetimes as EFTs with the typical EFT that results from taking the low-energy limit of general relativity (see, *e.g.*, Donoghue 1995). The latter imposes diffeomorphism invariance from the outset. One first notes that the Einstein-Hilbert Lagrangian density that produces the Einstein equations is proportional to the scalar curvature, and as such is the simplest diffeomorphism-invariant Lagrangian density that contains derivatives of the metric (which must be included for the metric to be a dynamical field in the theory). The effective Lagrangian density is constructed by including all other powers of the curvature, consistent with diffeomorphism invariance. These extra terms then serve to cancel infinities at all orders that arise in the quantization process. (To make the connection with low-energy limits discussed in this essay, note that renormalization is analogous to integrating out high-energy terms.)

The best moral perhaps is that drawn by Barceló, *et al.* (2004) who suggest that acoustic spacetimes simply demonstrate that some phenomena typically associated with general relativity really have nothing to do with general relativity:

> Some features that one normally thinks of as intrinsically aspects of gravity, both at the classical and semiclassical levels (such as horizons and Hawking radiation), can in the context of acoustic manifolds be instead seen to be rather generic features of curved spacetimes and quantum field theory in curved spacetimes, that have nothing to do with gravity *per se*. (Barceló, *et al.* 2004, pg. 2.)

This takes some of the initial bite out of Volovik's solution to the cosmological constant problem. If acoustic spacetimes really have nothing to do with general relativity, their relevance to reconciling the latter with quantum field theory is somewhat diminished. On the other hand, Volovik's solution to the cosmological constant problem is meant to carry over to other analogues of general relativity besides superfluid ^4He. In particular, it can be run for the case of the superfluid ^3He-A, which differs significantly from ^4He in that fields other than massless scalar fields arise in the low-energy limit. The fact that these fields model the dynamics of the Standard Model perhaps adds further plausibility to Volovik's solution. To investigate further, I now turn to ^3He.

(B) Helium 3 and the Standard Model

The second way superfluid Helium can provide information about the nature of spacetime is via an analogy between superfluid ^3He and the Standard Model. This analogy is based ultimately on the derivation of effective Lagrangian densities for force fields by treating the latter as vacuum corrections to interactions between matter fields and potential fields. Such derivations are at the basis of the "induced electrodynamics" of Zeldovich (1967) and the "induced gravity" of Sakharov (1967).

The ground state of superfluid ^3He is believed to be a Bose condensate consisting of pairs of ^3He atoms forming a type of Cooper pair. In the case of superconductors, such Cooper pairs consist of electrons in spin singlet ($S = 0$) states. Recall that conventional superconductors have s-wave ($l = 0$) orbital symmetry, whereas high-T_c superconductors have d-wave ($l = 2$) symmetry. ^3He Cooper pairs are in spin triplet ($S = 1$) states with p- wave ($l = 1$) symmetry. Thus the ^3He Cooper pair wave function consists of 3 spin ($S_z = 0, \pm 1$) and 3 orbital ($l_z = 0, \pm 1$) substates. The corresponding order parameter is characterized by a 3×3 matrix with symmetry $SO(3) \times SO(3) \times U(1)$. The breaking of this symmetry leads to a number of distinct superfluid states. Volovik (2003, 2001) demonstrates that the low-energy EFT for the A-phase, denoted ^3He-A, reproduces (3+1)- dim QED and aspects of the Standard Model and general relativity.

To see how this comes about, note first that the Hamiltonian for ^3He Cooper pairs is a modified version of the BCS Hamiltonian (7), with the gap parameter $\Delta_{\alpha\beta}(\vec{k})$ now having spin- and k-dependent components. In the mean field approximation, it takes the form (Annett 2004, pg. 154; Volovik 2003, pg. 77),

$$\begin{aligned} H_{^3He} &= \sum_{k,\sigma}(\varepsilon_k - \mu)c^\dagger_{k\sigma}c_{k\sigma} - \sum_{k,\alpha,\beta}\left(\Delta_{\alpha\beta}(\vec{k})c^\dagger_{k\alpha}c^\dagger_{-k\beta} + \Delta^*_{\alpha\beta}(\vec{k})c_{-k\gamma}c_{k\delta}\right) \\ &= \sum_{k,\alpha,\beta}\chi^\dagger_{\alpha\beta}(\vec{k})\left((\varepsilon_k - \mu)\sigma_3 + (\sigma_\mu k_i A_{\mu i})\sigma_+ + (\sigma_\mu k_i A_{\mu i})^*\sigma_-\right)\chi_{\alpha\beta}(\vec{k}) \end{aligned} \tag{24}$$

where the kinetic energy and c-operators are now those for ^3He atoms. The gap parameter is given by $\Delta_{\alpha\beta}(\vec{k}) = \sum_{k'\gamma\delta} V_{\alpha\beta\gamma\delta}(k,k')\langle c_{k\gamma}c_{-k\delta}\rangle$, where V describes an appropriate interaction potential. In the second line, the gap parameter has been encoded in a 3×3 matrix $A_{\mu i}$ *via* $\Delta_{\alpha\beta}(\vec{k}) = A_{\mu i}(\sigma_\mu i\sigma_2)_{\alpha\beta}$, $\mu, i = 1, 2, 3.$[20]

In the superfluid phase ^3He-A, the $S_z = 0$ substates are absent and the zero-spin and orbital axes are constant and parallel.[21] This is conventionally encoded in a gap parameter of the form $A_{\mu i} = |\Delta_{\alpha\beta}|d_\mu(m_i + in_i)$, where d_μ is the unit zero- spin axis and $\hat{m}\times\hat{n} \equiv \hat{l}$ is the unit orbital axis. Substituting this into (24) yields,

$$H_{^3HeA} = \sum_{k,\alpha,\beta}\chi^\dagger_{\alpha\beta}(\vec{k})\left((\varepsilon_k - \mu)\sigma_3 + c_\perp(\sigma_\mu d_\mu)(m_i k_i\sigma_1 + n_i k_i\sigma_2)\right)\chi_{\alpha\beta}(\vec{k}) \tag{25}$$

with energy given by $E^2(\vec{k}) = (k^2/m - \mu)^2 + c_\perp^2(\vec{k}\times\hat{l})^2$, where $c_\perp = |\Delta_{\alpha\beta}|/k_F$ (Volovik 2003, pg. 82). The energy vanishes at the two Fermi points $k_i = q_a k_F l_i$, $q_a = \pm 1$,[22] and one can subsequently linearize the energy about these

[20] This can be viewed as a linear expansion of $\Delta_{\alpha\beta}(\vec{k})$ in the 2×2 basis $\sigma_\mu i\sigma_2$, where the 0th component may be taken as $-iI$, for the identity I. This 0th component encodes the spin singlet case and can hence be discarded. The "matrix" $A_{\mu i}$ transforms as an $SO(3)$ vector under both its spin index μ and its orbital index i. For discussion, see Volovik (2003, pp. 76-77), Annett (2004, pp. 155-156).

[21] To make this a bit more concrete, for $S_z = 0$, three of the possible nine ^3He Cooper pair substates are absent. Each of the remaining substates describes two ^3He atoms with spin axes pointing in the same direction and perpendicular to their orbital angular momentum axis. The zero-spin axis points in the direction of zero spin projection; *i.e.*, perpendicular to the spin axes.

[22] More precisely, the energy vanishes at these points near the Fermi surface where $\mu = k_F^2/2m$. The subscript "a" labels one of four species of quasiparticles. These are distinguished by their effective charges q_a and by their spin projection onto d_μ given by $S_{za} = (1/2)\sigma_\mu d_\mu = \pm 1/2$. The effective charge determines the chirality of the quasiparticle, in so far as chirality is determined by the sign of the determinate of the dreibein matrices (see below) sign$|e_b^{i(a)}| = -q_a$. For the interpretation of S_{za}, see the discussion below.

points in a manner similar to the procedure in Section 2.2. To second order, one obtains

$$
\begin{aligned}
E^2(k_i) &\approx 2\left(c_{\|} l_i(k_i - q_a A_i)\right)^2 + 2\left(c_\perp m_i(k_i - q_a A_i)\right)^2 + 2\left(c_\perp n_i(k_i - q_a A_i)\right)^2 \\
&\equiv g^{ij}(k_i - q_a A_i)(k_j - q_a A_j),
\end{aligned} \tag{26}
$$

where $A_i = k_F l_i$, and $c_{\|} = k_F/m$. In the second line the notation has been simplified by the introduction of the quantity $g^{ij} = c_{\|}^2 l^i l^j + c_\perp^2(\delta_{ij} - l^i l^j) \equiv e_b^i e_b^j$ ($b = 1, 2, 3$) for the "dreibein" $e_1^i = 2c_\perp m_i$, $e_2^i = -2c_\perp n_i$, $e_3^i = q_a c_{\|} l_i$ (Volovik 2003, pg. 106). Volovik interprets g^{ij} as the spatial part of an effective metric $g^{\mu\nu}$ describing the ${}^3He$-A superflow, with $g^{00} = -1$, $g^{0i} = -v_i$, and inverse given by

$$
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + g_{ij}(dx^i - v^i\, dt)(dx^j - v^j\, dt), \qquad \mu, \nu = 0, 1, 2, 3. \tag{27}
$$

This is similar to the metric arising in ${}^4He$. It is anisotropic, depending on the direction l_i, with $c_\perp$ and $c_{\|}$ being the velocities of quasiparticles in motion transverse to, and parallel to l_i, respectively.

The Hamiltonian corresponding to (26) is given by (Volovik 2003, pg. 105),

$$
H'_{{}^3HeA} = \sum_{k,\alpha,\beta} \chi^\dagger_{\alpha\beta}(\vec{k})\left(e_b^{i(a)}(k_i - q_a A_i)\sigma_b\right)\chi_{\alpha\beta}(\vec{k}), \qquad b = 1, 2, 3. \tag{28}
$$

and its corresponding Lagrangian density can be written as,

$$
\mathcal{L}'_{{}^3HeA} = \bar{\Psi}\gamma^\mu(\partial_\mu - q_a A_\mu)\Psi, \qquad \mu = 0, 1, 2, 3 \tag{29}
$$

where $\gamma^\mu = e_b^\mu e_b^\nu(\sigma_\nu \otimes \sigma_3)$ (σ_0 being the 2×2 identity), the Ψ's are 4-spinors, and $A_0 = k_F l_i v_i$.[23] This describes massless Dirac fermions interacting with a vector potential A_μ in a curved spacetime with metric $g_{\mu\nu}$. Note that we cannot yet identify A_μ with the electromagnetic potential, since (29) has no Maxwell term.

It turns out that a Maxwell term arises naturally as a vacuum correction to the coupling between the quasiparticles and the potential field A_μ. This is demonstrated by expanding (29) in small fluctuations in A_μ about the ground state, and then integrating out the high-energy fluctuations. The result is a term of the general form $\mathcal{L}[l_i, v_i]$ with a constant γ that depends logarithmically

[23] To motivate the form of A_0, Volovik (2003, pg. 106) notes that (26) is given in a frame moving with the superfluid. In the "environment" frame in which the superfluid moves with velocity v_i, the Hamiltonian is doppler shifted by $H'_{3HeA} \to H'_{3HeA} + k_i v_i = H'_{3HeA} + (k_i - q_a A_i)v_i + q_a k_F l_i v_i$.

on the cut-off energy.[24] For $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with A_μ the function of l_i and v_i given above, this term takes the form of the Maxwell Lagrangian density in a curved spacetime $\mathcal{L}_{Max} = (4\gamma)^{-1}\sqrt{-g}g^{\mu\nu}F_{\mu\alpha}F_{\nu\beta}$, where $g^{\mu\nu}$ is the ^3He-A effective metric. Combining this with (29), we obtain (3+1)-dim QED.

Volovik (2003, pp. 114-115) now indicates how this can be extended to include $SU(2)$ gauge fields, and hence, in principle, the dynamics of the Standard Model. The trick is to exploit an additional degree of freedom associated with the quasiparticles described by (29). In addition to their charge q_a and chirality $C_a = -q_a$, such quasiparticles are also characterized by the value of $S_{za} = \pm 1/2$, which determines the spin orientation of the initial ^3He atoms (see footnote 22). This last property can be interpreted as a quasiparticle $SU(2)$ isospin symmetry and incorporated explicitly into the effective Lagrangian density by means of $\mathcal{L}'_{3HeA} = \bar{\Psi}\gamma^\mu(\partial_\mu - q_a A_\mu - q_a \sigma_i W^i_\mu)\Psi$, where the new effective field W^i_μ can be identified as an $SU(2)$ potential field (*i.e.*, the potential for the weak force). Expanding this Lagrangian density in small fluctuations in the W-field about the ground state then produces to second order a Yang-Mills term. The general moral is that discrete degeneracies in the Fermi point structure of the energy spectrum induce local symmetries in the low-energy sector of the background liquid (Volovik 2003, pg. 116). For the discrete two- fold ($\mathbb{Z}_2$) symmetry associated with S_{za}, we obtain a low-energy $SU(2)$ local symmetry; and in principle for larger discrete symmetries $\mathbb{Z}_N$, we should obtain larger local $SU(N)$ symmetry groups. In this way the complete local symmetry structure of the Standard Model could be obtained in the low-energy limit of an appropriate condensed matter system.

Remarks: Volovik (2003, pg. 112) observes the following differences between QED and the effective Lagrangian density for ^3He-A:

(a) The vector field l_i is an observable for ^3He, but the potential field A_μ (formed from l_i at low energies) is not an observable for QED.

(b) The effective metric and effective gauge field(s) are mixed, due to their common dependence on l_i and v_i.

Arguably, (a) can be addressed at the expense of locality: Interpretations of QED (and Yang-Mills theory in general) in which the potential field is awarded observable status are possible, but they require a form of non-locality (*e.g.* Belot 1998). The second observation (b) is a problem simply because of the fact that the effective metric does not obey the Einstein equations. Hence its presence in the effective Lagrangian density contaminates what would otherwise be an EFT for QED. Note, however, that in the isotropic case in which $c_\perp = c_\parallel$, the metric "decouples" from the l_i-field, and hence from the effective electromagnetic field.

[24] See, e.g., Volovik (2003, pg. 112). A detailed derivation is given in Dziarmaga (2002). This method of obtaining the Maxwell term as the second order vacuum correction to the coupling between fermions and a potential field was proposed by Zeldovich (1967).

In this restricted case, QED emerges uncontaminated; but then we lose contact with modeling general relativistic spacetimes.

The contamination of the effective Lagrangian density due to the effective metric can be seen explicitly by applying the same procedure in deriving the Maxwell term to the case of the effective metric. In particular, one expands the Lagrangian density in small fluctuations in the effective metric $g_{\mu\nu}$ about the ground state and then integrates out the high-energy terms. This follows the procedure of what is known as "induced gravity", after Sakharov's (1967) derivation of the Einstein-Hilbert Lagrangian density as a vacuum correction to the coupling between quantum matter fields and the spacetime metric. In Sakharov's original derivation, the metric was taken to be Lorentzian, and the result included terms proportional to the cosmological constant and the Einstein- Hilbert Lagrangian density (as well as higher-order terms). In the case of the ^3He-A effective metric, the result contains higher-order terms dependent on the superfluid velocity v_i, and these terms dominate the Einstein- Hilbert term.[25] These terms are not diffeomorphic invariant, which is understandable, stemming, as they do, ultimately from the non-relativistic Galilei-invariant superfluid Lagrangian density. Volovik (2003, pg. 130) indicates that these terms originate from integrating over quasiparticles far from the Fermi points. The mechanism that would enforce diffeomorphism invariance in the EFT would thus be one that constrains the integration over quasiparticles to regions close to the Fermi points, where the effective metric is Minkowskian. To investigate such a mechanism, Volovik (2003, pg. 132) considers the limit $m \to \infty$, $v_i \to 0$, interpreted as an "inert vacuum". In this limit, it turns out that vacuum fluctuations of the effective metric do induce the Einstein- Hilbert term without contamination. Since this limit involves no superfluidity, Volovik's (2003, pg. 113) conclusion is that our "physical vacuum" cannot be completely modeled by a superfluid.

This approach to general relativity and the Standard Model views both as theories of low-energy phenomena induced by the ground state of a condensed matter system, although perhaps not a superfluid. In the context of general relativity, the literal interpretation would identify spacetime as this ground state. Matter fields and gauge potential fields would be identified as low-energy quasiparticle and collective bosonic excitations of spacetime, respectively, with gauge fields identified as "induced" vacuum corrections to the interactions between matter fields and potential fields. The viability of this conception of spacetime rests on the viability of Volovik's inert vacuum system. Given the nature of the $m \to \infty$ limit, it may appear doubful that there are physical examples of condensed

[25] See, *e.g.*, Volovik (2003, pg. 113). Sakharov's original procedure results in a version of semiclassical quantum gravity, in so far as it describes quantum fields interacting with a classical, unquantized spacetime metric. In the condensed matter context, the background metric is not a classical background spacetime, but rather arises as low-energy degrees of freedom of a quantized non-relativistic system (the superfluid). Hence one could argue this condensed matter version of induced gravity is not semiclassical.

matter systems for which the Einstein action can be induced in the low-energy limit. Even apart from this problem, there is the question of whether all the symmetries of the Standard Model can be expressed in such systems. From a more constructive point of view however, Volovik's discussion indicates that any system purporting to reproduce general relativity and the Standard Model in the low-energy limit minimally must have Fermi points in its energy spectrum, and in order to avoid superfluidity, such Fermi points should not be the consequence of symmetry breaking.

3.2 Spacetime and 4-dim Quantum Hall Liquids

Recently another condensed matter analogue of spacetime has been suggested by Sparling (2002). This analogue is based on Zhang and Hu's (2001) extension of the 2-dim QHE to 4 dimensions. Sparling's suggestion is that spacetime emerges from the edge states of Zhang and Hu's 4-dim QH liquid. To initially motivate this, recall from Section 2.3 that the edge states of a 2-dim QH liquid can be described by a (1+1)-dim EFT of massless relativistic fermions. This suggests that (3+1)- dim massless relativistic fermions may be obtainable from the edge states of a 4- dim QH liquid, and this is borne out (Zhang and Hu 2001, Hu and Zhang 2002). The connection to spacetime comes in the guise of the twistor formalism: It turns out that the 4-dim QHE can be formulated in terms of twistors, and one of the goals of the twistor programme is to construct spacetime from twistors. Thus the claim that spacetime emerges from 4-dim QHE edge states rests, in part, on the extent to which spacetime can be reconstructed from twistors. This will be fleshed out below, following a brief description of the 4-dim QHE.

QHE on S^2 and S^4

The key to extending the 2-dim QHE to four dimensions is Haldane's (1983) formulation of the 2-dim case in terms of spherical geometry. Haldane considered an electron gas on the surface of a 2-sphere S^2 with a $U(1)$ Dirac magnetic monopole at its center. The radial monopole field serves as the external magnetic field of the original setup. By taking an appropriate thermodynamic limit, the 2- dim QHE on the 2-plane is recovered. To motivate Zhang and Hu's (2001) extension to 4-dim, note that a Dirac monopole can be formulated as a $U(1)$ connection on a principle fiber bundle $S^3 \to S^2$, consisting of base space S^2 and bundle space S^3 with typical fiber $S^1 \cong U(1)$. This is known as the 1st Hopf bundle.[26] There is also a 2nd Hopf bundle $S^7 \to S^4$, consisting of the 4-sphere S^4 as base space, and the 7-sphere S^7 as bundle space with typical fiber $S^3 \cong SU(2)$. The $SU(2)$ connection on this bundle is referred to as a Yang monopole. Zhang and Hu's 4-dim QHE then consists of taking the appropriate

[26] See, *e.g.*, Nabor 1997, Chapter 0 for the relation between the Dirac monopole and the 1st Hopf bundle. The latter is essentially a way of mapping the 3-sphere onto the 2-sphere by viewing S^3 as a collection of "fibers", all isomorphic to a "typical fiber", and parameterized by the points of S^2.

thermodynamic limit of an electron gas on the surface of a 4-sphere with an $SU(2)$ Yang monopole at its center.

In just a bit more detail, the Hamiltonian for an electron with mass m moving on a 2-sphere S^2 with radius R in the presence of a Dirac monopole is given by,

$$H_{S^2QHE} = \frac{1}{2mR^2}\sum_{i<j}\Lambda_{ij}^2, \qquad i,j = 1,2,3. \tag{30}$$

This is an expression for the kinetic energy, where $\Lambda_{ij} = -i(x_i D_j - x_j D_i)$ is the orbital angular momentum, with $D_i = \partial_i + a_i$ being the covariant derivative associated with the monopole potential field a_i. This latter satisfies $\vec{\partial}\times\vec{a} = \vec{B}$, for the field strength $|\vec{B}| = I/eR^2$, with $2I$ being an integer (the Dirac quantization condition). The eigenstates of (30) belong to finite matrix representations of $SO(3)$, the symmetry group of S^2. Haldane (1983) obtained N-particle states as antisymmetric products of the lowest energy eigenstates, and showed that in the "thermodynamic" limit $N \to \infty$, $I \to \infty$, $R \to \infty$, holding I/R^2 constant, these states reproduce those in the original planar case.[27]

Zhang and Hu (2001) extended this treatment to 4-dim by taking the 2nd Hopf bundle $S^7 \to S^4$ as the starting point. The Hamiltonian for an electron moving on a 4-sphere S^4 with radius R with an $SU(2)$ monopole at its center is an extension of (30) given by

$$H_{S^4QHE} = \frac{1}{2mR^2}\sum_{a<b}\Lambda_{ab}^2, \qquad a,b = 1,2,3,4,5. \tag{31}$$

where $\Lambda_{ab} = -i(x_a D_b - x_b D_a)$ is the angular momentum with $D_a = \partial_a + A_a$ being the covariant derivative associated with the $SU(2)$ monopole potential field A_a. This latter takes values in the $SU(2)$ Lie algebra generated by $[I_i, I_j] = i\epsilon_{ijk}I_k$, with Casimir operator $I_i^2 = I(I+1)$, where I labels the dimension of the $SU(2)$ representation (and, it turns out, corresponds to the magnetic flux of the $SU(2)$ monopole; hence is the analogue of the "I" in the 2-dim case). In a manner similar to the 2-dim case, one obtains the eigenstates of (31) as finite matrix representations of $SO(5)$, the symmetry group of S^4, and a 4-dim QH liquid is constructed by taking the equivalent of Haldane's thermodynamic limit for S^4.

Finally, a low-energy Chern-Simons-Ginsberg-Landau effective topological field theory for such 4-dim QH liquids was constructed by Bernevig *et al.* (2002). This theory is based on statistical transmutations for extended objects ("branes"), as opposed to points, and thus side-steps the dimensional restrictions of the 2-dim CSGL theory.

[27] It turns out that the lowest energy level is $(N = 2I+1)$-fold degenerate. An N-particle system is obtained by filling all of these states. The $N \to \infty$ limit thus requires taking $I \to \infty$. To recover an incompressible QH liquid on the 2-plane, the density $N/4\pi R^2 \sim I/R^2$ is required to remain constant as $R \to \infty$.

Remarks: Some authors have imbued the interplay between algebra and geometry in the 4-dim QHE extension with ontological significance. These authors note that there are only four normed division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$.[28] It is then observed that these may be associated with the four Hopf bundles, $S^1 \to S^1$, $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$, in so far as the base spaces of these fiber bundles are the compactifications of the respective division algebra spaces $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{R}^8$. Finally, one notes that the typical fibers of these Hopf bundles are $\mathbb{Z}_2$, $U(1) \cong S^1$, $SU(2) \cong S^3$, and $SO(8) \cong S^7$, respectively. These patterns are then linked with the existence of quantum Hall liquids:

> One , two, and four dimensional spaces have the unique mathematical property that boundaries of these spaces are isomorphic to mathematical groups, namely the groups $\mathbb{Z}_2$, $U(1)$ and $SU(2)$. No other spaces have this property. (Zhang and Hu 2001, pg. 827.)
>
> The four sets of numbers [*viz.*, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$] are mathematically known as division algebras. The octonions are the last division algebra, no further generalization being consistent with the laws of mathematics... Strikingly, in physics, some of the division algebras are realized as fundamental structures of the quantum Hall effect. (Bernevig *et al.* 2003, pg. 236803-1.)
>
> Our work shows that QH liquids work only in certain magic dimensions exactly related to the division algebras... (Zhang 2004, pg. 688.)

Before we see nature unfolding its secrets in the forms of division algebras and Hopf bundles, we should pause and take stock. Note first that Zhang and Hu's statement should be restricted to the spaces S^1, S^2, S^4, and should include S^8 as well, the boundary of S^8 being S^7. Furthermore, the statements of Bernevig *et al.* and Zhang should refer to *normed* division algebras. Baez (2001, pg. 149) carefully distinguishes between $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ as the only normed division algebras, and division algebras in general, of which there are other examples. Baez (2001, pp. 153-156) indicates how the sequence $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ can in principle be extended indefinitely by means of the Cayley-Dickson construction. Starting from an n-dim $*$-algebra A (*i.e.*, an algebra A equipped with a conjugation map $*$), the construction gives a new $2n$-dim $*$-algebra A'.[29] The next member of the sequence after $\mathbb{O}$ is a 16-dim $*$-algebra referred to as the "sedenions". The point here is that the sedenions and all subsequent higher-dimensional constructions do not form division algebras; in particular, they have zero divisors. The

[28] A normed division algebra A is a normed vector space, equipped with multiplication and unit element, such that, for all $a, b \in A$, if $ab = 0$, then $a = 0$ or $b = 0$. $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are associative, whereas $\mathbb{O}$ is non-associative (see, *e.g.*, Baez 2001, pg. 149).

[29] The elements of A' are defined as pairs of elements of A, with multiplication in A' given by $(a, b)(c, d) = (ac - db*, a * d + cb)$, and conjugation in A' given by $(a, b)* = (a*, -b)$, for $a, b \in A$.

question therefore should be whether the absence of zero divisors in a normed $*$-algebra has physical significance when it comes to constructing QH liquids.

Zhang (2004, pg. 687) implicitly suggests it does. He identifies various quantum liquids with each Hopf bundle: 1-dim Luttinger liquids with $S^1 \to S^1$, 2-dim QH liquids with $S^3 \to S^2$, and 4-dim QH liquids with $S^7 \to S^4$. Bernevig *et al.* (2003) complete the pattern by constructing an 8-dim QH liquid as a fermionic gas on S^{15} with an $SO(8)$ monopole at its center. But whether this pattern is physically significant remains to be seen. It is not entirely clear, for example, how the bundle $S^1 \to S^1$ is essential in the construction of Luttinger liquids in general. In particular, while Luttinger liquids are necessarily 1-dim, it's not clear what role, if any, the trivial $\mathbb{Z}_2$ monopole associated with $S^1 \to S^1$ plays in their construction. Moreover, while Luttinger liquids arise at the edge of 2-dim QH liquids, this pattern does not carry over to higher dimensions: it is not the case that 2-dim QH liquids arise at the edge of 4-dim QH liquids, nor is it the case that 4-dim QH liquids arise at the edge of 8-dim QH liquids. Furthermore, and more importantly, Meng (2003) demonstrates that higher- dimensional QH liquids can in principle be constructed for any even dimension, and concludes that the existence of division algebras is not a crucial aspect of such constructions (see, also Karabali and Nair 2002). Hence, while the relation between Hopf bundles and normed division algebras on the one hand, and quantum liquids on the other, is suggestive, it perhaps should not be interpreted too literally.

So far in this discussion no mention of spacetime has been made. To see where spacetime comes in, we need to move to the edges.

Edge States for 4-dim QH Liquids and Twistors

Recall from Section 2.3 that low-energy edge states of a 2-dim QH liquid take the form of (1+1)-dim relativistic massless fermions. It turns out that edge excitations can be viewed as particle-hole dipoles formed by the removal of a fermion from the bulk to outside the QH droplet, leaving behind a hole (*e.g.*, Stone 1990). If the particle-hole separation remains small, such dipoles can be considered single localized bosonic particles. In 1-dim, in which relativistic massless particles move at speed c independently of their momentum, these dipoles are stable. In higher dimensions, the direction of velocity will in general depend on the momentum, hence the uncertainty principle should prevent stable dipoles from forming. Hu and Zhang (2002) determined that there is a subset of dipole states at the edge of a 4-dim QH liquid for which the isospin degrees of freedom associated with the $SU(2)$ monopole counteract the uncertainty principle. The main result of Hu and Zhang (2002) was to establish that these stable edge states satisfy the (3+1)-dim zero rest mass field equations, and hence can be interpreted as zero rest mass relativistic fields. These include spin-1 Maxwell fields, as well as spin-2 fields satisfying the sourceless linearized Einstein equations. On the other hand, the $I \to \infty$ limit required to recover a QH liquid leads to an "embarrassment of riches" (Zhang and Hu 2001, pg. 827) in which states with very large isospin I degrees of freedom occur. One consequence of this is that the stable dipole edge states include states corresponding to zero rest mass fields of all higher spins.

Sparling's (2002) insight was to see that these stable dipole states correspond to twistor representations of zero rest mass fields. In particular, he demonstrates that the edge of a 4-dim QH liquid corresponds to a particular region of twistor space $\mathbb{T}$. $\mathbb{T}$ is the carrying space for matrix representations of $SU(2,2)$ which is the double covering group of $SO(2,4)$. Elements Z^{α} of $\mathbb{T}$ are called twistors and are thus spinor representations of $SO(2,4)$. $\mathbb{T}$ contains a Hermitian 2-form $\Sigma_{\alpha\beta}$ (a "metric") of signature $(++--)$. This 2-form splits $\mathbb{T}$ into three regions $\mathbb{T}^{+}$, $\mathbb{T}^{-}$, $\mathbb{N}$, defined by $\Sigma_{\alpha\beta}Z^{\alpha}Z^{\beta} > 0$, $\Sigma_{\alpha\beta}Z^{\alpha}Z^{\beta} < 0$, $\Sigma_{\alpha\beta}Z^{\alpha}Z^{\beta} = 0$, respectively. The connection to spacetime is based on the fact that $SO(2,4)$ is the double covering group of $C(1,3)$, the conformal group of Minkowski spacetime. This allows a correspondence to be constructed under which elements of $\mathbb{N}$, "null" twistors, correspond to null geodesics in Minkowski spacetime, and 1-dim subspaces of $\mathbb{N}$ (*i.e.*, twistor "lines") correspond to Minkowski spacetime points.[30]

To make the identification of the edge of a 4-dim QH liquid with $\mathbb{N}$ plausible, note that the symmetry group of the edge is $SO(4) \cong S^3$ and that of the bulk is $SO(5) \cong S^4$. The twistor group $SO(2,4)$ has $SO(4)$ in common with $SO(5)$. Intuitively, the restriction of $SO(2,4)$ to $SO(4)$ can be induced by a restriction of twistor space $\mathbb{T}$ to $\mathbb{N}$.[31] With the edge identified as $\mathbb{N}$, edge excitations are identified as deformations of $\mathbb{N}$. In twistor theory, such deformations take the form of elements of the first cohomology group of projective null twistor space $\mathbb{PN}$, and these are in fact solutions to the zero rest mass field equations of all helicities in Minkowski spacetime (Sparling 2002, pg. 25).

Remarks: Sparling's twistorial formulation of the 4-dim QHE suggests that spacetime arises from the edge of a 4-dim QH liquid. In particular, the edge corresponds to null twistor space from which (compactified) Minkowski spacetime can be reconstructed. This story comes with two caveats.

First, there is a question of whether spacetime can be said to emerge from the edge of the Zhang/Hu liquid in the same sense that relativistic fields emerge from condensed matter systems. In the latter case, such fields emerge in a low-energy limit (*viz.*, low-energy approximation). On first blush, in the twistor formalism spacetime is reconstructed from the edge, which is identified with null twistor

[30] More precisely, the correspondence is between $\mathbb{PN}$, the space of null twistors up to a complex constant (*i.e.*, "projective" null twistors), and *compactified* Minkowski spacetime (*i.e.*, Minkowski spacetime with a null cone at infinity). This is a particular restriction of a general correspondence between projective twistor space $\mathbb{PT}$ and complex compactified Minkowski spacetime.

[31] Technically, this restriction corresponds to a foliation of the 4-sphere with the level surfaces of the $SO(4)$-invariant function $f(Z^{\alpha}) = \Sigma_{\alpha\beta}Z^{\alpha}Z^{\beta}$. These surfaces are planes spanned by null twistors (Sparling 2002, pp. 18-19, 22).

space, and no low-energy limit is assumed in this identification. However, there is a sense in which spacetime can be said to emerge, but in a different type of limit; namely, the thermodynamic limit.

It turns out that the edge ground state of the 4-dim QH liquid is $(2I+1)$-fold degenerate, implying that the bulk liquid has $(2I+1)$ boundary surfaces.[32] Sparling's preferred interpretation has each edge shrinking in the thermodynamic limit about a particular projective line in twistor space. This speculation is based on the fact that "... hyperquadrics lie arbitrarily close to any given projective line in projective three-space" (Sparling 2002, pg. 28). Under the basic twistor correspondence, twistor lines correspond to spacetime points; hence Sparling's suggestion is that the thermodynamic limit of each edge of the bulk liquid corresponds to a single spacetime point. The bulk liquid is then interpreted as "gluing" the points of spacetime together. One should note however that this is currently at best a speculation; in particular, the nature of the thermodynamic limit in the twistor formulation is still unknown (Sparling 2002, pg. 27).

The second caveat concerns the approach to spacetime in the twistor formalism in general. Even granted that the 4-dim QHE admits a thoroughly twistorial formulation, down to the thermodynamic limit, there is still the question of whether spacetime as currently described by general relativity can be recovered. It turns out that no consistent twistor descriptions have been given for massive fields, or for field theories in generally curved spacetimes with matter content. In general, only conformally invariant field theory, and those general relativistic spacetimes that are conformally flat, can be completely recovered in the twistor formalism. This is not to say that the twistor connection with the QHE is not a significant achievement. Advocates view twistors as a route to quantum gravity. As such, the twistor formulation of the 4-dim QHE points to similarities between two roads to quantum gravity, *via* twistors and *via* condensed matter systems, that were previously seemingly unrelated.[33]

4 Conclusion

To recap, the previous sections have considered a number of examples of EFTs in (2+1)- and (3+1)-dim arising in nonrelativistic condensed matter systems. If these examples are read literally, they suggest that relativistic matter fields and relativistic gauge potential fields are emergent phenomena, arising as low- energy fluctuations above the ground state of a nonrelativistic quantum liquid. More radically such examples suggest that spacetime itself is an emergent phenomenon

[32] There is one boundary for each isospin direction (Hu and Zhang 2002, pg. 66).

[33] Note further that the twistor emphasis on conformal invariance is not as restrictive as might first be thought, in so far as the verdict is still out on whether quantum field theory can be reformulated in a conformally invariant way. In general, the route to quantum gravity that stresses conformal structure over metrical structure should not be ignored by philosophers of spacetime.

that arises from a nonrelativistic quantum liquid in either a low- energy limit or an appropriate thermodynamic limit.

Viability of the Condensed Matter View of Spacetime
The extent to which this condensed matter view of spacetime is viable will depend, to begin with, on how one initially chooses to model spacetime in the context of contemporary theories. If, for instance, one is satisfied with modeling spacetime as the vacuum ground state for quantum field theory, then the examples of EFTs in ^3He-A may be taken to suggest that the ground state of the latter effectively models spacetime. If, however, one requires spacetime to be modeled by solutions to the Einstein equations of general relativity, then the examples of EFTs in superfluids ^4He and ^3He-A will be found wanting. In particular, we found that superfluid 4He only models aspects of curved spacetimes that are independent of the Einstein equations, and that the latter cannot fully be recovered either in ^4He or as an induced phenomenon in ^3He-A. Finally, if one is satisfied with modeling spacetime by conformally flat general relativistic spacetimes, then spacetime may be taken to arise, via twistors, in an appropriate thermodynamic limit of the edge of a 4-dim QH liquid. However, there remains the worry that the exact nature of this thermodynamic limit is still unknown.

Note that, while most philosophers of spacetime may seek to model spacetime by solutions to the Einstein equations, the other options suggested by the condensed matter view should not be entirely discounted. They may be viewed with an eye toward the eventual reconciliation of general relativity and quantum theory. Ultimately, it may be that this is where the real significance of the condensed matter view lies. In this context, the common element in all condensed matter analogues of spacetime is the claim that both spacetime and gravity are emergent phenomena of a quantum condensed matter system. This entails that there is no special relation between spacetime and gravity, and moreover, that gravity need not be quantized.

Significance to the Substantivalist/Relationalist Debate
Issues of viability to the side, one may ask how the condensed matter view of spacetime fits into the traditional debate between substantivalists and relationalists over the ontological status of spacetime. On first blush, the condensed matter view might appear to find a home in the substantivalist camp, in so far as identifying spacetime with a quantum liquid entails a claim that spacetime exists as a physical substance. On the other hand, typical substantivalists qualify this existence claim with an independence clause; namely, that spacetime exists as a physical substance independently of material constituents it may contain. On the surface, it may appear that the condensed matter view denies such independence, in so far as it views both spacetime and its material constituents as derivative, in an emergentist sense, of the underlying quantum liquid. There is, however, room to maneuver here, if we recall that the notion of emergence in the relevant examples involves adopting interpretations under

which the emergent and host structures are ontologically distinct. It may thus be possible for a condensed matter substantivalist to formulate a definition of emergence based on a notion of ontological distinctness strong enough to entail ontological independence. I leave it to such a substantivalist to flesh out the details. Minimally, however, the emergent aspect of the condensed matter view blurs some of the standard distinctions between traditional substantivalism and relationalism.

Universality and Structural Realism
Beyond the vagaries of the substantivalist/relationalist debate, the condensed matter view also suggests a structural realist approach to the nature of spacetime.

Note first that the fact that relativistic EFTs arise in the low-energy sector of many condensed matter systems is no coincidence. Volovik (2003) indicates that the type of EFT that emerges in the low-energy limit of a given theory is determined by the topology of the theory's momentum space; in particular, by the points, lines or surfaces at which the quasiparticle energy becomes zero (pg. 5). Much of Volovik (2003) is devoted to the topological classification of quantum vacua into "universality classes".[34] Briefly, one might say that momentum space topology encodes low-energy dynamics; hence common momentum space topology entails common low-energy dynamics. In the case of ^{3}*He-A* and the Standard Model, for instance, the common momentum space topology consists of the existence of topologically stable fermion zero modes in the vicinity of Fermi points (Volovik 2003, pg. 462).[35] This is an example of universality: The microscopic details of a particular theory do not affect its momentum space topology and hence its low-energy behavior. Such details only serve to fix the "fundamental constants" (*i.e.*, the speed of light, the speed of sound, superfluid density, particle mass, *etc.*), and these can always be rescaled (Volovik 2005, pp. 6-7).

There are many similarities between this account of universality and the standard account offered by Renormalization Group (RG) Theory. In typical RG analyses (*e.g.*, Saunders 2003, Huggett and Weingard 1995), universality classes are defined by the fixed points of the RG flow in the parameter space (*i.e.*, the space of Hamiltonians, or equivalently, coupling constants). A given fixed point corresponds to a low-energy theory (an EFT) and defines a universality class consisting of all theories with RG flows that terminate at the fixed point. Such theories share the same low-energy behavior. They also behave similarly at critical temperatures associated with phase transitions, if the latter exist. One can

[34] More precisely, the classification is based on the topological properties of the 2-particle correlation function (Feynman propagator) (Volovok 2003, Chapter 8).

[35] This assumes that the Standard Model can be viewed as a low-energy EFT, which is arguably justifiable in so far as its "quasiparticle" energies are extremely small compared to its cut-off energy (*viz.*, the Planck energy). For (3+1)-dim fermionic vacua, Volovik (2003, pp. 86-87) indicates that there are three universality classes: Vacua with Fermi surfaces, vacua with Fermi points, and vacua with a fully gapped energy spectrum.

thus associate a phase with a universality class, and a phase transition with a boundary in the parameter space that separates fixed point regions (*i.e.*, regions in which all flow lines terminate at a given fixed point). This analysis of critical phenomena improves on Ginzberg-Landau theory (*e.g.*, Saunders 2003), and, importantly, is more general: under the RG analysis, phase transitions are not necessarily characterized by spontaneously broken symmetries. In particular, there are examples of fixed point theories that share the same symmetries and yet are separated by a phase transition (see, *e.g.*, Wen 2005, pg. 118). Hence RG universality classes are not characterized essentially by symmetries.

This is a point also stressed by Volovik: Universality classes characterized by momentum space topology are not essentially distinguished by symmetries. Volovik makes this explicit by contrasting the topological classification of EFTs with the approach to QFTs motivated by Grand Unified Theories (GUTs). Under the GUT approach, the low energy (relative to the Planck energy cut-off) gauge symmetries of current QFTs are remnants of a larger unified GUT symmetry at high energies; and the transition from the GUT to current QFTs is characterized by spontaneous symmetry breaking. In a nutshell, the GUT slogan claims "...the higher the energy, the higher the symmetry".[36] The condensed matter view inverts this slogan and discards its emphasis on symmetry breaking. Under the condensed matter view, high energies (relative to an appropriate cut-off) are now associated with the less-perfect symmetries of the underlying condensed matter system, and low energies are associated with the more- perfect gauge symmetries of current QFTs, now conceived as EFTs. Importantly, such EFTs are now classified in terms of topology and not symmetry. Note that spontaneous symmetry breaking does occur in condensed matter systems: a quantum liquid (for instance) results from a spontaneously broken symmetry of a normal liquid at some critical temperature. However, the essential feature of the condensed matter view involves what happens *next*: it is in the subsequent low-energy sector of the quantum liquid that the gauge (and spacetime) symmetries of current QFTs emerge. The condensed matter view thus stresses topology as opposed to symmetry as the means of defining universality classes.[37] Indeed the condensed matter view involves a two-part rejection of the ontological prominence of symmetries in quantum field theory: First, it views the symmetries (both gauge and spacetime) of QFTs not as fundamental, but as emergent phenomena; and moreover the process of emergence itself is viewed as a process not essentially governed by spontaneous symmetry breaking.

The connection with structural realism may be made through the notion of a universality class. Again, such a class consists of systems that share a common

[36] Volovik (2003, pg. 1). In this context, a "high" GUT symmetry is a gauge symmetry represented by a single compact Lie group with a minimum of parameters. In contrast, the low-energy gauge symmetries of the Standard Model are represented by a product group structure $SU(3) \times SU(2) \times U(1)$ with a relatively large number of parameters.

[37] This emphasis on "topological order", as opposed to a classification of order based on symmetry is also stressed by Wen (2004, Chapter 8) in a slightly different context.

low-energy dynamics, irrespective of their "microscopic" details. Such universality classes might be identified as encoding a common dynamical structure that may be manifested in different systems via different "individuals-based" ontologies. In fact, Saunders (2003) has recently proposed such a view based on RG universality classes. Volovik's topological classification of universality classes applied to condensed matter analogues of spacetime extends this view to spacetime structure. The resulting structural realist interpretation of spacetime would do away with the underlying quantum liquid of such analogues (i.e., its microscopic details) in favor of the universality class it belongs to. Of course, given the qualifications above (in particular, the problem of modeling full-blown general relativity in a condensed matter system), just what the universality class best associated with spacetime structure is, is still unknown at present. Nevertheless, the condensed matter view of spacetime in this context deserves further consideration, if only because it offers, in the form of structural realism, a tertium quid between traditional substantivalism and relationalism.

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