Studies in History and Philosophy of Modern Physics 66 (2019) 24–33

Contents lists available at ScienceDirect

Studies in History and Philosophy of Modern Physics

journal homepage: www.elsevier.com/locate/shpsb

Non-locality in intrinsic topologically ordered systems

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ARTICLE INFO

Article history:
Received 8 November 2017
Received in revised form
19 October 2018
Accepted 29 October 2018
Available online 7 December 2018

ABSTRACT

Intrinsic topologically ordered (ITO) condensed matter systems are claimed to exhibit two types of non-locality. The first is associated with topological properties and the second is associated with a particular type of quantum entanglement. These characteristics are supposed to allow ITO systems to encode information in the form of quantum entangled states in a topologically non-local way that protects it against local errors. This essay first clarifies the sense in which these two notions of non-locality are distinct, and then considers the extent to which they are exhibited by ITO systems. I will argue that while the claim that ITO systems exhibit topological non-locality is unproblematic, the claim that they also exhibit quantum entanglement non-locality is less clear, and this is due in part to ambiguities associated with the notion of quantum entanglement. Moreover, any argument that claims some form of "long-range" entanglement is necessary to explain topological properties is incomplete if it fails to provide a convincing reason why mechanistic explanations should be favored over structural explanations of topological phenomena.

1. Introduction

This essay is concerned with two notions of non-locality that are claimed to be present in intrinsic topologically ordered (ITO) condensed matter systems.[1] I'll call one type “topological non-locality", and the other type “quantum entanglement non-locality". The interplay between the two is supposed to be profound: An ITO system is supposed to allow information to be encoded in quantum entangled states in a topologically non-local way that protects it against local errors. This could provide a basis for fault-tolerant quantum memory and, ultimately, for topological quantum computation (Terhal, 2015). An ITO system is supposed to make this possible because its ground states are *locally indistinguishable*, in the sense that no operator that represents a localized observable can distinguish them, but *non-locally distinct*, in the sense that there are operators that represent topological non-localized observables that can distinguish them. Some authors use the phrase “topological quantum order” (TQO) to refer to this characteristic (Bravyi, Hastings, & Verstraete, 2006). Quantum entanglement non-locality enters the picture insofar as (i) the TQO property entails that ITO ground states must be entangled (Preskill, 1999), but (ii) any correlation they exhibit between local observables with support on disjoint sets decays exponentially as a function of separation distance (due to another property they possess; namely, the presence of an energy gap). This complication has motivated various authors to suggest that, since ITO ground states must be entangled, but can only exhibit “short-range" correlations, they must exhibit a form of “long-range" entanglement, and ultimately this is what explains their topological properties.

In the following sections, I will assess this claim. My conclusion is that, while topological non-locality can be made distinct from quantum entanglement non-locality, and ITO systems can unproblematically be said to exhibit topological non-locality, the extent to which they exhibit quantum entanglement non-locality is less clear, and this is due in part to ambiguities associated with the notion of quantum entanglement. Moreover, any argument that claims some form of “long-range" entanglement is necessary to explain topological properties is incomplete if it fails to provide a convincing reason why mechanistic explanations should be favored over structural explanations of topological phenomena.

Section 2 attempts to make the distinction between topological non-locality and quantum entanglement non-locality more precise. Section 3 considers the property of topological quantum order (TQO) as a potential link between these two notions of non-locality

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[1] Intrinsic topological order (see, *e.g.*, Wen, 2013) should be made distinct from what is referred to as symmetry-protected topological order. Examples of the former include fractional quantum Hall systems and Kitaev's (2003) toric code. Examples of the latter include integer quantum Hall systems, topological insulators, and topological superconductors. The current essay is only concerned with the former.

https://doi.org/10.1016/j.shpsb.2018.10.009
1355-2198/

and finds it wanting. On the one hand, a system that possesses TQO need not exhibit topological non-locality; on the other hand, while TQO entails quantum entanglement non-locality with respect to certain *local* observables, there is no inherent reason why there should be a relation between the latter and those *non-local* observables that potentially possess topological non-locality. Section 4 then considers an intrinsic topologically ordered (ITO) system as another potential link between topological non-locality and quantum entanglement non-locality, and, again, finds it wanting. An ITO system is characterized, in part, by observables that exhibit topological non-locality, and these entail that an ITO system also exhibits TQO, and hence quantum entanglement non-locality. But the observables that exhibit the latter are not the same as those that exhibit the former. Moreover, an ITO system is also characterized by at most "short-range" correlations between local observables, and this has motivated some authors to insist that an ITO system must exhibit some form of "hidden", "long-range" entanglement. Section 5 assesses a claim to this effect found in Chen, Gu, and Wen (2010). Here I argue that this claim falters, due in part to ambiguities associated with the notion of quantum entanglement, and to an implicit assumption that explanations of topological phenomena must be mechanistic.

2. Two types of non-locality

To set the stage for the discussion of non-locality, I'd first like to consider the notion of *localization*.[2] This is the requirement that observables must be localized in finite regions of space.[3] One way this can be satisfied is if observables are represented by *local operators*. Let $\mathscr{H}^{(n)} = \otimes^{\nu}{}_{i=1}V_i$ be the Hilbert space for a finite many-body condensed matter system consisting of n subsystems arranged on an $L \times L$ lattice, where V_i represents the state space of the ith subsystem. An *ℓ-local operator* O_{loc} on $\mathscr{H}^{(n)}$ acts non-trivially only on some set X of lattice sites of diameter $\ell < L$:

$$O_{loc} = \{\otimes_{i \notin X} I_i\} \otimes \{\otimes_{j \in X} O_j\} \quad (1)$$

where I_i and O_j are the identity and a single-subsystem operator, respectively, on site i.[4] One says that O_{loc} has support on X.

One way localization can fail results in what I will call *topological non-locality*:

Topological non-locality occurs when the observables of interest are, or encoded in, topological properties.

A topological property is invariant under continuous deformations of the system. An example is the number of distinct types of non-contractible loops on a surface.[5] The 2-dimensional surface of a torus, for instance, contains two types of non-contractible loop, one encircling the hole, and the second type encircling the handle. Consider a system composed of subsystems arranged on a lattice on a torus, and define an operator O_{loop} on the corresponding Hilbert space that acts as the identity on all lattice sites except for those that form a non-contractible loop γ, on which it acts as a single-subsystem operator O_j:

$$O_{loop} = \{\otimes_{i \notin \gamma} I_i\} \otimes \{\otimes_{j \in \gamma} O_j\} \quad (2)$$

O_{loop} might be called a non-contractible loop operator. Any observable it represents exhibits topological non-locality. In particular, it is not localized in a finite region of the lattice, but rather has support on a non-contractible loop that encircles the entire lattice space.

A second notion of non-locality is what I'll call *quantum entanglement non-locality*. This notion is supposed to be unique to quantum systems. Again, the goal of topological quantum memory is to use this unique aspect of quantum systems, together with topologically non-local properties, to encode information in a novel way. With this in mind, I will adopt the following definition:

Quantum entanglement non-locality occurs when the observables of interest exhibit distant correlations that violate a Bell inequality.

Suppose the observables of interest are represented by operators A and B. To say they are *correlated* is to say there is a state $|\psi\rangle$ with respect to which expectation values fail to satisfy a cluster decomposition condition:

$$\langle\psi|AB|\psi\rangle \neq \langle\psi|A|\psi\rangle\langle\psi|B|\psi\rangle \quad (3)$$

This encodes statistical dependence.[6] Note that quantum entanglement non-locality cannot be encoded in statistical dependence alone, which certainly is not unique to quantum systems.

The observables represented by A and B exhibit a *Bell inequality-violating correlation* with respect to a state just when they are conditionally statistically dependent with respect to a random variable λ.[7] The conditional statistical independence of (the observables represented by) A and B with respect to λ entails that the correlation between A and B satisfies a Bell inequality. Hence a violation of the latter entails a violation of the former. Naively, conditional statistical independence with respect to λ entails that λ screens A off from B, and *vice-versa*. More provocatively, when conditional statistical dependence holds, λ acts like a common cause that can explain the correlation between A and B. Hence (to be provocative), a Bell inequality-violating correlation cannot be explained by a common cause explanation.[8]

Now suppose A and B are local operators with support on sets X and Y. To say (the observables represented by) A and B are *distant*

[2] Earman and Valente (2014, p. 3) identify a number of distinct notions of locality, one of which is localization, and another of which is the obverse of what below is referred to as quantum entanglement non-locality. I will follow their lead in keeping these notions of locality distinct from one associated with independence or separability conditions for a composite system with respect to its subsystems, and another associated with a prohibition on superluminal signals and/or superluminal propagation.

[3] Insofar as a finite region of space need not be an arbitrarily small neighborhood of a point, this notion of localization is weaker than what might be called a "pointillist" notion under which the physical quantities of interest must be localized in arbitrarily small neighborhoods of points. Such a pointillist notion of *localization* should be made distinct from pointillist notions of *separability*, which are statements about the independence of the state of a composite system with respect to the states of its subsystems. An example of the latter is Butterfield's (2011, pg. 350) "localism", which is the requirement that " … the state assigned by a physical theory to (the systems within) a spatial or spacetime region R is determined by (supervenient upon) the states assigned to the elements of a covering of R consisting of arbitrarily small open sets". Other examples of pointillist notions of separability include Myrvold's (2011, pg. 425) "patchy separability", Healey's (2007, pg. 46) "weak separability", and Belot's (1998, pg. 540) "synchronic locality".

[4] See, *e.g.*, Hastings (2012, pp. 174–5, 186). The diameter diam(X) of a set X of lattice sites can be defined by $\mathrm{diam}(X) = \max \mathrm{dist}(i, j)$, $\forall i, j \in X$, where dist(i, j) is an appropriate lattice metric.

[5] A non-contractible loop is a loop (*i.e.*, a closed curve) that cannot be continuously deformed into a point. One non-contractible loop is distinct from another if they share a point in common and one cannot be continuously deformed into the other.

[6] Under the Born rule, (3) translates into $p_\psi(a,b|A,B) \neq p_\psi(a|A)p_\psi(b|B)$, which relates the joint probability of obtaining the values a and b of A and B, respectively, in the state $|\psi\rangle$, to the marginal probabilities of obtaining these results separately.

[7] A and B are conditionally statistically independent with respect to $|\psi\rangle$ and λ just when $p_\psi(a,b|A,B,\lambda) = p_\psi(a|A,\lambda)p_\psi(b|B,\lambda)$. In the context of Bell inequalities, this is referred to as Bell locality.

[8] This provocative way of speaking is advocated by, *e.g.*, Bub (2016, p. 74).

correlated is to say they are correlated and the distance between X and Y exceeds an appropriate bound on causal signal propagation. More precisely, let dist(X, Y) be the spatial distance between X and Y. Then we require dist(X, Y) $> v\Delta t$, where v is a bound on signal propagation and Δt is a relevant time interval. Intuitively, A and B are distant correlated just when they are so far apart that their correlation cannot be due to a causal signal that propagates over the intervening distance. In more provocative terms, a distant correlation cannot be explained by a direct cause explanation. In the relativistic context, the appropriate bound v is given by the speed of light c, and one can say that A and B exhibit a distant correlation just when they are correlated and X and Y are spacelike separated. This entails no signal traveling at a speed less than or equal to c can connect X and Y. For a non-relativistic condensed matter system, an appropriate bound is a bit more nuanced. Such a system, for instance, cannot be expected to be decomposable into subsystems that are spacelike separated, simply due to its scale. Moreover, if the spatiotemporal structure of a non-relativistic condensed matter system is encoded in Galilean spacetime, then there *is* no kinematic constraint on the speed of signal propagation. On the other hand, a dynamical constraint is possible. For instance, if the dynamics of the system is governed by an exponentially decaying local Hamiltonian, then an appropriate constraint on signal propagation can be derived with the help of a Lieb–Robinson bound.[9] Given this technology, the notion of a distant correlation between physical quantities can be made coherent for both relativistic and non-relativistic systems.

One can show that (3) holds for any A, B and any state $|\psi\rangle$ *if and only if* $|\psi\rangle$ is not a product state. If an entangled state is defined as a non-product state, then any A and B are correlated with respect to an entangled state. On the other hand, the condition that $|\psi\rangle$ be an entangled state in the non-product state sense is obviously not sufficient for A and B to exhibit quantum entanglement non-locality. If $|\psi\rangle$ is entangled in the non-product state sense, then A and B are correlated with respect to $|\psi\rangle$, but not necessarily distant correlated, nor Bell inequality-violating correlated.[10]

An example of quantum entanglement non-locality is exhibited by some observables associated with a maximally entangled 2-qubit Bell state. An example of the latter is represented by the density operator,

$$\rho = (1/2)(|0_1 0_2\rangle + |1_1 1_2\rangle)(\langle 0_1 0_2| + \langle 1_1 1_2|) \quad (4)$$

where "maximally entangled" means that the reduced density operator of one of the subsystems with respect to the other is a multiple of the identity: $\rho^{(1)} = \mathrm{Tr}_2(\rho) = I/2$. The observables represented, for instance, by the local 2-qubit operators $Z_1 \otimes I_2$ and $I_1 \otimes Z_2$, where Z and I are the Pauli Z operator and the identity on $\mathbb{C}^2$, exhibit a Bell inequality-violating correlation (for certain axes of spin), and if we assume the subsystems are separated by a sufficiently large spatial distance (and that they are governed by a local Hamiltonian dynamics), then distant correlation holds, too.

Finally, a note on the ambiguity of entanglement seems appropriate at this point. As Earman (2015, pg. 305) points out, entanglement is relative to a decomposition of a system into subsystems (or more precisely, a decomposition of a system's algebra of observables into subsystem algebras). Thus a state of a composite system may be maximally entangled with respect to one decomposition, but a product state with respect to another (see, *e.g.*, Zanardi, Lidar, & Lloyd, 2004, p. 1). Obviously, for quantum entanglement non-locality to be meaningful, we need to first have identified the relevant observables of the system, and this entails among other things having identified the relevant degrees of freedom. Ascribing quantum entanglement non-locality to a system thus may become difficult if the system is associated with a set of observables that admits alternative descriptions in terms of distinct degrees of freedom (more provocatively, if the system admits incompatible yet empirically indistinguishable descriptions). As we'll see below in Section 5, an example of this occurs in a fractional quantum Hall system.

3. A possible link?

The previous section identified two distinct notions of non-locality. Topological non-locality is a particular way of violating localization. On the other hand, quantum entanglement non-locality is consistent with localization. Intuitively, the observables of interest can be localized in finite regions of space or spacetime, yet still exhibit distant correlations that violate a Bell inequality.[11] So, in general, non-locality in the sense of a violation of localization does not entail quantum entanglement non-locality. If topological quantum memory is predicated on the existence of a link between these notions of non-locality, the question then becomes, under what conditions can such a link be established?

One way such conditions might arise involves a property of quantum states that Bravyi et al. (2006, pg. 3) refer to as "topological quantum order"[12]:

Topological quantum order (ℓ-TQO). A state $|\psi_1\rangle \in \mathscr{H}^{(n)}$ has ℓ-TQO *if and only if* there is another state $|\psi_2\rangle$ orthogonal to it such that, for any ℓ-local operator O_{loc}.

(i) $\langle\psi_1|O_{loc}|\psi_2\rangle = 0$, and
(ii) $\langle\psi_1|O_{loc}|\psi_1\rangle = \langle\psi_2|O_{loc}|\psi_2\rangle$.

Condition (i) says ℓ-local operators cannot map ℓ-TQO states into each other, and condition (ii) says ℓ-local operators act on ℓ-TQO states in the same way. Both conditions are typically interpreted as saying that ℓ-TQO states cannot be locally distinguished.[13] According to Bravyi et al. (2006, p. 3), this makes ℓ-TQO states "natural candidates for protecting quantum information from decoherence". The intuition is that, since ℓ-TQO states have exactly the same local properties, by encoding information in a superposition of ℓ-TQO

[9] See, *e.g.*, Hastings (2012, pp. 176–7). Let A_X be a local operator with support on set X, and let $B_\lambda(X)$ be the set of lattice sites i such that dist(i, X) $\le \lambda$. Let $A^\lambda{}_X$ be a local operator with support on $B_\lambda(X)$. Then one form of a Lieb–Robinson bound states that $||A^\lambda{}_X(t) - A_X(t)||$ decays exponentially for $\lambda > v_{LR}t$, where time evolution is governed by a local Hamiltonian describing an exponentially decaying interaction, and v_{LR} is a constant that characterizes the latter. In other words, the support of a time-evolved local operator, with initial support on X, under an exponentially decaying local Hamiltonian is approximately restricted to lie within an effective light-cone B_λ(X), where $\lambda = v_{LR}t$ (as opposed to spreading out indefinitely). For comparison, note that in the relativistic case, the restriction is to the lightcone $B_\lambda(X)$, for $\lambda = ct$, hence it is independent of the dynamics.

[10] For instance, a classically mixed state is not a product state, but does not violate a Bell inequality. See, *e.g.*, Brunner, Cavalcanti, Pironio, Scarani, and Wehner (2014, p. 437). Of course whether or not an entangled state *simpliciter* exhibits quantum entanglement non-locality depends on how one defines an entangled state. Earman (2015) identifies four distinct notions, the weakest of which is the failure to be a product state, while the strongest involves a violation of a Bell inequality.

[11] In the algebraic formalism, for instance, localization is built into the notion of a net of local algebras of observables, yet (in the relativistic context) Bell inequality-violating correlations are endemic in typical states of the global algebra generated by the net.

[12] See, also, Bravyi, Hastings, and Michalakis (2010, pg. 6), Hastings (2011, pg. 1; 2012, pg. 186), Terhal (2015, pg. 340).

[13] Conditions (i) and (ii) define the elements of a matrix $\mathbf{M}$ with entries $M_{ij} = \langle\psi_i|O_{loc}|\psi_j\rangle$, i, $j = 1, 2$, which is a constant times the identity, $\mathbf{M} = \alpha\mathbf{I}$, where $\alpha = \langle\psi_i|O_{loc}|\psi_i\rangle$. Thus two orthogonal states have ℓ-TQO just when any ℓ-local operator projected onto the subspace they span is a multiple of the identity.

states, one protects it from decoherence due to local environmental interactions. This intuition might then suggest that, since ℓ-TQO states are locally indistinguishable, yet still distinct, they must differ in some non-local property. In particular, if local operators cannot distinguish them, they might still be distinguished by non-local operators, and in particular, by topologically non-local operators. Thus we might say that ℓ-TQO states exhibit topological non-locality just when they can only be distinguished by topologically non-local operators.[14]

The link with quantum entanglement non-locality rests on the following claim found in Preskill (1999).[15]:

Claim. Let $|\psi\rangle \in \mathscr{H}^{(n)}$ be an element of an $(\ell+1)$-distance nondegenerate quantum error correction code (QECC). Then it is maximally entangled with respect to the decomposition $\mathscr{H}^{(n)} = \mathscr{H}^{(n-\ell)} \otimes \mathscr{H}^{(\ell)}$.

A QECC is a procedure for encoding information in the elements of a subspace $\mathscr{C} \subset \mathscr{H}^{(n)}$ of an n-qubit Hilbert space in such a way that errors, represented by local operators, can be detected and corrected. In an $(\ell+1)$-distance nondegenerate QECC, the basis states of $\mathscr{C}$ have ℓ-TQO.[16] Preskill (1999, pp. 15–6) shows that if a state represented by density operator ρ is an element of such a $\mathscr{C}$, then the reduced density operator $\rho^{(\ell)}$ obtained by tracing over $n-\ell$ subsystems is a multiple of the identity: $\rho^{(\ell)} = \mathrm{Tr}_{(n-\ell)}(\rho) = I/2^{\ell}$. To demonstrate this, Preskill first notes that the ℓ-TQO property requires $\langle i|O^{(n)}|j\rangle = 0$, for n-qubit basis states $|i\rangle$, $|j\rangle$ and any ℓ-local, n-qubit operator $O^{(n)}$. One can then show that this entails $\mathrm{tr}(\rho^{(\ell)}O^{(\ell)}) = 0$, where $\rho^{(\ell)}$ is the density operator for an ℓ-qubit state, and $O^{(\ell)}$ is the restriction of $O^{(n)}$ to those ℓ-qubits on which it acts non-trivially.[17] Preskill then notes that $\rho^{(\ell)}$ can be expanded in the Pauli operator basis as $\rho^{(\ell)} = I/2^{\ell} + \sum^{3}{}_{a=1} c_a E_a$, where I is the identity, c_a is a constant, and E_a is one of the three ℓ-qubit (*i.e.*, $2^{\ell} \times 2^{\ell}$) Pauli operators X, Y, Z. Thus $\mathrm{tr}(\rho^{(\ell)}E_b) = \mathrm{tr}(E_b/2^{\ell}) + \sum_a c_a \mathrm{tr}(E_a E_b) = 2^{\ell} c_b$, where the last equality holds since $\mathrm{tr}(E_b) = 0$, and $\mathrm{tr}(E_a E_b) = 2^{\ell}\delta_{ab}$. Since the ℓ-TQO property requires this expression to vanish, one has the result $c_b = 0$ (for any b), and hence $\rho^{(\ell)} = I/2^{\ell}$.

The implication is that if an n-qubit state has ℓ-TQO, then it is maximally entangled with respect to the decomposition $\mathscr{H}^{(n)} = \mathscr{H}^{(n-\ell)} \otimes \mathscr{H}^{(\ell)}$. Recall that this means there are physical quantities that are represented by local operators associated with the bipartite decomposition $\mathscr{H}^{(n-\ell)} \otimes \mathscr{H}^{(\ell)}$, and that exhibit Bell inequality-violating correlations, and, charitably, distant correlations (assuming sufficient separation distance and an appropriately local dynamics).

Note that Preskill's result holds for any n-partite state that has the ℓ-TQO property with respect to ℓ-local, n-partite operators that can be expanded in a traceless operator basis. Certainly it holds for a composite system of qubits (*i.e.*, subsystems characterized by Hilbert spaces of the form $\mathbb{C}^D$, $D \in \mathbb{Z}$, on which act operators that can be expanded in the Pauli basis), but it is less clear if it holds for composite systems with subsystems that possess more complex degrees of freedom.[18] Thus ℓ-TQO is suggestive, but not definitive, of a link between topological non-locality and quantum entanglement non-locality. Beyond the limitations of Preskill's result, it is not definitive for two reasons:

(1) First, as noted above, the existence of ℓ-TQO states need not entail the existence of topologically non-local operators that distinguish them. In other words, ℓ-TQO states can be made distinct by operators that fail to be ℓ-local, and such operators need not be topologically non-local.

(2) Second, in cases in which ℓ-TQO states *are* made distinct by topologically non-local operators, it seems clear that the corresponding topologically non-local observables cannot be the same as those with respect to which an ℓ-TQO state is maximally entangled. These latter observables are represented by ℓ-local and $(n-\ell)$-local operators that evidently satisfy localization. Thus if a link between topological non-locality and quantum entanglement non-locality is to be forged, it remains to be shown what relation the topologically non-local observables have to the entangled ℓ-local and $(n-\ell)$-local observables. Moreover, that such entangled local observables exist for the decomposition $\mathscr{H}^{(n-\ell)} \otimes \mathscr{H}^{(\ell)}$ does not entail that they are physically relevant (*i.e.*, measurable), and not just a formal property of the system.

As we will see in the next section, the notion of an *intrinsic topologically ordered* system strengthens, but does not completely secure, this suggestive link between topological non-locality and quantum entanglement non-locality. In particular, an intrinsic topologically ordered system is characterized by ground states that possess ℓ-TQO by virtue of topologically non-local observables, and this addresses Concern (1) above. On the other hand, as we shall see, Concern (2) remains.

4. Intrinsic topological order

Systems that exhibit ITO (intrinsic topological order) are claimed to possess a type of order that cannot be characterized by a local order parameter, and undergo phase transitions that cannot be characterized by spontaneously broken symmetries. Rather, in an ITO system, order is characterized, in some sense, by topological properties.[19] The qualification "in some sense" is important, since there is no consensus in the physics literature on how best to define ITO, as some authors acknowledge (*e.g.*, Nayak, Simon, Stern, Freedman, & Das Sarma, 2008, p. 1103; Terhal, 2015, p. 340). For the purposes of this essay, the importance of ITO systems relates to how they are supposed to support topological quantum computation.[20]

A quantum computation consists of an operation performed on a set of qubits. A major concern with realizing such operations in a physical system is the fact that they typically require the system to be in an entangled state, and such states are extremely sensitive to environmental interactions (*i.e.*, "noise"). A topological quantum

[14] The phrase "topological quantum order" as defined by Bravyi et al. (2006) thus has nothing fundamentally to do with topology. Bravyi et al. (2006, pg. 3) state that it " … arises most frequently on systems with a nontrivial topology, such as a torus".

[15] See also Scott (2004, p. 3).

[16] The ℓ-TQO property is sometimes referred to as the QECC conditions (*e.g.*, Terhal, 2015, pg. 317). One can show that the ℓ-TQO property is necessary and sufficient for the existence of a QECC that detects and corrects errors represented by ℓ-local operators.

[17] Let $O^{(n)}$ be an ℓ-local, n-qubit operator; *i.e.*, $O^{(n)} = \{\otimes_{i=1}^{n-t} I_i\} \otimes \{\otimes^{\ell}{}_{j=1} O_j\}$. Let the ℓ-restriction of $O^{(n)}$ be the ℓ-qubit operator $O^{(\ell)} = \{\otimes^{\ell}{}_{j=1} O_j\}$. If ${}_{(n)}\langle k|O^{(n)}|i\rangle_{(n)} = 0$, for all n-qubit basis states $|k\rangle_{(n)}$, $|i\rangle_{(n)}$, and any ℓ-local, n-qubit operator $O^{(n)}$, then ${}_{(\ell)}\langle k|O^{(\ell)}|i\rangle_{(\ell)} = 0$, for all ℓ-qubit basis states $|k\rangle_{(\ell)}$, $|i\rangle_{(\ell)}$, and the ℓ-restriction $O^{(\ell)}$ of $O^{(n)}$. Now suppose ${}_{(n)}\langle k|O^{(n)}|i\rangle_{(n)} = 0$, where $O^{(n)}$ is an ℓ-local n-qubit operator. Let $\rho^{(\ell)}$ and $O^{(\ell)}$ be ℓ-qubit operators, where the latter is the ℓ-restriction of $O^{(n)}$. Then $\mathrm{tr}(\rho^{(\ell)}O^{(\ell)}) = \sum_i \langle i|\rho^{(\ell)}O^{(\ell)}|i\rangle = \sum_{i,j,k} c_{jk}\langle i|j\rangle\langle k|O^{(\ell)}|i\rangle = 0$, where $|i\rangle$, $|j\rangle$, $|k\rangle$ are ℓ-qubit basis states, and c_{jk} are constants.

[18] Granted, this concern may be dissolved if we restrict our investigation of potential links between topological non-locality and quantum entanglement non-locality to those systems capable of supporting topological quantum memory and/or the enterprise of topological quantum computation. On the other hand, such a restriction may be limiting when it comes to investigating potential links in the context of intrinsic topological order, if our view is that the latter is a concept that has use beyond applications to quantum computation.

[19] For more on how intrinsic topological order differs from standard notions of order, see, *e.g.*, Bain (2017).

[20] See, *e.g.*, Nayak et al. (2008) and Pachos (2012). For semi-popular accounts of topological quantum computation, see, *e.g.*, Simon (2010) and Read (2012).

computation is supposed to address this concern by encoding qubit operations in a topological feature of the system. The main theoretical candidates for this proposal are 2-dimensional composite systems that exhibit low-energy excitations that obey fractional statistics. The latter entails that when any two such excitations are repeatedly exchanged, the state of the entire system picks up a non-trivial phase that depends on the number of exchanges. Encoding qubit operations in such global "braiding" operations then, in principle, makes them immune to local environmental perturbations. One example of a physical system that exhibits these properties is a fractional quantum Hall (FQH) system.[21] An FQH system consists of a 2-dim conductor carrying a current in a strong magnetic field. At low temperatures, the longitudinal resistance (in the direction of the current) vanishes, and the transverse ("Hall") resistance is quantized in fractional factors of h/e^2, where h is Planck's constant and e is the electron charge.[22] The ground state of the system is characterized by a uniform-density quantum liquid, and low-energy excitations are predicted to obey fractional statistics. Another family of theoretical models intended to support topological quantum computation are topological quantum error correction codes, the simplest of which is Kitaev's (2003) toric code.[23] This consists of a collection of qubits arrayed on the edges of a square lattice on the surface of a torus. One can identity a set of "loop observables" with such a system (represented by operators that take the general form of (2)), and construct a corresponding Hamiltonian for which low-energy excitations take the form of anyonic quasiparticles, as in the FQH example; and one can then subsequently model qubit operations by braiding operations.

Intrinsic topological order (ITO) is supposed to be a way of characterizing what these systems all have in common. With the examples of FQH systems and the toric code in mind, the following is a list of properties that have been used by various authors to characterize an ITO system:

Intrinsic topological order (ITO). An n-body system possesses ITO just when:

(a) It exhibits a *ground state degeneracy* that depends on its topology.
(b) It exhibits *low-energy quasiparticle excitations that obey fractional statistics.*
(c) There is a finite *energy gap* between its ground states and its excited states.
(d) Its ground states exhibit *ℓ-TQO.*

Again, it should be noted that not all authors agree on all the items in this list. For instance, Wen (2013) only identifies the first three properties, whereas other authors focus on property (d).[24] Property (a) characterizes an ITO system as possessing more than one ground state, with the number depending on the topology of the system.[25] With respect to property (b), an n-particle system obeys fractional statistics just when, upon exchange of any two single-particle subsystems, the state $|\psi\rangle$ of the system changes by $|\psi\rangle \to e^{i\theta}|\psi\rangle$, where $\theta \in (0, \pi)$ is the statistical phase. Such a state is a carrier of a 1-dimensional representation of the braid group B_n.[26] In $(2+1)$-dimensions, different values of θ, so-constrained, encode the number of times two particles "wind" about each other in the process of exchange and correspond to different elements of B_n (*i.e.*, different braid transformations on an n-particle state). This "winding number" associated with particle exchanges in $(2+1)$-dim is another example of a topological property.[27]

One can now show that there is a sense in which properties (a) and (b) entail property (d), at least for the two paradigm examples of ITO systems; namely, the toric code (Kitaev, 2003), and fractional quantum Hall (FQH) systems (Wen & Niu, 1990). For these systems, the physically relevant observables are represented by loop operators (*i.e.*, operators that have support on contractible and non-contractible loops),[28] such that.

(I) ground states exhibit properties (a) and (b) with respect to non-contractible loop operators; and
(II) ground states cannot be distinguished by contractible loop operators, but can be distinguished by non-contractible loop operators.

To demonstrate Claim (I), one can show that the non-contractible loop operators commute with the system's Hamiltonian, but not among themselves. The former condition entails that the system's ground state must be an eigenvector of these operators, and the latter condition entails that there must be one such ground state for each such operator, with the number of ground states depending on the number of distinct non-contractible loops. Moreover, the commutation relations these operators satisfy depend on the statistical phase exhibited by the system's anyonic quasiparticle excitations.[29]

[21] Chen et al. (2010, pp. 1–2) also identity chiral spin liquids and Z_2 spin liquids as other physical examples, although only FQH systems have been realized experimentally. For discussions of FQH systems in the philosophical literature, see, *e.g.*, Bain (2016), Lancaster and Pexton (2015), Shech (2015), and Lederer (2015).

[22] More precisely, the Hall resistance is given by $h/e^2\nu$, where the filling factor $\nu = N/D$ gives the number of filled energy levels, where N is the number of electrons per unit area, and D is the degeneracy of states per area in a given energy level.

[23] Discussions of the toric code and its extensions can be found in Pachos (2012, pp. 83–101).

[24] Examples of the latter include Bravyi et al. (2006), Bravyi et al. (2010), Hastings (2011, 2012), and Terhal (2015). To be precise, Wen (2013, pg. 13) suggests that an ITO system can be characterized by its topology-dependent gapped ground state degeneracy, and the non-Abelian Berry phases exhibited by its ground states that encode their low-energy anyonic excitations. More recently, Zeng and Wen (2015) have proposed a characterization of ITO systems in terms of "stable gapped quantum liquids".

[25] An FQH system with filling factor $1/q$, where q is an odd integer, exhibits a q^g-fold ground state degeneracy, where g is the genus of the surface on which the system exists (Wen & Niu, 1990). The toric code exhibits a 4^g-fold ground state degeneracy (Kitaev, 2003).

[26] Bose–Einstein and Fermi–Dirac statistics are defined by $\theta = 0$ and π, respectively, and the corresponding n-particle state is a carrier of a 1-dim representation of the permutation group S_n. Note that an n-particle state can also be a carrier of a higher-dimensional representation of either B_n or S_n. In the former case, one says the system obeys non-Abelian fractional statistics; in the latter case, one says the system obeys parastatistics.

[27] Another way to make a topological connection with property (b) is to note that a $(2+1)$-dim system that exhibits low-energy quasiparticles that obey fractional statistics can be described by an effective field theory that takes the form of a topological quantum field theory (TQFT). In particular, one can transform the Lagrangian density that describes a $(2+1)$-dim non-interacting system of n-particles that exhibits fractional statistics, into a Lagrangian density that describes an interacting system of n-particles that exhibits standard statistics (Bose–Einstein or Fermi–Dirac) coupled to a Chern–Simons gauge field. The latter Lagrangian density is an example of a TQFT. In fact, some authors take the distinguishing characteristic of an ITO system to be that it admits a low-energy effective field theory in the form of a TQFT (*e.g.*, Nayak et al., 2008, p. 1107).

[28] In both examples, physical relevance is informed by gauge-invariance. In the FQH case, the physically relevant observables are represented by gauge-invariant Wilson loop operators $W_s(\gamma) = \exp[is\int_\gamma a]$ defined in terms of a loop γ, a Chern–Simons gauge field a, and a quasiparticle charge s (see, *e.g.*, Witten, 2016, p. 361). In the toric code case, the physically relevant observables are represented by Z_2-invariant loop operators that act as the Pauli X operator on lattice sites arrayed along loops, and as the Pauli Z operator on dual lattice loops (Kitaev, 2003).

[29] In the case of a torus, there are two non-contractible loop operators U_1, U_2, that satisfy the constraint $U_1U_2 = U_2U_1e^{i2\theta}$, where θ is the statistical phase. For an FQH system, $\theta = \pi/q$, where $1/q$ is the filling factor (Wen & Niu, 1990, p. 9388). For the toric code, $\theta = \pi/2$ (Kitaev, 2003, p. 10).

Claim (II) is based on the following additional properties of the loop operators:

(i) The ground state subspace is closed under the actions of loop operators.
(ii) Contractible loop operators act trivially on ground states.
(iii) Non-contractible loop operators act non-trivially on ground states.

Conditions (i) and (ii) entail the ground states cannot be distinguished by contractible loop operators. Conditions (i) and (iii) entail that ground states can be distinguished by non-contractible loop operators.

One might be tempted to conclude from Claims (I) and (II) that ITO ground states possess TQO, and this is due to the presence of topologically non-local observables (represented by non-contractible loop operators). One might then appeal to Preskill's result to conclude that ITO ground states must be entangled. Hence, on first blush, it appears as if the topological non-locality of an ITO system entails its quantum entanglement non-locality. But this rush to judgment is a bit too fast for four reasons.

First, we are still faced with the limitation of Preskill's result to composite qubit systems mentioned in Section 3.

Second, in our rush, we've only addressed Concern (1) of Section 3; *i.e.*, establishing a link between topological non-locality and the ℓ-TQO property. Concern (2) remains; *i.e.*, we are no closer to understanding the relation between the topologically non-local observables with respect to which ITO ground states exhibit ℓ-TQO, and the local observables with respect to which ITO ground states, as ℓ-TQO states, are maximally entangled.

Third, strictly speaking, the ℓ-TQO property introduced in Section 3 was defined in terms of ℓ-local operators; *i.e.*, operators with support on sets of lattice sites of diameter $\ell < L$, where L is the size of the system. If ITO ground states are claimed to possess ℓ-TQO in virtue of their indistinguishability with respect to contractible loop operators, then the latter must be deemed ℓ-local. On the one hand, the notion of localization adopted in Section 2 is general enough to allow this; for instance, one might consider the support of an ℓ-local contractible loop operator to be the area (of diameter ℓ) enclosed by the loop. On the other hand, a received view among philosophers and physicists deems observables represented by loop operators as non-local. Care should be taken here since there are two senses of non-locality associated with this received view. One sense is associated with separability; *i.e.*, a general requirement that the state of a composite system be independent of the states of its subsystems. Thus interpretations of local gauge theories that take the observables to be represented by holonomies (*i.e.*, loop operators defined in terms of gauge potentials) are typically deemed non-local in the sense of violating some version of separability (*e.g.*, Belot, 1998; Healey, 2007; Myrvold, 2011). However, the issue of whether a loop operator is ℓ-local involves the notion of localization, and not separability. With respect to localization, the concern is that the relevant loop observables for ITO systems are metric-independent.[30] This entails they cannot be associated with a notion of localization that depends on a distance relation (like the diameter of a finite region of space, for instance).

In light of this, one might be tempted to back away from the claim that contractible loop operators are ℓ-local. One might, for instance, allow that contractible loop operators are non-local and replace the ℓ-TQO property (d) of ITO ground states with a weaker property that only requires ITO ground states to be distinguishable up to contractible loop operators, as opposed to ℓ-local operators. But this puts Preskill's result in jeopardy. Recall that this result implies that an ℓ-TQO state $|\psi\rangle \in \mathscr{H}^{(n)}$ is maximally entangled with respect to the decomposition $\mathscr{H}^{(n)} = \mathscr{H}^{(n-\ell)} \otimes \mathscr{H}^{(\ell)}$. This requires the notion of ℓ-locality insofar as it is based on assigning a value of ℓ to the range of the operators that fail to distinguish the relevant states (codewords in a quantum error correction code, or ground states in an ITO system). Thus in weakening ℓ-local indistinguishability to contractible loop indistinguishability, we lose the connection between topological non-locality and quantum entanglement non-locality based on Preskill's result.[31]

In any event, there is an additional complication, and this is the fourth reason to be cautious about seeing a link between topological non-locality and quantum entanglement non-locality in ITO systems based simply on Claims (I) and (II). This complication is the fact that the gap property (c) entails that ITO ground state correlations between local observables decay exponentially, and this seems to put the entangled nature of these states into question. The complication is due to the following:

Exponential Clustering Theorem. Let $|\psi\rangle$ be a ground state with gap Δ of a reasonably local non-relativistic Hamiltonian, and let A, B be local observables with support on disjoint sets X, Y. Then

$$|\langle\psi|AB|\psi\rangle - \langle\psi|A|\psi\rangle\langle\psi|B|\psi\rangle| \leq C(\Delta)e^{-\mu \mathrm{dist}(X,Y)} \qquad (5)$$

where $\mu > 0$ is a constant, $\mathrm{dist}(X,Y)$ is the distance between X and Y, and $C(\Delta)$ is a constant that depends on the gap Δ.[32]

Thus, for a reasonably local non-relativistic gapped Hamiltonian, ground state correlations of local observables with support on disjoint sets decay exponentially as a function of separation distance. The implication is that ITO ground states, being gapped, approximate product states across sufficiently separated sets of lattice sites. This entails ITO ground states cannot be entangled in the non-product sense across sufficiently separated subsystems. This creates an apparent conundrum: Where did the quantum entanglement non-locality go that was supposed to underwrite the performance of ITO systems as topological quantum memories? Of course the obvious response is that it was never there to begin with.

Some authors, however, insist on finding a notion of quantum entanglement non-locality in ITO systems. Their intuition is voiced by Chen et al. in the context of a fractional quantum Hall system:

In FQH systems, the correlation of any local operators are short ranged. This seems to imply that FQH states are 'short sighted' and they cannot know the topology of space which is a global and long-distance property. However, the fact that ground-state degeneracy does depend on the topology of space implies that FQH states are

[30] In an FQH system, this is expressed by the fact that the relevant loop observables (footnote [22]) $W_s(\gamma)$ are invariant under continuous deformations of the loop γ. In other words, $W_s(\gamma) = W_s(\gamma')$ *iff* γ and γ' can be continuously deformed into each other (see, *e.g.*, Witten, 2016, p. 361). More generally, metric independence is expressed by the fact that an ITO system can be characterized by a low-energy effective field theory that takes the form of a topological quantum field theory (footnote [21]).

[31] There is still some wiggle room for an advocate of the claim that ITO systems exhibit both topological and quantum entanglement non-locality. One could, for instance, maintain that, while contractible loop operators are ℓ-local (and hence, Preskill's result holds for ℓ-local indistinguishable ITO ground states), the observables they represent are not: One might claim that the representation relation is a many-one map from equivalence classes of ℓ-local contractible loop operators to metric-independent loop observables.

[32] This theorem is due to Nachtergaele and Sims (2006). A "reasonably local" Hamiltonian is a sum of local operators that describes an exponentially decaying interaction. This secures the use of an appropriate Lieb–Robinson bound in the proof of the theorem. The particular form of the inequality (5) is for the case of a unique ground state; a slightly modified form also holds for degenerate ground states. The theorem is a non-relativistic version of a similar clustering theorem in relativistic quantum field theory due to Fredenhagen (1985).

not short sighted and they do find a way to know the global and long-distance structure of space. So, despite the short-range correlations of any local operators, the FQH states must contain certain hidden long-range correlation (Chen et al.,2010, p. 2.).

Similarly, in a discussion of the toric code, Kitaev states:

We see that the ground state degeneracy depends on the surface topology … On the other hand, there is a finite energy gap between the ground state and excited states, so all spatial correlation functions decay exponentially. This looks like a paradox–how do parts of a macroscopic system know about the topology if all correlations are already lost as small scales? The answer is that there is long-range entanglement which cannot be expressed by simple correlation functions (Kitaev, 2003, p. 7.).

In other words, "hidden long-range correlation" or "long-range entanglement" is required to explain the topology-dependent degeneracy of FQH and toric code (and by implication, ITO) ground states, in the absence of an explanation underwritten by ground state correlations between local observables. With this as motivation, Chen, et al. construct an argument that claims that ITO ground states are "long-range entangled", and to this I now turn.

5. ITO and long-range entanglement

Chen et al. (2010) claim that an ITO ground state is "long-range entangled". By the latter, they mean the following (Chen et al., 2010, p. 4):

Short-/Long-Range Entanglement. A non-product state $|\psi\rangle$ is *short-range entangled* (SRE) *if and only if* $|\psi\rangle = U|\psi_{\text{prod}}\rangle$, where U is a local unitary evolution, and $|\psi_{\text{prod}}\rangle$ is a product state. Otherwise, $|\psi\rangle$ is *long-range entangled* (LRE).

Two aspects of this definition perhaps need a bit more clarification. First, a *local unitary evolution* is a unitary transformation generated by a local Hamiltonian over a finite time. More precisely, $U = T\{\exp[i\int_0^1 dg\tilde{H}(g)]\}$ where T is the path-ordering operator, and $\tilde{H}(g) = \Sigma_i O_i(g)$ is a sum of ℓ-local operators (Chen et al.2010, pp. 3–4). Second, it will become relevant in the following to recall that the notion of a product state of a composite system is relative to a decomposition of the system into subsystems. Thus, to say a state is short-/long-range entangled is to say it is short-/long-range entangled with respect to a particular decomposition of the composite system into subsystems.

Before assessing Chen et al.'s claim that ITO ground states are LRE, the importance of this claim in the literature should perhaps be noted. Chen et al. go on to use LRE as the basis for a general theory of ordered systems, claiming that long-range entanglement is what distinguishes intrinsic topologically ordered systems from other types of ordered systems, both those described by the Landau–Ginsburg theory of local order parameters and spontaneously broken symmetries, as well as other types of non-Landau–Ginsburg order similar to, but distinct from, intrinsic topological order (what are called "symmetry-protected topological orders"). Wen (2013) claims that long-range entanglement is the microphysical mechanism that underlies intrinsic topological order. Lancaster and Pexton (2015) have appealed to long-range entanglement as the physical mechanism that warrants a description of fractional quantum Hall systems as exhibiting emergence. Finally, a significant body of literature has arisen over how best to quantify long-range entanglement, including the notion of topological entanglement entropy (*e.g.*, Kitaev & Preskill, 2006; Levin & Wen, 2006).

Now on to the argument. Note, first that Chen et al.'s claim that ITO ground states are LRE is distinct from the discussion in Section 3 of ℓ-TQO and its relation to quantum entanglement. There, it was noted that a state that has ℓ-TQO must be a maximally entangled state (with respect to an ℓ ⊗ $(n-\ell)$ decomposition). Chen et al.'s claim is a bit different; namely, they claim that a ground state of a system that has ITO must be a *long-range* entangled state, where, assumedly, ℓ-TQO is necessary but not sufficient for a state to be an ITO ground state.

To forge a link between ITO ground states and long-range entanglement, Chen, et al. first prove the following Lemma (Chen et al., 2010, pp. 26–7):

Lemma. Two gapped ground states $|\psi_0\rangle$, $|\psi_1\rangle$ are in the same quantum phase if and only if $|\psi_0\rangle = U|\psi_1\rangle$, where U is a local unitary evolution.

Apart from the definition of a quantum phase, the proof of the Lemma depends essentially on the gapped property of the states.[33] The motivation for introducing the notion of a quantum phase is the assumption that distinct ITO systems are in distinct quantum phases. For example, fractional quantum Hall systems (again, the paradigm of ITO systems) characterized by different filling factors are assumed to be in distinct quantum phases of matter.

By the definition of short-range entanglement, a gapped ground state that is SRE can be mapped to a product state by a local unitary evolution. Technically, the latter is an adiabatic transformation (i.e., one that does not affect the energy of the state it acts on); thus, under a local unitary evolution, an SRE gapped ground state can be mapped into a product gapped ground state. Thus, according to the Lemma, an SRE gapped ground state is in the same quantum phase as a product gapped ground state. Chen et al. now state:

Since a direct-product state is a state with trivial topological order [i.e., ITO], we see that a state with a short-range entanglement also has trivial topological order. This leads us to conclude that a nontrivial topological order is related to long-range entanglement (Chen et al., 2010, pg. 4.).

The argument can be reconstructed in the following way:

1. An SRE gapped ground state and a product gapped ground state belong to the same quantum phase. (Definition of SRE and Lemma.)
2. If two ground states are in the same quantum phase, then if one is not an ITO ground state, neither is the other.[34]
3. A product state with respect to an ℓ ⊗ (n−ℓ) decomposition does not have ℓ-TQO, hence it is not an ITO ground state (Preskill, 1999 and definition of ITO.).
4. Therefore, an SRE gapped ground state with respect to an ℓ ⊗ (n−ℓ) decomposition cannot be an ITO ground state.

The conclusion can be restated as, if a state is an ITO ground state, then (since the latter are gapped), it cannot be SRE with respect to an ℓ ⊗ (n−ℓ) decomposition. If it is not SRE, then it is either a product state with respect to an ℓ ⊗ (n−ℓ) decomposition, or it is LRE. Since an ITO ground state cannot be a product state with respect to an ℓ ⊗ (n−ℓ) decomposition, it must therefore be LRE with respect to an ℓ ⊗ (n−ℓ) decomposition.

As it stands, this reconstruction is certainly a valid argument; but just what it says about the relation between topological non-locality and quantum entanglement non-locality in ITO systems is less clear. Here are four concerns.

[33] Chen et al. (2010, p. 3) define a quantum phase in the following way. Let $H(g)$ be a local Hamiltonian that depends smoothly on a parameter g, and let $|\psi(g)\rangle$ be a ground state of $H(g)$. The system described by $H(g)$ undergoes a quantum phase transition at $g = g_c$ just when the ground state expectation value $\langle\psi(g)|O_{loc}|\psi(g)\rangle$ of some local operator O_{loc} has a non-analyticity at g_c in the thermodynamic limit. A quantum phase is an equivalence class of ground states $|\psi(g)\rangle$ of all local Hamiltonians $H(g)$ that can be connected by a smooth adiabatic path that exhibits no quantum phase transitions.

[34] This follows from the definition of a quantum phase (see previous footnote).

The first concern is simply that the appeal to long-range entanglement only seeks to address one of the problems outlined in Section 4 of viewing the topological non-locality of an ITO system as entailing its quantum entanglement non-locality; namely, the problem associated with the exponential clustering theorem. Long-range entangled observables are supposed to explain how it is possible for there to be correlations present in a gapped system that exhibits topological non-locality (in particular, topology-dependent ground state degeneracy). Regardless of the success of this explanation, there still remain the first three problems of Section 4; namely, the limitations of Preskill's result, the question of how an ITO's topological observables relate to its quantum entangled observables, and issues related to treating loop operators and the observables they represent as local.

A second concern is over the nature of the correlations that an ITO ground state, as an LRE state, is supposed to exhibit. It cannot be the case that an ITO ground state exhibits distant correlations between local observables that violate a Bell inequality. Simply put, the exponential clustering theorem entails that there can be no correlations between local observables in an ITO ground state for sufficiently large separation distance (regardless of whether such correlations are, in addition, distant correlations that violate a Bell inequality). So the fact that an ITO ground state is LRE must be indicative of another type of correlation, perhaps between (topologically) non-local observables represented by non-contractible loop operators. But the nature of this type of correlation, and in particular the extent to which it is a distant correlation that violates a Bell inequality, and hence supports quantum entanglement non-locality, remains murky. In particular, if an essential characteristic of a topological observable is metric-independence, then there can be no separation distance between distinct topological observables, and hence such observables cannot be distant correlated.

A third concern stems from an interplay between the conventionality of fractional statistics and the ambiguity of entanglement. Recall that an ITO system is characterized, in part, by low-energy excitations that obey fractional statistics. One can show that in 2-dimensions, a composite system whose subsystems exhibit fractional statistics can be redescribed as a composite system whose subsystems exhibit standard statistics and are coupled to a Chern–Simons (CS) gauge field (see, e.g., Fradkin, 2013, Chapter 10). In the latter case, the CS interaction can be interpreted as attaching some number of magnetic fluxes to each subsystem, and this effectively serves to modify the fractional statistics of the subsystems. Depending on the strength of the CS interaction, one can change the statistics to either Bose–Einstein or Fermi–Dirac statistics. A prime example of this is a fractional quantum Hall system. In the low-energy sector in which such a system displays the relevant observables, the system admits three alternative descriptions[35]:

(1) A system of strongly interacting electrons coupled to an external magnetic field.
(2) A system of non-interacting composite fermions coupled to a modified external magnetic field.
(3) A system of non-interacting composite bosons in the absence of an external magnetic field.

A composite fermion is an electron with an even number of attached magnetic fluxes, and a composite boson is an electron with an odd number of attached magnetic fluxes. In accounts (2) and (3), the attached fluxes are described in terms of an internal coupling with a CS gauge field, and they serve to both modify the external magnetic field[36] and change the fractional statistics of account (1) to Fermi–Dirac and Bose–Einstein, respectively. Importantly, all three accounts reproduce the relevant observables, and in this sense are empirically equivalent. Only the first electron account, however, involves degrees of freedom that obey fractional statistics. Moreover, while the electron account (A) exhibits the properties of an ITO system, the composite particle accounts (2) and (3) do not.[37]

One implication of these examples is that, for a fractional quantum Hall system, the relevant observables fail to determine if the system exhibits ITO. More simply put, the relevant observables fail to determine a unique decomposition of the system's state space, and hence fail to determine if its ground states are entangled (let along long-range entangled). Advocates of ITO might respond by claiming that the decomposition of the system into strongly interacting electron subsystems (as opposed to non-interacting composite particle subsystems) is more fundamental, and hence the ITO account is to be preferred. But if the concern is over the physical significance of intrinsic topological order, and in particular, the physical significance of long-range entanglement, then claiming fundamentality for the electron decomposition seems to beg the question. Note finally that, given the conventionality of fractional statistics, this type of concern should extend from fractional quantum Hall systems to systems claimed to exhibit ITO in general, including the toric code.

A fourth and final concern is with Chen et al.'s motivation for introducing the notion of LRE. The exponential clustering theorem entails that ITO ground state correlations between local observables decay exponentially. This complicates an explanation of the presence of non-local topological observables (that encode the topology-dependent ground state degeneracy) in terms of local observables. Chen et al.'s solution is to argue that the ground states are LRE, and this, assumedly, means that while local observables can only possess short-range correlations, they are nevertheless related in a "hidden" non-local way. But why seek an explanation of non-local observables in terms of local observables in the first place? One possible reason is an implicit assumption that an understanding of a phenomenon must be underwritten by a mechanistic (al beit perhaps non-causal) explanation. Wen (2013, pg. 14) for instance, views LRE as the microphysical mechanism that underwrites ITO. The concern then is that there are alternative accounts of explanation that do not make non-local observables as mysterious as perhaps mechanism-based accounts.

In a bit more detail, a mechanistic explanation of a phenomenon seeks an account of the phenomenon in terms of a microphysical mechanism; i.e., a particular collection of entities and activities that are organized in such a way that they realize a regularity, law,

[35] The relevant observables are the vanishing of the longitudinal resistance, and plateaus in the Hall resistance (see, *e.g.*, Bain, 2016, p. 30).

[36] In the composite fermion case, they reduce it to values associated with the integer quantum Hall effect, while in the composite boson case, they completely nullify it.

[37] For instance, it is not clear if non-contractible loop operators with the attendant properties can be constructed for the composite particle accounts, and hence it is not clear if the ground states of the latter are topologically degenerate and possess ℓ-TQO. (In fact, insofar as account (2) describes an FQH system as an integer quantum Hall system of composite fermions, and integr quantum Hall systems have unique ground states, account (2) does, too.) Moreover, Shi (2004, p. 6815) has argued that the ground states of the composite accounts cannot be entangled. Shi observes that the composite particle accounts admit single-particle densities that exhibit off-diagonal long-range order (ODLRO), and argues that ODLRO is best understood in terms of disentanglement of the condensate mode from other modes. In contrast, the electron single-particle density does not exhibit ODLRO.

principle, etc. (Weber et al.,2013, pg. 59).[38] In Chen et al.'s view, assumedly, an FQH system consists of electron subsystems interacting with an external magnetic field, and its topology-dependent ground state degeneracy is explained in terms of the long-range entanglement of these electrons. This constitutes a mechanistic explanation insofar as it explains a phenomenon (topology-dependent ground state degeneracy) in terms of a microphysical mechanism; namely, a collection of entities and activities (electrons and their long-range entanglement) that are organized in such a way that they realize a regularity/law/principle; the latter as encoded in the electron FQH Hamiltonian, say. In contrast, a structural explanation of a phenomenon seeks an account of the phenomenon " … by showing how the (typically mathematical) structure of the theory itself limits what sort of objects, properties, states, or behaviors are admissible within the frame-work of that theory, and then showing that the explanandum is in fact a consequence of that structure" (Bokulich, 2011, pg. 40). On this account, the topology-dependent ground state degeneracy of an FQH system is explained in terms of the topological structure of the system's lattice: This lattice is a non-simply connected surface that takes the form of a torus with two or more distinct families of non-contractible loops (depending on how many "holes" the torus has), and it is this topological feature of the system that explains why its ground states are degenerate. Thus, under a structural explanation, the topological nature of the system has explanatory efficacy, and it needs no further explanation in terms of a microphysical mechanism.

My point here is not to argue in favor of structural explanations over mechanistic ones in general. Rather, my point is just to note that Chen et al. owe us a reason for their preference for a mechanistic explanation of the topological properties of ITO systems, not just because there are alternative accounts available, but also because their preferred mechanistic account faces the other concerns outlined above.

6. Conclusion

This essay has considered three attempts to find a link between topological non-locality and quantum entanglement non-locality: an attempt based on the ℓ-TQO property, an attempt based on the properties of an ITO system, and an attempt based on the claim that ITO ground states are long-range entangled. I've argued that all three attempts are limited in their scope. The attempt based on the ℓ-TQO property faces three limitations:

(i) Preskill's (1999) result that ℓ-TQO states are maximally entangled is limited to composite qubit systems (and their extensions).
(ii) ℓ-TQO states need not be topologically distinguishable.
(iii) For ℓ-TQO states that are topologically distinguishable, there is no reason to expect a link between the topological observables responsible for distinguishing ℓ-TQO states on the one hand, and the distinct quantum entangled observables that Preskill's result entails, on the other hand.

The attempt based on the properties of an ITO system addresses (ii), but (i) and (iii) remain. In addition, this attempt faces the following limitations:

(iv) To apply Preskill's result to an ITO system requires addressing the issue of the locality of contractible loop operators.
(v) The exponential clustering theorem for gapped ground states entails there can be no correlations between local observables on scales needed to explain the topological features of an ITO system.

Finally, the attempt based on the claim that ITO ground states are long-range entangled is intended to address (v), but (i), (iii), and (iv) remain. In addition, this attempt faces the following questions:

(vi) In what sense do long-range entangled observables exhibit distant Bell-inequality-violating correlations?
(vii) What privileges the decomposition of an ITO composite system with respect to which ITO observables can be considered long-range entangled?
(viii) Why require a local mechanistic explanation of topologically non-local observables, as opposed to a structural explanation?

These considerations suggest that, while topological non-locality can be made distinct from quantum entanglement non-locality, and ITO systems can unproblematically be said to exhibit topological non-locality, the extent to which they exhibit quantum entanglement non-locality remains unclear.

[38] Perhaps it should be noted that there is a second way of understanding a mechanism; namely, as a general physical process, or organizing principle, that can be instantiated by any of a number of distinct microphysical processes (mechanism in the first sense).

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