Recursive Undecidability and Incompleteness of $\mathcal{N}$

Prop: If $\mathcal{N}$ is consistent and recursively decidable, then any recursively enumerable subset of $\mathbb{N}$ is recursive.

Proof: Show that if $\mathcal{N}$ is consistent and recursively decidable, it can be used to construct a method that decides membership in any recursively enumerable subset of $\mathbb{N}$.

Suppose A is a recursively enumerable subset of $\mathbb{N}$.

Then: There is a recursive function f such that $A = \{n \in \mathbb{N} : n = f(m),\ m \in \mathbb{N}\}$.

Thus: Since f is recursive and therefore expressible in $\mathcal{N}$, there is a *wf* $\mathcal{A}(x_1, x_2)$ such that
(i) If $f(m) = n$, then $\vdash_{\mathcal{N}} \mathcal{A}(0^{(m)}, 0^{(n)})$ *or* If $\nvdash_{\mathcal{N}} \mathcal{A}(0^{(m)}, 0^{(n)})$, then $f(m) \neq n$
(ii) If $f(m) \neq n$, then $\vdash_{\mathcal{N}} \sim\mathcal{A}(0^{(m)}, 0^{(n)})$ *or* If $\nvdash_{\mathcal{N}} \sim\mathcal{A}(0^{(m)}, 0^{(n)})$, then $f(m) = n$

Now: If $\mathcal{N}$ is recursively decidable, then for any *wf* $\mathcal{A}$, there is an effective method that determines if $\mathcal{A}$ is or is not a theorem (by Church's Thesis).

So: (i) and (ii) determine, for any $n \in \mathbb{N}$, if n is or is not in the set A. For any n, n is not in A just when a certain *wf* is not a theorem of $\mathcal{N}$. And n is in A just when the negation of this *wf* is not a theorem.

Note: If $\mathcal{N}$ is consistent, then this method will always work: there will be no n that both *is* and *is not* in A, since for any *wf* $\mathcal{A}$, we cannot have both $\nvdash_{\mathcal{N}} \mathcal{A}$ and $\nvdash_{\mathcal{N}} \sim\mathcal{A}$.

Corollary: If $\mathcal{N}$ is consistent, then it cannot be recursively decidable.

Proof: Suppose $\mathcal{N}$ is consistent and recursively decidable.

Then: Any recursively enumerable set is recursive (above Prop).

But: K is a recursively enumerable set that is not recursive. (Prop. 7.30.)

Alternative Proof of recursive undecidability of $\mathcal{N}$:

Suppose $\mathcal{N}$ is consistent and recursively decidable.

Now: Enumerate all *wfs* of $\mathcal{L}_{\mathcal{N}}$ with one free variable: $\mathcal{A}_0(x)$, $\mathcal{A}_1(x)$, ...

Next: Define a 1-place relation D on $\mathbb{N}$ by:

$D(n)$ holds *iff* $\vdash_{\mathcal{N}} \sim\mathcal{A}_n(0^{(n)})$

Then: Since $\mathcal{N}$ is assumed to be recursively decidable, there is an effective method that determines if $D(n)$ holds; namely, $D(n)$ holds if and only if the *wf* $\sim\mathcal{A}_n(0^{(n)})$ is a theorem of $\mathcal{N}$.

So: By Church's Thesis, D is recursive.

Thus: D is expressible in $\mathcal{N}$, say by the *wf* $\mathcal{A}^D(x)$ such that
(i) If $D(n)$ holds, then $\vdash_{\mathcal{N}} \mathcal{A}^D(0^{(n)})$
(ii) If $D(n)$ doesn't hold, then $\vdash_{\mathcal{N}} \sim\mathcal{A}^D(0^{(n)})$

Now: $\mathcal{A}^D(x)$ must appear in the list of *wfs* with one free variable, say $\mathcal{A}^D(x) = \mathcal{A}_m(x)$, $\in \mathbb{N}$.

So: For the case $n = m$, we have:
(1) *If* $D(m)$ holds, *then* $\vdash_{\mathcal{N}} \sim\mathcal{A}_m(0^{(m)})$ (definition of D)
(2) *If* $\vdash_{\mathcal{N}} \sim\mathcal{A}_m(0^{(m)})$, *then* $D(m)$ holds (definition of D)
(3) *If* $D(m)$ holds, *then* $\vdash_{\mathcal{N}} \mathcal{A}_m(0^{(m)})$ (expressibility of D (i))
(4) *If* $D(m)$ doesn't hold, *then* $\vdash_{\mathcal{N}} \sim\mathcal{A}_m(0^{(m)})$ (expressibility of D (ii))

Now: If $\mathcal{N}$ is consistent, then (1) and (3) entail $D(m)$ cannot hold.

But: If $D(m)$ doesn't hold, then (2) entails $\nvdash_{\mathcal{N}} \sim\mathcal{A}_m(0^{(m)})$, whereas (4) entails $\vdash_{\mathcal{N}} \sim\mathcal{A}_m(0^{(m)})$.

So: If $\mathcal{N}$ is consistent, it cannot be recursively decidable.

Prop. If a first order system S is complete, then it is recursively decidable.

Proof: Suppose S is complete.

Then: If S is inconsistent, it is recursively decidable (the set of S-theorems will be identical to the set of all *wfs*, which is recursively decidable).

So: Suppose S is consistent.

Then: The following is an effective method to determine if a *wf* $\mathcal{A}$ is a theorem of S:

1. Enumerate the theorems of S.
2. Search list until either $\mathcal{A}$ or $\sim\mathcal{A}$ is found.
3. If $\mathcal{A}$ is found, it is a theorem of S. If $\sim\mathcal{A}$ is found, $\mathcal{A}$ is not a theorem of S (by completeness).

Thus: By Church's Thesis, the characteristic function for the set of G-numbers of S-theorems is recursive; hence the set of G-numbers of S-theorems is recursive; hence S is recursively decidable.

Comment: Recall that the set of theorems of any (recursively axiomatizable) first order system S is recursively enumerable. And this entails that, for any *wf* $\mathcal{A}$, if $\mathcal{A}$ is an S-theorem, then it will occur somewhere in the list of theorems. But if $\mathcal{A}$ is *not* an S-theorem, no effective search of the list will halt. If S is complete, then one can search for either $\mathcal{A}$ or $\sim\mathcal{A}$; and such a search is guaranteed to halt eventually.