Important topics in Chapter 7

Church's Thesis. A partial function is computable by algorithm *iff* it is a recursive partial function.

Def. A set is **effectively enumerable** if there is an effective procedure by which its members may be listed.

Def. 7.7. A subset of $\mathbb{N}$ is **recursively enumerable** if it is the range of a recursive function, or if it's empty.

Comment: Church's Thesis entails "effectively enumerable" is equivalent to "recursively enumerable".

Prop. The set of theorems of $\mathcal{N}$ is recursively enumerable.

Prop. 7.8. Every recursive set is recursively enumerable.

Def. 7.11. A class of questions is **recursively unsolvable** if there is no single algorithm which provides answers for all the questions in that class (assuming Church's Thesis holds).

Props. 7.20, 7.21. The set of Turing machines may be effectively enumerated T_0, T_1, ... in such a way that each suffix determines effectively and completely the instructions for the corresponding machine.

Def. 7.23. A (partial) function defined on $\mathbb{N}$ is **Turing computable** if there is a Turing machine which computes its values.

Turing's Thesis. A partial function is computable by algorithm *iff* it is Turing computable.

Prop. 7.25. A partial function defined on $\mathbb{N}$ is Turing computable *if and only if* it is a recursive partial function. (This makes Turing's Thesis equivalent to Church's Thesis.)

Prop. 7.29. The Halting Problem for Turing machines is unsolvable; *i.e.*, there is no algorithm which provides answers to questions from the set $\{$does machine T_m halt with input n? $\mid m, n \in \mathbb{N}\}$.

Prop. 7.30. The set $K = \{n \in \mathbb{N} : T_n$ halts with input $n\}$ is recursively enumerable but not recursive.

Def. 7.9. A formal system is **recursively undecidable** if the set of Gödel numbers of its theorems is not recursive.

Some results:

(i) The following systems are recursively decidable:
- (a) The formal system of statement calculus L.
- (b) The formal system of predicate calculus $K_{\mathcal{L}}$, where $\mathcal{L}$ contains no function letters or constants, and only one-place predicate letters.
- (c) First order arithmetic $\mathcal{N}$ without multiplication.

(ii) The following systems are recursively undecidable:
- (a) The formal system of predicate calculus $K_{\mathcal{L}}$, where $\mathcal{L}$ contains at least on two-place function letter, one two-place predicate letter, and an infinite list of constants.
- (b) First order arithmetic $\mathcal{N}$ (provided $\mathcal{N}$ is consistent).
- (c) First order set theory ZF.