Prop. (Gödel's Second Theorem)
Let S be a consistent recursively axiomatizable extension of $\mathcal{N}$. There is no proof in S of the consistency of S.

Proof Outline

1. Construct a *wf* $\mathcal{C}$ of $\mathcal{L}_{\mathcal{N}}$ that asserts that $\mathcal{N}$ is consistent.
2. Demonstrate $\nvdash_{\mathcal{N}} \mathcal{C}$.
3. This demonstrates the Prop for $\mathcal{N}$. For any recursively axiomatizable extension of $\mathcal{N}$, a corresponding version of $\mathcal{C}$ can be constructed.

Step 1.

Consider the *wf* $A_1^2(a_1, f_1^1(a_1))$ of $\mathcal{L}_{\mathcal{N}}$, abbreviated by $0 = 0'$.

Note: $\vdash_{\mathcal{N}} \sim(0 = 0')$ (axioms N1, K5)

So: $\mathcal{N}$ is consistent *iff* $\nvdash_{\mathcal{N}} 0 = 0'$.

Now: On $\mathbb{N}$, there's a recursive relation $\text{Pf}(m, n)$ that holds just when m is the G-number of a proof of the *wf* whose G-number is n. It is expressed in $\mathcal{N}$ by the *wf* $\mathcal{P}f(x_1, x_2)$.

Thus: The assertion $\nvdash_{\mathcal{N}} 0 = 0'$ can be expressed in $\mathcal{N}$ by the *wf* $\sim(\exists x_1)\mathcal{P}f(x_1, 0^{(g(0 = 0'))})$. This *wf* expresses the consistency of $\mathcal{N}$; call it $\mathcal{C}$. (In N, it says: "There is no $m \in \mathbb{N}$ such that m is the G-number of a proof of the *wf* $0 = 0'$." In other words, "There is no proof in $\mathcal{N}$ of the *wf* $0 = 0'$.")

Step 2.

Lemma: $\vdash_{\mathcal{N}} \mathcal{C} \rightarrow \mathcal{U}$, where $\mathcal{U}$ is the Gödel Sentence for $\mathcal{N}$.

Immediate Consequence: If $\vdash_{\mathcal{N}} \mathcal{C}$, then $\vdash_{\mathcal{N}} \mathcal{U}$. But $\nvdash_{\mathcal{N}} \mathcal{U}$ (by the Incompleteness Theorem). So $\nvdash_{\mathcal{N}} \mathcal{C}$.

Proof of Lemma:

Notation: Let $\exists\mathcal{P}(\mathcal{A})$ abbreviate the *wf* $(\exists x_1)\mathcal{P}f(x_1, 0^{(g(\mathcal{A}))})$. ("There's a proof in $\mathcal{N}$ of the *wf* $\mathcal{A}$.") $\mathcal{C}$ is then abbreviated by $\sim\exists\mathcal{P}(0 = 0')$.

Claim: The following hold. For any *wfs* $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{L}_{\mathcal{N}}$,

I: If $\vdash_{\mathcal{N}} \mathcal{A}$, then $\vdash_{\mathcal{N}} \exists\mathcal{P}(\mathcal{A})$
II: $\vdash_{\mathcal{N}} \exists\mathcal{P}(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\exists\mathcal{P}(\mathcal{A}) \rightarrow \exists\mathcal{P}(\mathcal{B}))$
III: $\vdash_{\mathcal{N}} \exists\mathcal{P}(\mathcal{A}) \rightarrow \exists\mathcal{P}(\exists\mathcal{P}(\mathcal{A}))$
IV: If $\vdash_{\mathcal{N}} \mathcal{A} \rightarrow \mathcal{B}$, then $\vdash_{\mathcal{N}} \exists\mathcal{P}(\mathcal{A}) \rightarrow \exists\mathcal{P}(\mathcal{B})$ (from I and II)

Ex. Proof of I. Suppose $\vdash_{\mathcal{N}} \mathcal{A}$. Then $Pf(m, g(\mathcal{A}))$ holds in $\mathbb{N}$ for some $m \in \mathbb{N}$. So $\vdash_{\mathcal{N}} \mathcal{P}f(0^{(m)}, 0^{(g(\mathcal{A}))})$. Hence $\vdash_{\mathcal{N}} (\exists x_1)\mathcal{P}f(x_1, 0^{(g(\mathcal{A}))})$ (since $\vdash_{\mathcal{N}} \mathcal{A}(t) \rightarrow (\exists x_i)\mathcal{A}(x_i)$, for t free for x_i in $\mathcal{A}(x_i)$).

Now:	$\vdash_{\mathcal{N}} \mathcal{U} \leftrightarrow \sim\exists\mathcal{P}(\mathcal{U})$	(defintion of $\mathcal{U}$)
So:[1]	$\vdash_{\mathcal{N}} \exists\mathcal{P}(\mathcal{U}) \rightarrow \exists\mathcal{P}(\sim\exists\mathcal{P}(\mathcal{U}))$	(IV and definition of $\leftrightarrow$)
Note:[2]	$\vdash_{\mathcal{N}} \exists\mathcal{P}(\mathcal{U}) \rightarrow \exists\mathcal{P}(\exists\mathcal{P}(\mathcal{U}))$	(III)
Note:[3]	$\vdash_{\mathcal{N}} \sim\exists\mathcal{P}(\mathcal{U}) \rightarrow (\exists\mathcal{P}(\mathcal{U}) \rightarrow (0 = 0'))$	(since $\sim\mathcal{A} \rightarrow (\mathcal{A} \rightarrow \mathcal{B})$ is logically valid)
So:[4]	$\vdash_{\mathcal{N}} \exists\mathcal{P}(\sim\exists\mathcal{P}(\mathcal{U})) \rightarrow \exists\mathcal{P}(\exists\mathcal{P}(\mathcal{U}) \rightarrow (0 = 0'))$	(IV)
Note:[5]	$\vdash_{\mathcal{N}} \exists\mathcal{P}(\exists\mathcal{P}(\mathcal{U}) \rightarrow (0 = 0'))$ $\rightarrow (\exists\mathcal{P}(\exists\mathcal{P}(\mathcal{U})) \rightarrow \exists\mathcal{P}(0 = 0'))$	(II)
So:[6]	$\vdash_{\mathcal{N}} \exists\mathcal{P}(\mathcal{U}) \rightarrow (\exists\mathcal{P}(\exists\mathcal{P}(\mathcal{U})) \rightarrow \exists\mathcal{P}(0 = 0'))$	(HS, 1, 4, 5)
Thus:	$\vdash_{\mathcal{N}} \exists\mathcal{P}(\mathcal{U}) \rightarrow \exists\mathcal{P}(0 = 0')$	(2, 6, and $\{\mathcal{A} \rightarrow \mathcal{B}, \mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})\} \vdash_{\mathcal{N}} \mathcal{A} \rightarrow \mathcal{C}$)
Or:	$\vdash_{\mathcal{N}} \sim\exists\mathcal{P}(0 = 0') \rightarrow \sim\exists\mathcal{P}(\mathcal{U})$	
Or:	$\vdash_{\mathcal{N}} \mathcal{C} \rightarrow \mathcal{U}$	